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On the Characteristics of Wing with Tip Clearance*

(Part 2, The Case in which the Tip Clearance is Contained
within a Uniform Flow)

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The aerodynamic characteristics of a wing with tip clearance are studied. A theoretical investigation of wings of semi-infinite or finite span is made on the lifting-line theory. The author tried to introduce as many as possible images of the wing in order to satisfy the boundary condition at the wall facing the wing tip. Experimental results obtained on a 70 mm chord wing (aspect ratio 4.15) with tip clearance are also reported. Chordwise pressure distribution measured as a function of the angle of incidence is obtained at various spanwise positions and for various tip clearances. The experimental and calculated characteristics of the wing are compared and their discrepancies are explained quantitatively. The effect of the flow pattern of the main stream in the neighborhood of the wing tip on the wing characteristics is also investigated by experiment.

1. Introduction

Many theoretical investigations which are developed for an inviscid, incompressible, uniform flow, have been made to obtain the aerodynamic performances of a wing with tip clearance. These analyses based on the lifting-line theory^{(1)~(4) (10)} and conformal-transformation method^{(5) (6)} have an incompleteness due to the simplifying assumptions which are introduced to facilitate the analyses. B. Lakshminarayana and J. H. Horlock⁽²⁾ and B. Lakshminarayana⁽³⁾ replaced a plane wall, which faces the wing tip, by a mirror image of the wing, but not with many mirror images. K. Nickel⁽⁴⁾ assumed the uniform downwash along the span in addition to the case of Refs. (2) and (3). H. Glauert⁽¹⁾ assumed a system of the tip vortices leaving both a wing and an infinite number of its mirror images. In the lifting-line analyses cited above the effect of side walls on the aerodynamic performances of the wing is introduced insufficiently, and the effect of trailing vortices is also inadequately taken into account. T. Nishiyama and T. Takagi⁽¹⁰⁾ proposed a new idea in series expression for the spanwise circulation distribution. This idea also can not predict successfully the aerodynamic performances of the wing at small tip clearances and the performances in the neighborhood of the wing tip.

Many experimental investigations^{(2) (6) (7) (8)} have

been made of aerodynamic performances of a stationary, isolated wing with tip clearance as a function of various factors. However, neither the effects of a flow pattern of the main stream on the performance of the wing nor the effects of the angle of incidence have been studied.

In the present paper aerodynamic performances are considered for rectangular wings of the high aspect ratio. An attempt is made theoretically to predict performances for wings of semi-infinite or finite span which have tip clearance and which shed trailing vortices all along the span. The boundary condition to be satisfied at the plane wall facing the wing tip is taken into consideration as precisely as possible.

On the other hand, the measurement of pressure distributions over a wing with tip clearance is made to obtain aerodynamic performances of the wing, the neighborhood of the wing tip being in the uniform flow. Data obtained from the present experiment are compared with those on the case where the neighborhood of a wing tip is in the non-uniform flow^{(7) (8)}.

Symbols

α : geometrical angle of incidence [see Fig. 4]
 $\bar{\alpha}$: geometrical angle of incidence measured from the attitude of no lift
 $\bar{\alpha}_0$: value of $\bar{\alpha}$ at mid-span
 $\bar{\alpha}_e$: effective angle of incidence measured from the attitude of no lift
 Γ : local circulation

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- Γ_0 : value of Γ at mid-span
 $\Gamma_\infty = \sigma U_0 c_0 \bar{\alpha}_0$, circulation for wing of infinite span
 $A = b/c$, aspect ratio
 $A_0 = b/c_0$
 $\bar{\eta} = y/(b/2)$
 $\bar{\eta}_1$: value of $\bar{\eta}$ at $y = y_1$
 η : co-ordinate [see Fig. 4]
 ρ : density of fluid medium
 $\sigma = (1/2)(dc_{L\infty}/d\bar{\alpha})$
 $a = 2(b+2s)b$
 $a_1, a_3, \dots, a_m$: Fourier coefficients [see Eqs. (3) and (9)]
 b : span
 c : chord length
 c_0 : value of c at mid-span
 $c_D = d/(\rho U_0^2 c_0/2)$
 $C_{Di} = D_i/(\rho U_0^2 c_0 b/2)$
 $c_L = l/(\rho U_0^2 c_0/2)$
 $c_{L\infty}$: lift coefficient for wing ($c = c_0$) of infinite span
 $c_{Li} = l/(\rho U_0^2 c_0/2)$
 $C_L = L/(\rho U_0^2 c_0 b/2)$
 $c_p = (p - p_0)/(\rho U_0^2/2)$
 d : local drag per unit span
 D_i : total induced drag
 l : local lift per unit span
 L : total lift
 n : number of mirror images of wing
 p : pressure on wing
 p_0 : reference pressure^{(7) (8)}
 P_T : total pressure of stream
 s : size of tip clearance [see Fig. 4]
 $U_0 = \sqrt{2(P_T - p_0)/\rho}$
 U_t : local velocity of stream
 w : induced velocity
 w_0 : value of w at mid-span

Fig. 1 Lifting lines of semi-infinite span with tip clearance

$w_A(y_{t1}), [w_B(y_{t1})]$: velocity induced at y_{t1} by trailing vortices which are shed along lifting line A, [B] respectively

x, y : chordwise and spanwise co-ordinates [see Fig. 3]

y_1 : station on the y -axis [see Fig. 3]

y_t : spanwise distance measured from wing tip [see Fig. 1]

y_{t1} : station on the y_t -axis [see Fig. 1]

2. Analyses based on the lifting-line theory

The following analyses in this chapter are made for wings placed in a uniform flow.

