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Natural Heat Convection in a Vertical Circular Tube*

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Pure natural heat convection fields in the entrance region of a heated vertical circular tube, in which the temperature and velocity profiles were developing simultaneously, were solved by a numerical method of finite difference. And then the characteristics of heat transfer were obtained. The analysis presented here was based on the assumption that the inlet velocity profile is uniform.

It was verified analytically and numerically that, for the case of uniform wall temperature, the inlet velocity U_1 has an asymptotic value $1/8$ as the tube length approaches infinity. As for the case of uniform heat flux at the wall, U_1 is directly proportional to $\sqrt{L/P_r}$ over the range $L > L_e$, where L_e is the entrance length. The relationship between the dimensionless pipe length L and dimensionless suction velocity U_1 was also obtained, using $P_r = 0.72$, for both the wall conditions.

In the case that the pipe length is semi-infinite and the wall temperature is uniform, a relationship between P_r and thermal, as well as hydrodynamic, entrance length was obtained for $P_r = 0.01, 0.1, 0.72, 1.0, 10$ and 100 , and some interesting irregularities of developing velocity profiles were also observed except for the case $P_r = 1.0$.

1. Introduction

When a circular tube which is held vertically in a stationary fluid is heated, a buoyancy field is formed in the tube. The fluid in it is able to flow owing to the buoyancy, independently of the pressure difference between the inlet and the outlet. Both the fields of buoyancy and velocity are affected by each other, therefore the governing differential equations become simultaneous equations.

All of the existing solutions for laminar flow heat transfer in a circular tube are based on the following assumptions.

(a) One method is to solve the only energy equation on the assumption of either an established parabolic velocity profile or a uniform velocity profile throughout the tube. The solutions based on a parabolic profile are associated either with such fluids that Prandtl number is much higher than unity or with the case that heating begins sufficiently far from the entrance. It may be inferred⁽¹⁾ that the usefulness of the solution which was obtained by Graetz⁽²⁾ on the assumption that the velocity profile and wall temperature are uniform, is restricted to such fluids that Prandtl number is much lower

than unity. Recently, J. R. Sellars, M. Tribus and J. S. Klein⁽³⁾ solved the classical Graetz problem for several wall heating conditions.

(b) The second method of solving the energy equation in the entrance region is to employ the velocity profile which has ever been obtained on the assumption that the flow is isothermal. This method is applicable solely to the forced convection flows of high flow rate, because the effect of buoyancy is neglected. W. M. Kays⁽⁴⁾, P. Goldberg⁽⁵⁾ and D. H. Ulrichson⁽⁶⁾ have used Langhaar's⁽⁷⁾ velocity distribution and R. K. McMordie and A. F. Emery⁽⁸⁾ have done so taking into account the effect of axial heat conduction. For fluids of which physical properties

Nomenclature

r_0, d, l : radius, diameter and length of tube
 u, v : axial velocity, radial velocity
 t : temperature
 $\vartheta = t - t_1, \Delta\vartheta = t_w - t_1$
 p : absolute pressure
 c_p : specific heat at constant pressure
 λ : thermal conductivity
 α : thermal diffusivity
 ρ : mass density
 α : heat transfer coefficient
 P_r : Prandtl number
 N_u : Nusselt number
 G_r : Grashof number

Subscripts

1: tube inlet
 w : tube wall
 ∞ : ambient fluid
 x : based on axial distance from $x=0$

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vary significantly with temperature, P. M. W. Schmidt and G. Leppert⁽⁹⁾ have solved the temperature field on the assumptions that the inlet velocity profile is parabolic and the properties, such as specific heat, thermal conductivity and viscosity can be taken as power laws.

(c) For the case that the flow involves an internal heat generation, T. M. Hallman⁽¹⁰⁾, E. M. Sparrow and R. Siegel⁽¹¹⁾ have analyzed the characteristic of heat transfer in a vertical circular tube under the condition of forced and free convection being combined, assuming that the heat generation per unit volume is uniform.

A number of experimental results of this problem have been published. Velocity profile and temperature profile in the entrance region were investigated experimentally by Shimizu⁽¹²⁾ and Hishida⁽¹³⁾ respectively. G. A. Kemeny and E. V. Somers⁽¹⁴⁾ obtained Nusselt number and pipe friction coefficient using water and oils, and W. G. Brown⁽¹⁵⁾ investigated the flow under the condition of forced and free convection in combination.

All of these published theoretical solutions were worked out solely for the cases of either a forced convection flow or a buoyancy-affected forced convection flow. No solution for the pure natural convection flow of which characteristics are affected remarkably by buoyancy is found. The purpose of the present study is to solve the fundamental equations of the pure natural convection flow in a vertical circular tube using such a numerical method that J. R. Bodoia and J. F. Osterle⁽¹⁶⁾ employed for the pure natural convection flow between two vertical flat plates. The fields of velocity and temperature were solved simultaneously for two wall condition, uniform wall heat flux and uniform wall temperature, on the assumption that the flow is steady. Consequently, both the local and mean Nusselt numbers, velocity entrance length, thermal entrance length and a relationship between the inlet velocity U_1 and the tube length L are obtained.

In the case that the tube length is semi-infinite and the wall temperature is uniform, the dimensionless inlet (suction) velocity U_1 is proved analytically to become $1/8$ independently of the Prandtl number P_r . As for this case, calculation is worked out varying P_r as a parameter over the range from 0.01 to 100. In the case of finite length the problem was solved only for $P_r=0.72$, varying the suction velocity U_1 as a parameter. All of the solutions of this paper are based on the assumptions that the inlet edge of the tube is bell-mouthed and the inlet velocity profile is uniform, but solely for the case of infinite length, using $P_r=0.72$, the inlet velocity profile was assumed to be parabolic.