2.1 Effects of tip clearance on the wing of semi-infinite span

The effect of the aerodynamic interference merely due to a plane wall (which faces wing tip) on performances of a wing can be studied by placing the wing of semi-infinite span as shown in Fig. 1. Induced velocities are given by

$$\left. \begin{aligned} w_A(y_{t1}) &= \frac{1}{4\pi} \oint_0^\infty \frac{d\Gamma}{dy_t} \frac{dy_t}{y_{t1} - y_t} \\ w_B(y_{t1}) &= w_A(-\{y_{t1} + 2s\}) \\ &= -\frac{1}{4\pi} \int_0^\infty \frac{d\Gamma}{dy_t} \frac{dy_t}{y_{t1} + y_t + 2s} \end{aligned} \right\} \dots\dots\dots (1)$$

where the sign of integration $\oint$ means that the "Cauchy principal value" is taken. Putting

$$y_t = c_0 \tan \theta \dots\dots\dots (2)$$

$$\Gamma(\theta) = \Gamma_\infty \sum_{m=1,3,5,\dots}^\infty a_m \sin m\theta \dots\dots\dots (3)$$

and substituting Eqs. (2) and (3) into Eq. (1), we can get the total induced velocity $w(\theta_1)$ at θ_1 as

$$\begin{aligned} w(\theta_1) &= w_A(\theta_1) + w_B(\theta_1) = \frac{\Gamma_\infty}{4\pi c_0} \sum_{m=1,3,5,\dots}^\infty a_m m \left[-\cos \theta_1 \left\{ \cos \theta_1 \cos m\theta_1 \ln \left(\frac{\cot(\theta_1/2) - 1}{\tan(\theta_1/2) + 1} \right) \right. \right. \\ &\quad + \sum_{p=1}^{(m-1)/2} (2p-1)^{-1} [(-1)^{p+1} \sin(2p-m)\theta_1 - \cos(2p-m)\theta_1] \\ &\quad + \sum_{p=1}^{(m+1)/2} (2p-1)^{-1} [(-1)^{p+1} \sin(2p-m-2)\theta_1 - \cos(2p-m-2)\theta_1] \Big\} \\ &\quad - (1+t^2)^{-1/2} \left\{ \cos \varphi \cos m\varphi \ln \frac{1+\cot(\varphi/2)}{1-\tan(\varphi/2)} \right. \\ &\quad + \sum_{p=1}^{(m-1)/2} 2(2p-1)^{-1} \sin \left[(2p-1) \frac{\pi}{4} \right] \sin \left[(m-2p)\varphi - (2p-1) \frac{\pi}{4} \right] \\ &\quad \left. \left. + \sum_{p=1}^{(m+1)/2} 2(2p-1)^{-1} \sin \left[(2p-1) \frac{\pi}{4} \right] \sin \left[(m-2p+2)\varphi - (2p-1) \frac{\pi}{4} \right] \right\} \right] \dots\dots\dots (4) \end{aligned}$$

where $t = \tan \theta_1 + (2s/c_0)$, $\varphi = \tan^{-1} t$.

Substituting the expression $\Gamma(\theta_1) = \sigma U_0 c \bar{\alpha}_e = \Gamma_\infty \sum_{m=1,3,5,\dots}^\infty a_m \sin m\theta_1$ into the relation

Fig. 2 Numerical examples of spanwise circulation and downwash for lifting lines of semi-infinite span at various tip clearances

Fig. 3 Lifting lines of finite span with tip clearances ($r = b + 2s$)

$$y = -(b/2) \cos \theta \dots \dots \dots (7)$$

$$\Gamma(\theta) = \Gamma_{\infty} \sum_{m=1,3,5,\dots}^{\infty} a_m \sin m\theta \dots \dots \dots (8)$$

and substituting Eqs. (8) and (9) into Eq. (7), we get the induced velocity $w(\theta_1)$ at a point $\theta = \theta_1$ as

$$w(\theta_1) = \frac{\Gamma_{\infty}}{2b} \sum_{m=1,3,5,\dots}^{\infty} a_m \left\{ m \frac{\sin m\theta_1}{\sin \theta_1} + \bar{f}(\theta_1, m) + \bar{g}(\theta_1, m) \right\} \dots \dots \dots (9)$$

where

$$\bar{f}_1(\theta_1, m) = \sum_{n=1}^{\infty} m \left\{ \sqrt{(\cos \theta_1 + na)^2 - 1} - (\cos \theta_1 + na) \right\}^m / \sqrt{(\cos \theta_1 + na)^2 - 1},$$

$$\bar{g}(\theta_1, m) = \bar{f}(\pi - \theta_1, m)$$

Using Eq. (10) and the same relation as Eq. (5), we get the fundamental equation for determining a_m ,

$$\sum_{m=1,3,5,\dots}^{\infty} a_m \left[\left\{ 1 + \frac{\sigma}{2A_0} \frac{c}{c_0} \frac{m}{\sin \theta_1} \right\} \sin m\theta_1 + \frac{\sigma}{2A_0} \frac{c}{c_0} \left\{ \bar{f}(\theta_1, m) + \bar{g}(\theta_1, m) \right\} \right] = \frac{c\bar{\alpha}}{c_0\bar{\alpha}_0} \dots \dots \dots (11)$$

After evaluating a_m from Eq. (11), we can get C_L and C_{Di} from the following equations

$$C_L = \left(\rho U_0 \int_{-b/2}^{b/2} \Gamma dy \right) / (\rho U_0^2 c_0 b/2) = c_{L\infty} (\pi/4) a_1 \dots \dots \dots (12)$$

$$C_{Di} = \left(\rho \int_{-b/2}^{b/2} w \Gamma dy \right) / (\rho U_0^2 c_0 b/2) = \frac{C_L^2}{\pi A_0} \left[\sum_{m=1,3,5,\dots}^{\infty} m \left(\frac{a_m}{a_1} \right)^2 + \frac{2}{\pi} \frac{1}{a_1^2} \int_0^{\pi} \left\{ \sum_{m=1,3,5,\dots}^{\infty} a_m [\bar{f}(\theta_1, m) + \bar{g}(\theta_1, m)] \right\} \left(\sum_{m=1,3,5,\dots}^{\infty} a_m \sin m\theta_1 \right) \sin \theta_1 d\theta_1 \right] \dots \dots \dots (13)$$

The integral in Eq. (13) can be reduced to elliptic integrals. (Laborious calculations are omitted here.) With an untwisted, rectangular wing, the above equations are simplified by relations $c = c_0$ and $\bar{\alpha} = \bar{\alpha}_0$.

2.3 Effects of tip clearance on the wing of finite span for the elliptic distribution of circulation⁽¹¹⁾

We shall now examine the effects of tip clearance on the aerodynamic performance of the wing for the

$$\bar{\alpha}_e = \bar{\alpha} - \{w(\theta_1)/U_0\} \dots \dots \dots (5)$$

we can show that

$$\frac{c_0 \bar{\alpha}_0}{c} \sum_{m=1,3,5,\dots}^{\infty} a_m \sin m\theta_1 = \bar{\alpha} - \frac{w(\theta_1)}{U_0} \dots \dots \dots (6)$$

Equation (6) gives a_m in Eq. (3).

With an untwisted wing of a uniform chord length, substituting Eq. (4) into Eq. (6), together with $c = c_0$ and $\bar{\alpha} = \bar{\alpha}_0$, we get a fundamental equation for determining a_m . The fundamental equation (6) holds at all points of the wing. In order to simplify the calculation of a_m , we shall assume Eq. (6) to be satisfied at a finite number of points of the wing. This leads Eq. (6) to simultaneous equations with finite unknowns. Evaluating a_m , we get the circulation and induced velocity distribution along the span, using Eq. (3) and Eq. (4) respectively. Figure 2 shows a numerical example for the wings of semi-infinite span. (Ten coefficients $a_1, a_3, \dots, a_{19}$ have been retained in the calculations.)

2.2 Effects of the tip clearance on the wing of finite span

Glauert's type of analysis is made for a wing system as shown in Fig. 3. The plan form of the wing can be taken into account in this analysis.

Referring to Fig. 3, and in a similar manner to the derivation of Eq. (1), we can show that the total induced velocity at a point y_1 is given by

$$w(y_1) = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \int_{-b/2}^{b/2} \frac{d\Gamma}{dy} \frac{dy}{y_1 - y - n(b+2s)} \dots \dots \dots (7)$$

Using the following equation (8) and (9), similar to the expression in Glauert's analysis,

$$\dots \dots \dots (8)$$

$$\dots \dots \dots (9)$$

case of an elliptic circulation. We suppose that the elliptic distribution of the spanwise circulation is obtained as a result of being affected by the aerodynamic interference due to side wall.

Substituting Eq. (8) into Eq. (9) and retaining only $m=1$, we have the elliptic distribution of circulation

$$\Gamma(\bar{\eta}_1) = \Gamma_0 \sqrt{1 - \bar{\eta}_1^2}, \quad (\Gamma_0 = a_1 \Gamma_\infty) \dots \dots \dots (14)$$

The expression for the induced velocity $w(\bar{\eta}_1)$, which is obtained from Eqs. (10), (8), and (14), can be rewritten by the binomial series expansion of terms corresponding to $n \geq 2$. The final result is given by

$$\begin{aligned} w(\bar{\eta}_1) = & (\Gamma_0/2b) \left[-\{(a - \bar{\eta}_1)/\sqrt{(a - \bar{\eta}_1)^2 - 1}\} \right. \\ & - \{(a + \bar{\eta}_1)/\sqrt{(a + \bar{\eta}_1)^2 - 1}\} + 3 \\ & - (0.6449a^{-2} + 0.0617a^{-4} + 0.0108a^{-6} \\ & + 0.0022a^{-8} + \dots) - (0.2470a^{-4} + 0.1301a^{-6} \\ & + 0.0535a^{-8} + \dots)\bar{\eta}_1^2 - (0.0867a^{-6} \\ & + 0.1070a^{-8} + \dots)\bar{\eta}_1^4 - (0.0285a^{-8} + \dots)\bar{\eta}_1^6 \dots \left. \right] \\ = & (\Gamma_0/b)f(a, \bar{\eta}_1) \dots \dots \dots (15) \end{aligned}$$

The induced velocity $w_{iw}(\bar{\eta}_1)$ only due to the image wings, which are introduced to satisfy the boundary condition at plane walls facing the tip, can be obtained by replacing $3(\Gamma_0/2b)$ by $2(\Gamma_0/2b)$ in the right hand side of Eq. (15), because the induced velocity due to the wing (not the image wing) is $\Gamma_0/2b$.