2. Theoretical analysis

2.1 Fundamental equations and boundary conditions

The purpose of this paper is to obtain the combined velocity and temperature fields in a uniformly heated vertical circular tube and to obtain the heat transfer characteristics of the flow on the following assumptions.

(1) The fluid enters the pipe with ambient temperature and uniform velocity profile. (2) The viscous dissipation terms can be neglected. (3) All physical properties of the fluid are constant, except for the density in the buoyancy term.

Let the origin be chosen at the lowest point of the tube, and let the x -axis be chosen upward along the center line of the tube, as shown in Fig. 1. Then the ambient pressure $p_{\infty x}$ at any point x is expressed as follows.

$$p_{\infty x} = p_{\infty 0} - \rho_1 g x \quad \dots\dots\dots (1)$$

The conservation equations for momentum, energy and continuity are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -g - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} \right) \quad \dots\dots\dots (2)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial x^2} - \frac{v}{r^2} \right) \quad \dots\dots\dots (3)$$

$$\rho g c_p \left(u \frac{\partial t}{\partial x} + v \frac{\partial t}{\partial r} \right) = \lambda \left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial x^2} \right) \quad \dots\dots\dots (4)$$

$$\frac{\partial(vr)}{r \partial r} + \frac{\partial u}{\partial x} = 0 \quad \dots\dots\dots (5)$$

In this study the following three cases are investigated, and the boundary conditions for each

Fig. 1 Configuration of a vertical tube

case are given as follows.

(a) The length is semi-infinite and the wall temperature is uniform. (This case is hereinafter referred to as the case of semi-infinite length.)

$$\begin{aligned} x \geq 0, r = r_0 & ; u = 0, v = 0, t = t_w \\ x \geq 0, r = 0 & ; \partial u / \partial r = 0, \partial t / \partial r = 0, v = 0 \\ x = 0, 0 \leq r < r_0 & ; u = u_1, v = 0, t = t_1 \\ x = 0 & ; p = p_{\infty 0} \end{aligned}$$

(b) The length is finite and the wall temperature is uniform. (The case of finite length.)

It may fairly be presumed⁽¹⁷⁾ that at the tube outlet the inner pressure is equal to the ambient one, because the flow is pure natural convection. For this case the following boundary condition also must be added.

$$x = l ; \int_0^{r_0} p r dr = p_{\infty 1} r_0^2 / 2$$

(c) The length is finite and the wall heat flux is uniform. (The case of uniform heat flux.)

$$\begin{aligned} x \geq 0, r = r_0 & ; u = 0, v = 0, (\partial t / \partial r)_{r=r_0} = q_w / \lambda \\ x \geq 0, r = 0 & ; \partial u / \partial r = 0, v = 0, \partial t / \partial r = 0 \\ x = 0, 0 \leq r < r_0 & ; u = u_1, v = 0, t = t_1 \\ x = 0 & ; p = p_{\infty 0} \\ x = l & ; \int_0^{r_0} p r dr = p_{\infty 1} r_0^2 / 2 \end{aligned}$$

Gravity field is converted into the buoyancy field by the following definition

$$p' = p - p_{\infty x} \quad (6)$$

Combination of equations (1) and $\rho_1 / \rho = 1 + \beta(t - t_1)$ leads to

$$-g - \frac{1}{\rho} \frac{\partial p}{\partial x} = g\beta(t - t_1) - \frac{1}{\rho} \frac{\partial p'}{\partial x} \quad (7)$$

2.2 Dimensionless forms of the equations

The following expressions are employed for each case.

(a) The case of semi-infinite length

$$G_{r,i} = \frac{g\beta\Delta t r_0^3}{\nu^2}, \quad R = \frac{r}{r_0}, \quad X = \frac{x}{G_{r,i} r_0},$$

$$U = \frac{u r_0}{G_{r,i} \nu}, \quad V = \frac{v r_0}{\nu}, \quad \Theta = \frac{\vartheta}{\Delta t},$$

$$P = \frac{p' r_0^2}{\rho G_{r,i}^2 \nu^2}, \quad X^* = \frac{X}{P_r}, \quad R_{a,i} = P_r G_{r,i}$$

(b) The case of finite length

The following are different from the case (a).

$$G_{r,f} = \frac{g\beta\Delta t r_0^4}{l\nu^2}, \quad X = \frac{x}{G_{r,f} l}, \quad U = \frac{u r_0^2}{G_{r,f} \nu l},$$

$$P = \frac{p' r_0^4}{\rho G_{r,f}^2 \nu^2 l^2}, \quad R_{a,f} = P_r G_{r,f}$$

(c) The case of uniform heat flux

The following are different from the case (a).

$$G_{r,*} = \frac{g\beta q_w r_0^4}{\lambda \nu^2}, \quad X = \frac{x}{G_{r,*} r_0}, \quad U = \frac{u r_0}{G_{r,*} \nu},$$

$$\Theta = \frac{\vartheta \lambda}{q_w r_0}, \quad P = \frac{p' r_0^2}{\rho G_{r,*}^2 \nu^2}$$

When the notation G_r corresponds to the Grashof

numbers of all cases, the fundamental equations for all cases are expressed similarly in such dimensionless forms as follows, independently of boundary conditions.