Substituting Eqs. (8) and (12) into Eq. (13), retaining only $m=1$ and knowing that

$$\begin{aligned} I & \equiv \int_{-1}^1 \frac{(a + \bar{\eta}_1)\sqrt{1 - \bar{\eta}_1^2}}{\sqrt{(a + \bar{\eta}_1)^2 - 1}} d\bar{\eta}_1 \\ & \text{and } \int_{-1}^1 \frac{(a - \bar{\eta}_1)\sqrt{1 - \bar{\eta}_1^2}}{\sqrt{(a - \bar{\eta}_1)^2 - 1}} d\bar{\eta}_1 \\ & = -a \left[\{(\mu^2 - 1)/\mu\} F(\pi/2, k) - \mu E(\pi/2, k) \right], \\ & \quad (\text{see Appendix}) \end{aligned}$$

where $\mu^2 = \{(a + \sqrt{a^2 - 4})^2 - 4\} / \{4 - (a - \sqrt{a^2 - 4})^2\}$ with $k^2 = \mu^{-4}$, and where $F(\pi/2, k)$, $E(\pi/2, k)$ are the complete elliptic integral of first and second kind respectively, we can show that the total induced drag is given by

$$\begin{aligned} D_i & = (\rho \Gamma_0^2/4) \{ 2a \{ [(\mu^2 - 1)/\mu] F(\pi/2, k) \\ & \quad - \mu E(\pi/2, k) \} + (3\pi/2) - (1.0130a^{-2} \\ & \quad + 0.1940a^{-4} + 0.0851a^{-6} + 0.0490a^{-8} + \dots) \} \\ & \dots \dots \dots (16) \end{aligned}$$

The induced drag $D_{i_{wall}}$ only due to the image wings indicating the effect of the plane walls can be obtained by replacing $(3\pi/2)(\rho \Gamma_0^2/4)$ with $\pi(\rho \Gamma_0^2/4)$ in the right hand side of Eq. (16), because the induced drag due to the wing (not the image wing) is $(\rho \Gamma_0^2/4) \int_{-1}^1 \sqrt{1 - \bar{\eta}_1^2} d\bar{\eta}_1 = \pi \rho \Gamma_0^2/8$.

When we estimate the induced drag of the wing with tip clearance, we can use the induced drag of an isolated wing in a free stream, added by the

correction $D_{i_{wall}}$.

The induced angle of incidence $\Delta \alpha_{i_{wall}}$ corresponding to $D_{i_{wall}}$ is given by $\Delta \alpha_{i_{wall}} = D_{i_{wall}}/L$, where $L = (\rho U_0^2 c_0 b/2)(c_{L\infty} \pi a_1/4) = \pi b \rho U_0 \Gamma_0/4$ [using Eqs. (12) and (14)]. Due to the upwash, the total induced angle of incidence becomes smaller by an amount of $\Delta \alpha_{i_{wall}}$ than that for the wing in a free stream.

Substituting Eqs. (14) and (15) into the same relation as Eq. (5) and putting $\bar{\eta}_1=0$, we can get the following equation for an untwisted wing,

$$\Gamma_0/\Gamma_\infty = \{1 + (\sigma/\Lambda_0)[f(a, \bar{\eta}_1)]_{\bar{\eta}_1=0}\}^{-1} \dots \dots \dots (17)$$

3. Experiments

3.1 Experimental apparatus and method

The wind tunnel and wing models used here are the same as those in Ref. (8). The arrangement of wings, however, at the working section of the tunnel differs from that in Ref. (8). Let us refer to Fig. 4. For the purpose of placing the neighborhood of the wing tip within a uniform stream, the author used a dummy or a second wing ③' other than the wing ③ having pressure holes. The chord length of the wing ③ is 70 mm and its aspect ratio approximately 4.15. The wing ③' has the same dimension as the wing ③. These wings are installed symmetrically about and perpendicularly to the vertical plane of symmetry of the tunnel. The gap between the wings ③ and ③' is equally divided by the vertical plane of symmetry. We can operate the dummy wing ③' in the same way as the wing ③.

The holes which are drilled at the center of the side walls in the upstream of $1.5c$ measured from the leading edge of those wings, can be used to obtain some pieces of information concerning the side wall boundary layers. The velocity distribution of the side wall boundary layer is the same as that of Fig. 5 in Ref. (8). The Reynolds number referred to the chord length is kept approximately at 1.4×10^5 ($U_0 = 30$ m/sec) in the working section throughout the experiment.

① Bleed-off, ②, ②' Guides, ③ Tested wing, ③' Dummy wing
Fig. 4 Sketch of working section, together with nomenclature

The experiments were made over a range of angles of incidence, covering $\alpha^\circ = -6, -4, -2, 0, 3, 6$; and a range of tip clearances, covering s mm $= 0, 1, 2, 3, 5, 7, 10$. The thickness of the side wall boundary layer was about 16 mm [shown in Fig. 5 of Ref. (8)], and the aspect ratio of the wing used was about 4.15.

Then c_p , c_L , c_D is obtained in the same way as in Ref. (8), using experimental data of the chordwise pressure distribution on the wing.

In the case $c = c_0$ [Eqs. (18) ~ (22)], knowing that

$$l = (\rho U_i^2 / 2) c (dc_{L\infty} / d\alpha) \{\bar{\alpha} - (w/U_i)\} \quad \text{.....(18)}$$

$$c_L = c_{L_i} (U_i/U_0)^2 \quad \text{.....(19)}$$

we can get the following relation

$$w = U_i \{\bar{\alpha} - c_L (U_0/U_i)^2 (dc_{L\infty} / d\alpha)^{-1}\} \quad \text{.....(20)}$$

A total induced drag D_i which is caused by the clearance flow due to the tip clearance and by the secondary flow due to the side wall boundary layer, is given by

$$C_{Di} = \int_{-b/2}^{b/2} \frac{l}{(\rho U_0^2 c b / 2)} \frac{w}{U_i} dy \quad \text{.....(21)}$$

in the form of the induced drag coefficient.