$$\begin{aligned} U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial R} \\ = \Theta - \frac{\partial P}{\partial X} + \left(\frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} + \frac{1}{G_r^2} \frac{\partial^2 U}{\partial X^2} \right) \end{aligned} \quad (8)$$

$$\begin{aligned} U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial R} \\ = -G_r^2 \frac{\partial P}{\partial R} + \left(\frac{\partial^2 V}{\partial R^2} + \frac{1}{R} \frac{\partial V}{\partial R} - \frac{V}{R^2} \right) \\ + \frac{1}{G_r^2} \frac{\partial^2 V}{\partial X^2} \end{aligned} \quad (9)$$

$$\begin{aligned} U \frac{\partial \Theta}{\partial X} + V \frac{\partial \Theta}{\partial R} \\ = \frac{1}{P_r} \left(\frac{\partial^2 \Theta}{\partial R^2} + \frac{1}{R} \frac{\partial \Theta}{\partial R} \right) + \frac{1}{P_r G_r^2} \frac{\partial^2 \Theta}{\partial X^2} \end{aligned} \quad (10)$$

$$\frac{\partial(VR)}{R \partial R} + \frac{\partial U}{\partial X} = 0 \quad (11)$$

The boundary conditions are also expressed in dimensionless form.

$$X \geq 0, R = 1 ; U = 0, V = 0 \quad (12)$$

$$\text{for the cases (a) and (b)} ; \Theta = 1 \quad (12)$$

$$\text{for the case (c)} ; \partial \Theta / \partial R = 1 \quad (13)$$

$$X \geq 0, R = 0 ; \partial U / \partial R = 0, \partial \Theta / \partial R = 0, V = 0 \quad (14)$$

$$X = 0, 0 \leq R < 1 ; U = U_1, V = 0, \Theta = 0 \quad (15)$$

$$X = 0 ; P = 0 \quad (16)$$

For the cases (b) and (c) the following condition is also necessary.

$$X = L ; \int_0^1 P R dR = 0 \quad (17)$$

2.3 Integration of the fundamental equations

Integrating Eq. (8) with respect to R , holding X fixed, and using Eq. (11) and the boundary conditions, leads the following

$$\begin{aligned} \frac{d}{dX} \int_0^1 R U^2 dR \\ = \int_0^1 R \Theta dR - \frac{d}{dX} \int_0^1 R P dR + \left(\frac{\partial U}{\partial R} \right)_{R=1} \end{aligned} \quad (18)$$

Similarly, Eqs. (10) and (11) become,

$$\begin{aligned} \frac{d}{dX} \int_0^1 R U \Theta dR \\ = \frac{1}{P_r} \left(\frac{\partial \Theta}{\partial R} \right)_{R=1} + \frac{1}{P_r G_r^2} \frac{d^2}{dX^2} \int_0^1 R \Theta dR \end{aligned} \quad (19)$$

$$\frac{d}{dX} \int_0^1 R U dR = 0 \quad (20)$$

Equation (20) integrates directly with $U = U_1$ at $X = 0$ to

$$\int_0^1 R U dR = \frac{U_1}{2} \quad (20)'$$

If $U = 2U_2(1 - R^2)$ at $X = 0$, Eq. (20) integrates to

$$\int_0^1 RU dR = \frac{U_1^2}{2} \dots\dots\dots(20)''$$

Integrating Eqs. (18) and (19) with respect to X from 0 to X , leads to;

$$\int_0^1 R(U^2 + P) dR = \frac{U_1^2}{2} + \int_0^X \left\{ \int_0^1 R \theta dR + \left(\frac{\partial U}{\partial R} \right)_{R=1} \right\} dX \dots\dots(21)$$

$$\int_0^1 RU \theta dR = \frac{1}{Pr} \int_0^X \left(\frac{\partial \theta}{\partial R} \right)_{R=1} dX + \frac{1}{Pr Gr^2} \frac{d}{dX} \int_0^1 R \theta dR \dots\dots(22)$$

2.4 Analysis of established flow region

The velocity profile is uniform at the inlet, but as the flow progresses, it varies with the axial position. Farther downstream, it may be assumed that the velocity profile no longer varies, namely, it has a convergent profile. Let us analyze the characteristics of this region on several assumptions.

(a) The case of uniform wall temperature

Now, we define the established region (or fully developed region) as the region where the following expressions are satisfied.

$$\frac{\partial U}{\partial X} = 0, \quad \frac{\partial \theta}{\partial X} = 0 \dots\dots\dots(23)$$

Substitute Eq. (23) into Eq. (11) to obtain

$$V = 0 \dots\dots\dots(24)$$

Substitute Eq. (24) into Eq. (9) to obtain

$$\frac{\partial P}{\partial R} = 0 \dots\dots\dots(25)$$

Substituting the equations $\partial \theta / \partial X = 0$ and $V = 0$ into Eq. (10), the established temperature profile in this region is obtained as follows.

$$\theta = 1 \dots\dots\dots(26)$$

Introducing all of these results into Eqs. (8) and (18), gives respectively

$$\frac{d^2 U}{dR^2} + \frac{1}{R} \frac{dU}{dR} = \frac{dP}{dX} - 1 \dots\dots\dots(27)$$

$$\frac{1}{2} \frac{dP}{dX} = \int_0^1 R \theta dR + \left(\frac{dU}{dR} \right)_{R=1} = C \dots\dots\dots(28)$$

The right side of Eq. (28) can be written as a constant C , because each term of it has been found to be a constant. Substituting Eq. (28) into Eq. (27), and then solving the equation (27) for U , it is found as follows.

$$U = (1/4 - C/2)(1 - R^2) \dots\dots\dots(29)$$

Substitute Eq. (29) into Eq. (20)' to obtain

$$C = 1/2 - 4U_1 \dots\dots\dots(30)$$

Assuming the thermal coefficient of volume expansion is positive, it results that $U_1 > 0$. Hence, from Eq. (30) the following inequality is obtained.

$$C < 1/2 \dots\dots\dots(31)$$

Now, we consider the value of C written by Eq. (30), first for the case of semi-infinite length.

Value of the dimensionless inlet velocity U_1 is uniquely determined by a given value of C from Eq. (30). And it may be presumed that the position X_e where the established region begins is determined finitely by that value of U_1 . And then, Eq. (28) reduces to $dP/dX = 2C$ in the region $X > X_e$, hence, if $C \neq 0$, then it will result that P increases or decreases indefinitely as X approaches infinity. Since this result contradicts the basic condition of natural convection flow, it must follow that

$$C = (1/2)(dP/dX) = 0 \dots\dots\dots(32)$$

Finally, for the case of semi-infinite length U_1 is determined from Eqs. (30) and (32) as follows.

$$U_1 = 1/8 \dots\dots\dots(33)$$

Combination of Eqs. (28) and (32) leads to

$$\int_0^1 R \theta dR + (dU/dR)_{R=1} = 0$$

This expression shows the fact that in this region the buoyancy force balances with pipe friction force.

As for the case of finite length, U_1 varies with the dimensionless pipe length, so let it be expressed as follows.

$$U_1 = f(L); \quad 0 < L < \infty$$

Again, let us consider the value of C for this case on the assumption that the function $f(L)$ is continuous all over the range. By solving numerically the case of semi-infinite length, and using the result (33), it was proved that the dimensionless modified pressure P varies with X -axis in such a pattern as a curve (1) in Fig. 2. According to this curve, if U_1 is allowed to increase by an infinitesimal amount from 1/8, then from Eq. (30) it will result that $C < 0$, in other words, the curve of P will not be able to have any point crossed by X -axis. One of the boundary conditions requires the point of P which meets X -axis, so the result mentioned above is not acceptable, and finally, U_1 can not be larger than 1/8. Hence,

$$U_1 < 1/8 \dots\dots\dots(34)$$

Finally, it is proved that in this case C must be greater than zero, namely, the curve of P should be such a curve (2) as shown in Fig. 2, and the position L where the curve meets X -axis can be regarded as the outlet of the tube.

Let mean Nusselt number be defined by the

Fig. 2 Variation of modified pressure

formula

$$N_u = \frac{\alpha x}{\lambda} = \frac{\int_0^x (\partial t / \partial r)_{r=r_0} dx}{\Delta \vartheta} \\ = G_r \int_0^X \left(\frac{\partial \theta}{\partial R} \right)_{R=1} dX \quad (35)$$

The limit value of N_u , as X approaches ∞ , is obtained from Eq. (22) as follows.

$$N_{u\infty} = P_r G_r \int_0^1 R U \theta dR = P_r G_r / 16 \quad (36)$$

(b) The case of uniform heat flux

Applying the boundary condition (13), Eq. (22) is reduced to

$$\int_0^1 R U \theta dR = \frac{1}{P_r} X + \frac{1}{P_r G_r^2} \frac{d}{dX} \int_0^1 R \theta dR \quad (37)$$

Hence, the dimensionless mixed mean temperature θ_b is obtained as follows.

$$\theta_b = \frac{\int_0^1 R U \theta dR}{\int_0^1 R U dR} \\ = \frac{2}{U_1 P_r} \left(X + \frac{1}{G_r^2} \frac{d}{dX} \int_0^1 R \theta dR \right) \quad (38)$$

Now, we presume that in this case $(\partial \theta / \partial R)$ has maximum value of its own at $R=1$, at any X . Then, the following inequality is obtained.

$$\theta_t = R + (\theta_w - 1)$$

where θ_t is a tangent to the temperature profile's curve at $R=1$, and θ_w is the dimensionless wall temperature. From this it follows that

$$\int_0^1 R \{R + (\theta_w - 1)\} dR < \int_0^1 R \theta dR < \int_0^1 R \theta_w dR$$

Hence, it results

$$\frac{1}{2} \theta_w - \frac{1}{6} < \int_0^1 R \theta dR < \frac{1}{2} \theta_w \quad (39)$$

Similarly, obtain the following inequality

$$\int_0^1 R U \{R + (\theta_w - 1)\} dR < \int_0^1 R U \theta dR \\ < \int_0^1 R U \theta_w dR$$

and the result is;

$$\theta_w - 1 < \frac{2}{U_1} \int_0^1 R U \theta dR < \theta_w \quad (40)$$

Combination of Eqs. (39) and (40) leads to

$$U_1 \left(1 - \frac{1}{\theta_w} \right) < \frac{\int_0^1 R U \theta dR}{\int_0^1 R \theta dR} < U_1 \left(\frac{1}{1 - 1/3\theta_w} \right)$$

The dimensionless wall temperature θ_w increases with X , and in the region where it is much greater than unity, this inequality transforms to

$$\int_0^1 R U \theta dR = U_1 \int_0^1 R \theta dR \quad (41)$$

Substituting Eqs. (20)' and (41) into Eq. (38), and then solving this equation for $\int_0^1 R \theta dR$, it is

found as

$$\int_0^1 R \theta dR = \frac{C_1}{U_1 P_r G_r^2} e^{U_1 P_r G_r^2 X} + \frac{1}{U_1 P_r} X + C_2 \quad (42)$$

where C_1 and C_2 are integration constants.