The induced drag for a wing segment of unit span has a relation $l(w/U_i) = \rho w \Gamma$. This relation, together with the alternative expression of the lift per unit span $l = c_L \rho U_0^2 c / 2$, gives

$$\Gamma = c U_0^2 c_L / 2 U_i = c U_i c_{L_i} / 2 \quad \text{.....(22)}$$

(a) $A=4.15$; rectangular wing; the main stream is uniform in the neighborhood of the wing tip

(b) $A=4.15$; rectangular wing; the main stream is non-uniform in the neighborhood of the wing tip⁽⁷⁾⁽⁸⁾

[— $\alpha=6^\circ$, --- $\alpha=3^\circ$, $\alpha=0^\circ$ in graphs of Figs. 5 (a) and (b)]

Fig. 5 Chordwise pressure distributions at various s , y_t and α , (1)

Thus we can obtain w , Γ , C_{Di} from Eqs. (18) ~ (22), using experimental data of c_L , U_0 , U_1 .

The present experiment was made without the removal of the side wall boundary layers. These layers lead to the formation of a secondary flow which affects the aerodynamic characteristics of the wing through the induced velocity caused thereby. Substituting experimental data for $s=0$ into the right hand side of Eq. (20), we can find the effect of the side wall boundary layers. From the fact that the velocity induced at a point by vortex filaments is obtained by adding together the contributions of the individual filaments, we can derive an idea mentioned below. In the flow field of the present experiment, there exist both the induced velocity due to the tip clearance whose neighborhood is within the uniform stream, and the induced velocity due to the boundary layer on the side walls to which the wings are fixed. In the above flow field, if we hope to remove the secondary induced velocity due to the side wall boundary layer from experimental data, we can achieve our aim by subtracting the secondary induced velocity for $s=0$ from the induced velocity for $s \neq 0$.

In the present experiment, the induced velocity w in Eqs. (18) and (21) corresponds to the sum of the induced velocity w_c only due to the tip clearance and the secondary induced velocity w_s only due to the side wall boundary layer. But we can replace w in Eqs. (18) and (21) with w_c , by use of the idea of the superposition of the induced velocities. Thus it follows that l and C_{Di} obtained from Eqs. (18) and (21) correspond to the values without the effect of the side wall boundary layer. The same holds good with Γ in Eq. (22).

In a part of experimental data given below the author removed the effect of the side wall boundary layer on the wing performances, by means of the idea of the superposition of the induced velocities.

3.2 Experimental results and considerations

We can not compare directly the present experiments with the experiments in Ref. (8), because the aspect ratio is 4.15 in the present investigation, while it is 8.43 in Ref. (8). However, there is the following phenomenon which enables us to compare Ref. (8) with the present paper. As for the spanwise variation of c_L and c_D at $s=10$ mm in the range about $y_t < (3/4)c$, Figs. 11 and 12 of Ref. (8) show quantitatively striking similarity to Figs. 6 and 7 in the present paper, respectively. On the other hand, considering the wing performance in the neighborhood of the wing tip at about $s < 10$ mm in the figures cited above, we can find that the effect of a non-uniform flow in the neighborhood of the wing tip decreases with an increase in s , while the

effect of the clearance flow is always dominant [see also Figs. 5 (a) and (b) in the present paper]. Therefrom, we can conclude that the effect of the geometrical aspect ratio on the wing performance is very small in the neighborhood of the wing tip. On the basis of above considerations, the author compares qualitatively experimental data in Ref. (8) with those in the present paper. The comparison is to be made in the later sections.

In the following subsections, we shall use the expression "the case of the uniform flow" to represent a case where the neighborhood of the wing tip is within a uniform flow, and similarly, "the case of the non-uniform flow" to represent a case where the neighborhood of the wing tip is within a non-

(c) $A=4.15$; rectangular wing; the main stream is uniform in the neighborhood of the wing tip
[--- $\alpha = -2^\circ$, — $\alpha = -4^\circ$, - - $\alpha = -6^\circ$]
Fig. 5 Chordwise pressure distributions at various s , y_t and α , (II)

uniform flow.

3.2.1 Chordwise pressure distributions

Figures 5 (a) and (b) show the effect of patterns of the main flow in the neighborhood of the wing tip on the pressure distribution. We can find that the effect of the flow pattern is dominant, provided s is smaller (about $s < 3$ mm) than the thickness of the side wall boundary layer which is approximately 16 mm in the upstream of $1.5c$ measured from the leading edge [see Fig. 5 of Ref. (8)], and that the effects of the flow pattern are masked with those of the clearance flow, provided s is larger [about $s > (7 \sim 10)$ mm] than the thickness of the side wall boundary layer. The effects of the flow pattern are remarkable in the following cases (i) and (ii):

(i) The top surface of the wing at $\alpha > 0$ [Figs. 5 (a) and (b)]: The pressure coefficient $|c_p|$ on the top surface is larger in "the case of the uniform flow" than in "the case of the non-uniform flow", because the stream velocity along the top surface is higher in the former than in the latter due to the side wall boundary layer. The increase in $|c_p|$ causes greater development of the clearance flow in "the case of the uniform flow" than in "the case of the non-uniform flow". This is proved by the graphs of Figs. 5 (a) and (b). In other words, comparing the pressure distributions at $y_t = 1 \sim 2$ mm in "the case of the uniform flow" with those in "the case of the non-uniform flow," we can find that the value of s at which a marked growth of the clearance flow appears, is slightly smaller in "the case of the uniform flow" than in "the case of the non-uniform flow", and that the above tendency becomes more pronounced as α increases.