Neglecting the axial heat conduction, Eq. (42) reduces to

$$\int_0^1 R \theta dR = \frac{1}{U_1 P_r} X + C_2 \quad (43)$$

In the range of great X , an expression $U = U_1 \times g(R)$ may be derived from the relation $\partial U / \partial X \ll \partial U / \partial R$. Therefore, the following expression is obtained.

$$-\left(\frac{\partial U}{\partial R} \right)_{R=1} \propto U_1 \quad (44)$$

The left sides of Eqs. (43) and (44) express the forces of buoyancy and pipe friction, respectively. And these terms should be equal, because the flow is pure natural convection. Hence, there results

$$U_1 \propto \frac{1}{U_1 P_r} X + C_2$$

Approaching X to infinity in this expression, it is derived that

$$X \propto U_1^2 P_r \quad (45)$$

Finally, it is proved that, when the tube length is sufficient, the suction velocity U_1 increases in proportion to $\sqrt{L}$ and decreases in proportion to $\sqrt{P_r}$.

3. Numerical solution

A numerical method of finite difference is applied to the problem. Being transformed into finite difference equations, the fundamental equations come to contain the Grashof number G_r in the form of $1/G_r^2$. When an established region exists, it is obvious from the preceding analysis that U_1 is determined independently of G_r , therefore, it should be presumed that both the axial and radial step-widths may be determined independently of G_r . Therefore, all terms containing $1/G_r^2$ vanish as G_r becomes greater than such a value that is determined solely by the Prandtl number P_r . Assuming that G_r is great enough, Eq. (9) is reduced to $\partial P / \partial R = 0$, and Eqs. (8) and (10) are transformed in such forms that contain no terms of $1/G_r^2$.

Finally, the fundamental equations are simplified to

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial R} = \theta - \frac{dP}{dX} + \left(\frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} \right) \quad (46)$$

$$U \frac{\partial \theta}{\partial X} + V \frac{\partial \theta}{\partial R} = \frac{1}{P_r} \left(\frac{\partial^2 \theta}{\partial R^2} + \frac{1}{R} \frac{\partial \theta}{\partial R} \right) \quad (47)$$

$$\frac{\partial(VR)}{R \partial R} + \frac{\partial U}{\partial X} = 0 \quad (48)$$

In these expressions there is not the momentum equation for R -direction, so V can not be contained

as an unknown. Assuming V as a known value, and using Eq. (20)' instead of Eq. (48), the simultaneous Eqs. (46) and (47) can be solved.

Transformation into a finite difference form is carried out for the obtained algebraic simultaneous equations to be one-order. Let the axial step-width and the radial step-width be ΔX and ΔR respectively as shown in Fig. 3, let the value at the representative mesh point $P_{j,k}$ be denoted by a suffix (j, k) , where j and k are integers and the coordinates of the point $P_{j,k}$ are $X = j \Delta X$, $R = k \Delta R$, and let the tube radius be divided into an even number $2n$. Then, the Eqs. (46), (47) and (48) reduce to

$$f_1(j, k)U(j+1, k-1) + f_2(j, k)U(j+1, k) + f_3(j, k)U(j+1, k+1) - \theta(j+1, k) + P(j+1, k)/\Delta X = f_4(j, k) \quad (49)$$

$$f_5(j, k)\theta(j+1, k-1) + f_7(j, k)\theta(j+1, k) + f_8(j, k)\theta(j+1, k+1) = f_6(j, k) \quad (50)$$

$$V(j+1, k+1) = \frac{k-1}{k}V(j+1, k)$$

$$-\frac{2k-1}{4k} \frac{\Delta R}{\Delta X} \{U(j+1, k) + U(j+1, k+1) - U(j, k) - U(j, k+1)\} \quad (51)$$

The Eq. (20)' reduces to

$$U(j+1, 1)R(1) + U(j+1, 2n+1)R(2n+1) + 4 \sum_{k=1}^n U(j+1, 2k)R(2k) + 2 \sum_{k=1}^{n-1} U(j+1, 2k+1) \times R(2k+1) = 3U_1/(2\Delta R) \quad (52)$$

where the coefficients R , f_1 , f_2 , ... are given as follows.

$$\left. \begin{aligned} R(k) &= (k-1)\Delta R \\ f_1(j, k) &= -\frac{V(j, k)}{2\Delta R} - \frac{1}{(\Delta R)^2} + \frac{1}{2R(k)\Delta R} \\ f_2(j, k) &= \frac{U(j, k)}{\Delta X} + \frac{2}{(\Delta R)^2} \\ f_3(j, k) &= \frac{V(j, k)}{2\Delta R} - \frac{1}{(\Delta R)^2} - \frac{1}{2R(k)\Delta R} \\ f_4(j, k) &= \frac{\{U(j, k)\}^2}{\Delta X} + \frac{P(j, k)}{\Delta X} \\ f_5(j, k) &= \frac{U(j, k)\theta(j, k)}{\Delta X} \\ f_6(j, k) &= -\frac{V(j, k)}{2\Delta R} - \frac{1}{P_r} \frac{1}{(\Delta R)^2} \\ &\quad + \frac{1}{P_r} \frac{1}{2R(k)\Delta R} \\ f_7(j, k) &= \frac{U(j, k)}{\Delta X} + \frac{1}{P_r} \frac{2}{(\Delta R)^2} \\ f_8(j, k) &= \frac{V(j, k)}{2\Delta R} - \frac{1}{P_r} \frac{1}{(\Delta R)^2} \\ &\quad - \frac{1}{P_r} \frac{1}{2R(k)\Delta R} \end{aligned} \right\} \quad (53)$$

Once all values at every mesh point up to j^{th} axial row are known, then the unknowns at $(j+1)^{\text{th}}$

axial row are all found by solving $4n+5$ simultaneous equations which are obtained by varying k , from 1 to $2n$, the Eqs. (49), (50) and (52). Then, $V(j+1, k)$ are found by Eq. (51), namely, all the unknowns at every mesh point of $(j+1)^{\text{th}}$ axial row are solved. By propagating the solution forward from one axial row to the next in such a step-by-step fashion as mentioned in the foregoing, both the fields of velocity and temperature can be solved throughout the tube.

For the case of semi-infinite length, since the dimensionless mixed mean temperature θ_b has been known by the preceding analysis to converge to unity as X approaches infinity, the calculation was extended until it reached 0.984. As for the case of finite length, just as in the case of uniform heat flux, it was extended until P become zero.

The terms containing $1/R$ comes to be without meaning at $R=0$, so the following formula⁽¹⁸⁾ was employed at this point.

$$\lim_{R \rightarrow 0} \{(1/R)(\partial/\partial R)\} = \partial^2/\partial R^2$$

The temperature gradient at $R=1$, which is necessary at each row to evaluate the heat transfer coefficient, is approximated by a tangent to a quadratic curve which is fit for the numerical results at $R=0.8, 0.9$, and 1.0 . And, the velocity gradient is also approximated by the same method. These can be expressed in such a computing form as

$$\left(\frac{\partial Y}{\partial R}\right)_{R=1} = -\frac{1}{2} \frac{1}{\Delta R} \{4Y(j+1, 2n) - Y(j+1, 2n-1) - 3Y(j+1, 2n+1)\} \quad (54)$$

The simultaneous equations associated with each row were solved by the inverted matrix elimination method, and all the integrations were calculated by Simpson's rule.

The radial step-width ΔR was specified at $1/10$ or $1/20$, while the axial step-width ΔX was specified at such 4 kinds values that increased with every

Fig. 3 Mesh network for difference equations

40 axial steps. For example, in the case of uniform heat flux, for $U_1=0.304$, it was specified at 10^{-5} , 10^{-4} , 10^{-3} and 10^{-2} in order.

The local Nusselt number, the mean Nusselt number and the dimensionless pipe friction coefficient are defined by the following formulas respectively.

$$N_{ux} = \alpha_x d / \lambda$$

for the case of uniform wall temp.

$$= \frac{\lambda(\partial t / \partial r)_{r=r_0}}{t_w - t_b} \frac{d}{\lambda} = \frac{2(\partial \Theta / \partial R)_{R=1}}{1 - \Theta_b} \quad \dots\dots(55)$$

for the case of uniform heat flux

$$= \frac{q}{t_{wx} - t_b} \frac{d}{\lambda} = \frac{2}{\Theta_{wx} - \Theta_b} \quad \dots\dots(56)$$

$$N_u = \alpha x / \lambda$$

In the case of uniform wall temperature, from Eqs. (35) and (22)

$$= P_r G_r \int_0^1 RU \Theta dR \quad \dots\dots(57)$$

For the case of uniform heat flux

$$N_u = \frac{\alpha d}{\lambda} = \frac{q_w}{t_{wx} - t_1} \frac{d}{\lambda} = \frac{2}{\Theta_{wx}} \quad \dots\dots(58)$$

$$F = \frac{2\tau_w}{\rho u_1^2} G_r = -2 \frac{(\partial U / \partial R)_{R=1}}{U_1^2} \quad \dots\dots(59)$$

4. Numerical results

4.1 The case of semi-infinite length

In this case, the Prandtl number P_r was varied as a parameter, and the solution was based on the assumptions that the inlet velocity profile and the

Fig. 4 Variation of local Nusselt number for the case of semi-infinite length, uniform wall temperature

Fig. 5 Variation of mean Nusselt number for the case of semi-infinite length, uniform wall temperature

wall temperature are uniform.

Variation of local Nusselt number N_{ux} is shown in Fig. 4. From this, it is noted that all the curves converge to 3.62 independently of P_r , as $X^* \rightarrow \infty$, whose convergent value is closely related to Kays' result for a forced convection flow. Variation of mean Nusselt number is, in the form of N_u/R_{a1} , shown in Fig. 5. All of the curves have a convergent value 6.25×10^{-2} at $X^* \approx 10^{-1}$, which agrees with the analytical result of Eq. (36). The broken curve in it is associated with the case of parabolic velocity profile at the tube inlet. Though this curve lies below the others all over the range, it has the same convergent value.

Figure 6 shows variation of P . It is noted from this that every curve converges. When $P_r \geq 1$, it converges at $X^* \approx 0.145$, while when $P_r < 1$, at the greater X^* , the lower P_r is. Since this phenomenon is associated with the entrance lengths, it will be explained in the succeeding discussion.

Relationship between P_r and the velocity entrance lengths L_{ev} , as well as the thermal entrance length L_{et} , is plotted in Fig. 7. The velocity

Fig. 6 Variation of modified pressure for the case of semi-infinite length, uniform wall temperature

Fig. 7 Relationship between P_r and entrance lengths for the case of uniform wall temperature

entrance length is defined by a distance from the inlet to where the numerical solution of $U(j+1, 3)$ reaches the analytical solution 0.24 of it. As for the thermal entrance length, it is defined by a distance from the inlet to where $\theta(j+1, 3)$ reaches 0.98. The curve of L_{et} is plotted in a straight line, whose slope is $1/2$, and may be expressed by a formula $L_{et} = (7/50)P_r$. The curve of L_{ev} is plotted in such a shape that, over the range $P_r \geq 1$, it meets the line of L_{et} , and, as P_r approaches zero, it converges to a limit. This phenomenon of convergency may be attributed to the fact that, when $P_r < 1$, since the temperature field develops more rapidly than the velocity field, in the region close to the inlet, the temperature field is able to become an almost established profile, and hence the flow mode comes to resemble an isothermal flow. The entrance lengths of isothermal flow have already been obtained by a number of investigators⁽¹⁹⁾, which are shown in broken lines in Fig. 7. It is noted that our convergent value is close to Bousinesq's solution. One other result that, when $P_r \geq 1$, both the entrance lengths are equal may be attributed to the following fact. In this case, though the velocity field develops more rapidly than the temperature field, the former, when it has developed