By the way, it should be noted that the pressure distributions on the bottom surface are scarcely affected by the flow patterns in the neighborhood of the wing tip [Figs. 5 (a) and (b)].

(ii) The bottom surface of the wing at $\alpha < 0$: Figure 8 (b) of Ref. (8) shows that a clearance flow from the top surface to the bottom surface occurs near the leading edge in the neighborhood of the wing tip at $\alpha = -4^\circ$.

Let us refer to Fig. 5 (c). The tip clearance s at which the clearance flow becomes predominant, is larger in "the case of the uniform flow" than in "the case of the non-uniform flow". This phenomenon comes from the fact mentioned below. The suction pressure on the top surface is more predominant in "the case of the uniform flow" than in "the case of the non-uniform flow" at a given (negative) angle of incidence, because the stream velocity in the neighborhood of the wing tip is higher in the former than in the latter. Therefore, the pressure

difference (which causes a clearance flow) between the top and the bottom surface diminishes in "the case of the uniform flow". This leads to the result that the tip clearance which exerts a marked effect of the clearance flow on the chordwise pressure distribution, becomes larger in "the case of the uniform flow" than in "the case of the non-uniform flow".

Let us consider, for example, the chordwise pressure distribution in the neighborhood of the wing tip for the case of $s=0$, $\alpha=-4^\circ$. Figure 8 (b) of Ref. (8) ("the case of the non-uniform flow") shows that the pressure distributions on the bottom surface have a sharp peak near the leading edge. Now, we shall refer to Fig. 5 ("the case of the uniform flow") in this paper. We can find from Fig. 5 (c) that the form of a peak is lower in height and less sharp in "the case of the uniform flow" than in "the case of non-uniform flow [Fig. 8 (b) of Ref. (8)]", and that the pressure distribution including the above

$A=4.15$; rectangular wing; the main stream is uniform in the neighborhood of the wing tip
Fig. 6 Spanwise lift distributions at various s and α

property of the peak, scarcely varies all along the wing span, except near the side wall to which the wing is attached. The discrepancy in geometrical property of the peak cited above comes from the fact that the wing surface boundary layer does not separate from the wing surface in the case of Fig. 8 (b) of Ref. (8), while it separates in the case of Fig. 5 (c) in this paper. Because, in the neighborhood of the wing tip, the wing surface boundary layer is disturbed by the non-uniform main flow in Fig. 8 (b) of Ref. (8), while it is not disturbed by the uniform main flow in Fig. 5 (c) [see 3.1.3 (b) in Ref. (8)].

3.2.2 c_L -distributions

Let us compare the graph of Fig. 11 of Ref. (8) ("the case of the non-uniform flow") with one of Fig. 6 in this paper ("the case of the uniform flow"). Firstly, considering "the case of the uniform flow" at $s=0$, we can find that spanwise c_L -distributions keep steady values even in the neighborhood of the wing tip. This is a marked difference between "the case of the uniform flow" and "the case of the non-uniform flow". Secondly, considering c_L -distributions at $s \neq 0$ in the neighborhood of the wing tip, we can find that a discrepancy in c_L -distribution due to the two patterns of the main stream in the neighborhood of the wing tip decreases as s increases, and that c_L -distributions in "the case of the uniform flow" at $s=10$ mm are very similar to those in "the case of the non-uniform flow" at $s=10$ mm. This seems to show that the effect of the side wall boundary layer on c_L -distributions is very small in the neighborhood of the wing tip, provided s is approxi-

mately equal to the thickness of the side wall boundary layer.

The gradient $\Delta c_L / \Delta s$ has a tendency to decrease with an increasing s as α approaches the angle of no lift. Especially, spanwise c_L -distributions at $\alpha = -2^\circ$ are approximately constant in the range of $s=5 \sim 10$ mm.

3.2.3 c_D -distributions

Figure 7 shows an example of spanwise c_D -distributions in "the case of the uniform flow". Let us compare the graph of Fig. 7 in this paper with one of Fig. 12 of Ref. (8). Some discrepancies in c_D -distributions due to the patterns of the main stream are recognized in the neighborhood of the wing tip. As compared with c_D in "the case of the uniform flow", (i) c_D at $s=0$ in "the case of the non-uniform flow" is very small in the neighborhood of the wing tip (exactly speaking, about $y_t < 3c/4$ at $\alpha=3 \sim 6^\circ$; about $y_t < c/4$ at $\alpha=0^\circ$), (ii) c_D at $s < (3 \sim 4)$ mm in "the case of the non-uniform flow" decreases slightly in the neighborhood of the wing tip (the region affected by the "non-uniform flow" is not clear, but approximately $y_t < c$), and the effect of the patterns of the main stream on c_D becomes less at $(3 \sim 4) \text{ mm} < s < 10$ mm in the neighborhood of the wing tip. The above phenomena (i) and (ii) are due to the effect of the velocity of the main stream rather than the effect of the clearance flow. Exactly speaking, these phenomena come from the fact that the (positive) pressure about the front stagnation point on the wing is higher in "the case of the uniform flow" than in "the case of the non-uniform flow", because the stream velocity in the neighborhood of the wing tip is higher in "the case of the uniform flow" than in "the case of the non-uniform flow".

Figure 8 shows polar diagrams of C_L vs. C_{Di} , where C_L was derived from the measured pressure on the wing, and C_{Di} was obtained from Eq. (21). We can find the following from Fig. 8. Firstly, for a given α , the gradient dC_L/dC_{Di} is almost constant despite variation of s in the range of $s < 10$ mm. The gradient increases with an increasing α . Sec-

$\alpha=3^\circ$; $A=4.15$; rectangular wing: the main stream is uniform in the neighborhood of the wing tip
Fig. 7 An example of spanwise drag distributions at various s

$A=4.15$; rectangular wing: uniform flow (Effect of side wall boundary layer was removed by the method of the subsection 3.1)

Fig. 8 Plots of C_L vs. C_{Di} from experimental data

only, the clearance s plays a similar rôle to the aspect ratio for an isolated wing in a free stream.

4. The comparison of the experiment with the theory

A numerical example of the analysis developed in the subsection 2.2 is given for an untwisted, rectangular wing ($b/2=145$ mm; $c=70$ mm; $A=4.15$). The wing has the same geometry as the one used in the experiment in the subsection 3.1 or Fig. 4. Here, 10 Fourier coefficients are retained in the circulation distribution [Eq. (9)], and 20 mirror images of the wing are taken into account in each side of the wing. Another numerical example of the analysis developed in the subsection 2.3 is given for an untwisted wing ($b/2=145$ mm, $c_0=70$ mm, $A_0=4.15$). In the above numerical examples, the author took the values of U_0 and σ such as to satisfy the condition of the experiment in the subsection

3.1. That is, $U_0=30$ m/sec, and σ was taken to be equal to the value of RAF 6 aerofoil of $0.1c$ max. thickness, i.e. $\sigma=2.6732$ rad $^{-1}$.

Figures 9~13 with caption including a parenthesis "(uniform flow)", show the plots of experimental data, after the effect of the boundary layer is removed, which is developed on the side wall where the wing is attached (see subsection 3.1).

Figure 9 shows a relation between Γ_0 and s . The circulation obtained from Eqs. (9) and (17) increases as s decreases. The circulation obtained from Eq. (9) has a similar tendency to the experimental data, in which the rate of increase in Γ_0 becomes large at about $s<2$ mm with a decreasing s . The theoretical circulation from Eq. (9) is approximately equal to the mean of experimental data measured at varying α , except in circulations at about $s<(4\sim5)$ mm. The theoretical circulations from Eq. (17) are 86~90 % of experimental data.

A comparison of the experimental circulation with the theoretical ones is made in Fig. 10. At $\alpha=-2^\circ$ and in the range of $s<10$ mm, the theoretical circulation from Eqs. (9) and (11) is very close to the experimental data. However, in the case where α deviates from the angle of no lift ($\alpha=-4.5^\circ$), i.e. in the case of $\alpha=0\sim6^\circ$, the discrepancy between the experimental and theoretical circulation distribution increases in the range of $s<10$ mm in the present investigation as s decreases (see also Figs. 12 and 13). An example for small s is given by case of $s=1$ mm, $\alpha=0\sim6^\circ$ (Fig. 10). Though the above discrepancy in circulation may be due to the effect of wall resistance on the clearance flow, this discrepancy will be affected also by mathematical models of the trailing vortex sheets, especially in the neighborhood of the wing tip.

Fig. 9 Experimental and theoretical variations of circulation at mid-span position with s (uniform flow)

Fig. 10 Experimental and theoretical variations of circulation distribution at various s and α (uniform flow)

At $s \approx 10$ mm the theoretical circulations are very close to the experimental circulations in the region of about $y_t/(b/2) > 0.1$. This fact is independent of α . The effect of the clearance flow on the circulation depends on α in the region of about $y_t/(b/2) < 0.1$.

Figure 11 shows the experimental and theoretical variations of the normal induced velocity with y_t , Eq. (20) being used to obtain the induced velocity from experimental data. The graphs of Fig. 11

indicate a similar tendency to ones of Fig. 10. Assuming the elliptic distribution of the spanwise circulation, we can find that the upwash is brought about by the trailing vortex sheets in the neighborhood of the wing tip (see curve ② in Fig. 11).

Figure 12 indicates the comparison of C_{Di} derived from the experimental data with C_{Di} calculated from Eqs. (16) and (13). The experimental and theoretical distributions of C_{Di} vs. s become flatter as α approaches the angle of no lift. The coefficient

① Eq. (10) ($A=4.15$, rectangular wing) ② Eq. (15) ($A_0=4.15$) ③ Isolated, rectangular wing ($s \rightarrow \infty$, $A=4.15$), Ref. (9) or Eq. (10)

Fig. 11 Experimental and theoretical variations of downwash distribution at various s and α (uniform flow)

Eq. (13) $A=4.15$, rectangular wing; [Eq. (16)] $(\rho U_0^2 c_0 b/2)$ $A_0=4.15$

Fig. 12 Experimental and theoretical variations of C_{Di} with s (uniform flow)

Eq. (12) $A=4.15$, rectangular wing

Fig. 13 Experimental and theoretical variations of C_L with s (uniform flow)

C_{Di} comes rapidly closer to a steady value at α near the angle of no lift, as s increases. The discrepancy between experiment and theory widens with an increase in α .

Figure 13 shows the experimental and theoretical variation of C_L vs. s . We can see from Fig. 13 that the experiment closely agrees with the theory.