Fig. 9 Developing fields of axial velocity and temperature for the case of semi-infinite length, uniform wall temperature, $P_r=0.72$

Fig. 8 Developing fields of axial velocity and temperature for the case of semi-infinite length, uniform wall temperature, $P_r=0.01$

Fig. 10 Developing fields of axial velocity and temperature for the case of semi-infinite length, uniform wall temperature, $P_r=100$

to some extent, is affected by the latter which is developing more slowly. Hence, it is not possible for the velocity field individually to develop fully. Finally, it is required that the temperature field be fully developed for the purpose that the velocity field is established, and it results that both the entrance lengths are equal for $P_r \geq 1$. Consequently, the distance from the inlet to where P converges comes to be equal to the velocity entrance length.

Three typical examples of the developing velocity field, as well as the temperature field, are shown in Figs. 8, 9 and 10. From these, it is noted that they are essentially different, according as $P_r < 1$ or $P_r = 1$ or $P_r > 1$. These differences are shown in the sketch in Fig. 11.

Figure 11 (a) is for $P_r > 1$. In this case, the velocity field develops more rapidly than the other, so, in the region very near the inlet, it happens that such a profile as designated 1 in this figure develops to some extent. But, as the flow progresses, it comes to develop inversely (1→2). This motion may be attributed to the fact that U is increased, first in the neighborhood of the wall, by the temperature field which begins to develop later. Farther downstream, when the temperature field has developed to some extent, the velocity profile stops developing inversely, and begins to develop normally into a parabolic profile (2→3). When $P_r < 1$, since the temperature profile develops more rapidly than the other, so, while the velocity profile is yet nearly uniform, the buoyancy in the neighborhood of the wall increases. Therefore, the velocity profile becomes, in the region near the inlet, such curves as shown in Fig. 11 (c), which curves present a mounting wave.

The height of this wave increases along the flow path, and having a maximum at a certain position, farther downstream it decreases to vanish. After this process it begins to develop normally. The height of wave is very small, for $P_r = 0.01$ it

is 0.22 %, for $P_r = 0.1$, 0.33 % at most to the amount of U at X -axis.

For the purpose of attributing the inlet velocity profile to the heat transfer characteristics, we tried, only for the case $P_r = 0.72$, assuming a parabolic velocity profile at the inlet, to solve the problem. For numerical calculation of this case, it is an only different technique in which instead of $U = U_1$, the parabolic profile $U = (1/4)(1 - R^2)$ is substituted into $U(1, k)$. The developments of fields are shown in Fig. 12. It is noted from this that the velocity field has such an inverse-developing region as observed in Fig. 11 (a). Effect of the inlet velocity profile on N_{uz} is shown in Fig. 13.

4.2 The case of finite length

For this case, the problem was solved, varying U_1 as a parameter over the range $0 < U_1 < 1/8$, on

Fig. 12 Developing fields of axial velocity and temperature for the case of parabolic entrance velocity profile, uniform wall temperature, for $P_r = 0.72$

Fig. 11 Variations of developing fields with P_r for the case of semi-infinite length, uniform wall temperature

Fig. 13 Effect of the inlet velocity profile on N_{uz}

the assumptions that the wall temperature is uniform and $P_r=0.72$. Variations of the local Nusselt number and the mean Nusselt number are shown in Figs. 14 and 15. They show that, when the dimensionless pipe length L is longer than 2.476×10^{-1} , the Nusselt numbers are not largely different from that of infinite length.

Fig. 14 Variation of local Nusselt number for the case of finite length, uniform wall temperature $P_r=0.72$

Fig. 15 Variation of mean Nusselt number for the case of finite length, uniform wall temperature, $P_r=0.72$

Fig. 16 Developing fields of axial velocity and temperature for $L=1.2 \times 10^{-2}$, uniform wall temperature, $P_r=0.72$

When L is longer than 3.45×10^{-2} , the developing profiles of velocity are not different, in the mode, from that of the infinite length, and it becomes a parabolic profile at the outlet. When L is shorter than 3.45×10^{-2} , U can not, at the outlet, reach the parabolic profile. One example of these cases is shown in Fig. 16.

When L is shorter than those mentioned above, the state of the flow field comes to resemble that of a vertical flat plate, and the velocity profile has, near the outlet, such a mounting wave as shown in Fig. 17. This wave's height is larger than that of Fig. 8. Relationship between L and U_1 is plotted in Fig. 18. It is noted from this that U_1 converges to $1/8$, and this value agrees with the result (33).

4.3 The case of uniform heat flux

The method of numerical calculation for this case is the same as section 4.2. Variations of the local Nusselt number, the mean Nusselt number and the friction coefficient are shown in Figs. 19, 20 and 21, respectively. In Fig. 19, all the curves have a convergent value 4.453, if any.

Two typical examples of developing fields are shown in Figs. 22 and 23. Figure 22 is for the case of sufficient pipe length. This shows that the profiles of both U and dimensionless modified temperature $(\theta_w - \theta)/(\theta_w - \theta_i)$ converge completely to certain

Fig. 17 Developing fields of axial velocity and temperature for $L=2.0 \times 10^{-3}$, uniform wall temperature, $P_r=0.72$

Fig. 18 Relationship between L and U_1 for uniform wall temperature, $P_r=0.72$

profiles, in the region of large X , but, more interestingly, the convergent profile of U is not parabolic. When L is longer than 0.24, both the profiles have such certain convergent profiles, but these convergent profiles vary with the pipe length L . When L is much shorter, the profile of U can not have the convergent profile but has such a mounting wave as shown in Fig. 23 in the region near the outlet. This shape resembles that of Fig. 11 (c), while the height of wave is remarkably large.

Relationship between L and U_1 is shown in Fig. 24. The tangent of this curve becomes $1/2$, as L

approaches 8×10^{-2} , which agrees with the result (45). From this curve the following expressions are obtained.

Fig. 19 Variation of local Nusselt number for the case of uniform heat flux, $P_r=0.72$

Fig. 20 Variation of mean Nusselt number for the case of uniform heat flux, $P_r=0.72$

Fig. 21 Variation of pipe friction coefficient for the case of uniform heat flux, $P_r=0.72$

Fig. 22 Developing fields of velocity and temperature for the case of uniform heat flux, $L=0.5504$, $P_r=0.72$

Fig. 23 Developing fields of velocity and temperature for the case of uniform heat flux, $L=1.245 \times 10^{-3}$, $P_r=0.72$

Table 1 Relationship between the number of developing fields and axial position.

(a) The case of semi-infinite length (Figs. 8, 9, 10 and 12)

$X^* = X/P_r$				
P_r	0.01	0.72	100	0.72
No.				Parabolic entrance velo. profile
1	1×10^{-4}	6.95×10^{-5}	1×10^{-5}	6.95×10^{-5}
2	"	1.395×10^{-4}	2 "	1.39×10^{-4}
3	4 "	2.78 "	4 "	2.78 "
4	7 "	5.56 "	7 "	5.56 "
5	1×10^{-3}	9.73 "	1×10^{-4}	9.73 "
6	2 "	1.39×10^{-3}	2 "	1.39×10^{-3}
7	4 "	2.78 "	4 "	2.78 "
8	7 "	5.56 "	7 "	5.56 "
9	1×10^{-2}	9.73 "	1×10^{-3}	9.73 "
10	2 "	1.39×10^{-2}	2 "	1.39×10^{-2}
11	4.2 "	2.78 "	4 "	2.78 "
12	7 "	5.56 "	7 "	
13	1.1×10^{-1}	9.73 "	1×10^{-2}	
14		1.39×10^{-1}	2 "	
15	Established region		4 "	
16		Established region	7 "	
17			1×10^{-1}	

(b) The cases of finite length and of uniform heat flux (Figs. 16, 17, 22 and 23)

(Figs. 16, 17, 22 and 23)

X/L

No.	L	Finite length		Uniform heat flux	
		1.2×10^{-2}	2.0×10^{-3}	5.504×10^{-1}	1.245×10^{-3}
1		8.34×10^{-8}	2.5×10^{-2}	9.08×10^{-5}	1.61×10^{-2}
2		1.67×10^{-2}	5.0 "	1.82×10^{-4}	2.41 "
3		3.33 "	1.0×10^{-1}	3.63 "	4.02 "
4		5.83 "	2.0 "	6.36 "	8.03 "
5		8.34 "	3.5 "	1.09×10^{-3}	1.61×10^{-1}
6		1.67×10^{-1}	5.0 "	1.82 "	2.81 "
7		3.33 "	<u>1.00</u>	3.63 "	4.82 "
8		5.84 "		6.36 "	8.03 "
9		8.34 "		1.16×10^{-2}	<u>1.00</u>
10		<u>1.000</u>		1.89 "	
11				3.71 "	
12				6.43 "	
13				1.17×10^{-1}	
14				1.90 "	
15				3.71 "

Fig. 24 Relationship between L and U_1 for the case of uniform heat flux, $P_r = 0.72$

$$\left. \begin{aligned} L &> 8 \times 10^{-2}; \log U_1 = (1/2) \log L - 0.931 \\ L &\leq 8 \times 10^{-2}; \log U_1 = (10/19) \log L + 0.184 \end{aligned} \right\} \dots\dots\dots (60)$$

The conditions of developing fields are collected in Table 1.

5. Conclusions

Pure natural convection field in a heated vertical circular tube was solved theoretically, to obtain the following conclusions.

(1) The dimensionless thermal entrance length L_{et} , for uniform wall temperature, is given as a function of P_r alone.

$$L_{et} = (7/50) P_r$$

(2) Relationship between P_r and the dimensionless entrance length L_{ev} , for uniform wall temperature, is given graphically by such a curve as shown in Fig. 7.

(3) In the case of semi-infinite length, for $P_r \neq 1$, there happens to be, at a certain range of X , such a region where the axial velocity U can not develop normally. The flow pattern of this region is different according as $P_r < 1$ or $P_r > 1$.

(4) For the case of uniform wall temperature, the profile of U converges to a parabola, and the temperature profile does to unity, as X approaches to infinity. As for the case of uniform heat flux, in the range $X > 0.24$, both the profiles converge to certain profiles, but these convergent profiles vary with the pipe length L . Hence, in this case, it can't define the entrance lengths.

(5) The local Nusselt number N_{ux} converges, for uniform wall temperature, to 3.62, and for uniform heat flux, to 4.45.

(6) For both the boundary conditions, relationships between L and U_1 are given graphically as shown in Figs. 18 and 24.

(7) When $L = 5 \times 10^{-3}$, U has a mounting wave in the neighborhood of the outlet, for both the wall conditions.

(8) When the inlet velocity profile is assumed as a parabola, both the local and mean Nusselt numbers are lower than that of uniform inlet velocity profile all over X range, but the convergent values of them are equal.

The numerical calculation of this study was performed on the digital computers HIPAC 103 and HITAC-5020 E.

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