Tables 1 (a) and (b) show the Fourier coefficients in the series for the circulation distribution at $s \rightarrow 0$ and $s \rightarrow \infty$ respectively. These coefficients were calculated for a rectangular wing using Eq. (11). With an increasing n , the coefficient a_1 attains the greatest rate of increase among the other coefficients a_m ($m=3, 5, 7, \dots$). And, in fact, the value of a_1 for $n=20$ is slightly greater than that of a_1 for $n=10$, i.e. $a_{1(n=20)} \approx (1.003 \sim 1.004)a_{1(n=10)}$. Thus, the variation of a_m ($m=3, 5, 7, \dots$) with an increasing n , is very small for $20 \geq n \geq 10$ [see Table 1 (a)]. If we evaluate the values of a_m ($m=1, 3, 5, \dots$) for $n > 20$, we can expect that the rate of increase of a_m ($m=1, 3, 5, \dots$) is much smaller than the above mentioned figures, i.e. $0.3 \sim 0.4\%$ of $a_{1(n=10)}$. Therefore, from Table 1 (a) (the case of aspect ratio 4.15), we conclude that the values of a_m for $n=10 \sim 20$ can be used in a good approximation. The effect of n ($>10 \sim 20$) on a_m is very small, especially at $m=19$ [see Table 1 (a)]. From the present author's, example [Table 1 (a)], it seems that it is preferable to increase the number of mirror images to calculate a_m in the analyses of Refs. (2) and (3).

5. Conclusions

(1) The lift and drag about the wing tip are markedly larger in the case where the neighborhood

of the wing tip is in the uniform flow than in the non-uniform flow formed by the side wall boundary layer. Namely, the effect of the pattern of the main stream on the wing performances is remarkable.

The increase in lift in the case where the neighborhood of the wing tip is in the uniform flow, is due to the pressure reduction on the top surface at a short distance in the chordwise direction from the leading edge. The increase in drag in the case where the neighborhood of the wing tip is in the uniform flow, is due to the pressure rise at the front stagnation point of the wing. The pressure reduction and increase mentioned above come from the fact that the stream velocity about the wing tip is higher in the case where the neighborhood of the wing tip is in the uniform flow than in the non-uniform flow.

(2) In the polar diagram of C_L vs. C_{Di} , and in the curve of C_L vs. α , the tip clearance plays a similar rôle to the aspect ratio for an isolated wing in a free stream. In the present investigation, for a given α , the gradient dC_L/dC_{Di} is almost constant despite variation of s in the range of $s < 10$ mm. The gradient increases with an increasing α .

(3) The spanwise circulation and downwash distribution in the range of the angles of incidence $\alpha < -2^\circ$ which covers the angle of no lift, and C_L in the range of about $-2^\circ < \alpha < 6^\circ$, can be evaluated using the method in the subsection 2.2.

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Table 1
(a) Values of Fourier coefficients a_m for several tip clearances [$A=4.15$, rectangular wing, Eq. (9)]

s mm	n	a_1	a_3	a_5	a_7	a_9	a_{11}	a_{13}	a_{15}	a_{17}	a_{19}
0.01	1	1.100 15	0.266 65	0.117 38	0.063 52	0.037 97	0.023 71	0.014 84	0.008 88	0.004 61	0.001 37
	10	1.161 36	0.282 90	0.124 23	0.067 20	0.040 17	0.025 08	0.015 70	0.009 40	0.004 88	0.001 45
	20	1.166 44	0.284 14	0.124 77	0.067 50	0.040 35	0.025 19	0.015 77	0.009 44	0.004 90	0.001 46
1.00	1	1.058 14	0.213 07	0.074 73	0.032 40	0.015 83	0.008 28	0.004 47	0.002 38	0.001 14	0.000 32
	10	1.113 61	0.225 45	0.078 86	0.034 17	0.016 69	0.008 73	0.004 71	0.002 51	0.001 20	0.000 34
	20	1.118 22	0.226 39	0.079 19	0.034 31	0.016 76	0.008 77	0.004 73	0.002 52	0.001 21	0.000 34
5.00	1	1.008 38	0.160 95	0.041 57	0.013 11	0.004 76	0.001 93	0.000 84	0.000 38	0.000 17	0.000 05
	10	1.055 53	0.169 42	0.043 64	0.013 75	0.005 00	0.002 02	0.000 88	0.000 40	0.000 17	0.000 05
	20	1.059 45	0.170 05	0.043 80	0.013 81	0.005 01	0.002 03	0.000 89	0.000 40	0.000 17	0.000 05
10.00	1	0.981 13	0.138 78	0.031 09	0.008 69	0.002 93	0.001 16	0.000 52	0.000 25	0.000 11	0.000 03
	10	1.022 51	0.145 40	0.032 49	0.009 07	0.003 06	0.001 21	0.000 54	0.000 26	0.000 12	0.000 03
	20	1.025 95	0.145 89	0.032 60	0.009 10	0.003 07	0.001 22	0.000 54	0.000 26	0.000 12	0.000 03

(b) Values of a_m for rectangular wing in free stream [$s \rightarrow \infty$, $A=4.15$, Eq. (9) or Ref. (9)]

a_1	a_3	a_5	a_7	a_9	a_{11}	a_{13}	a_{15}	a_{17}	a_{19}
0.867 55	0.093 14	0.018 15	0.005 00	0.001 78	0.000 76	0.000 36	0.000 18	0.000 08	0.000 02

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Appendix Evaluation of Integral I (page 545)

We shall evaluate the following integral:

$$I = \int_{-1}^1 \frac{\bar{\eta}_1^3 - a\bar{\eta}_1^2 - \bar{\eta}_1 + a}{\sqrt{(\bar{\eta}_1^2 - 2a\bar{\eta}_1 + a^2 - 1)(1 - \bar{\eta}_1^2)}} d\bar{\eta}_1 \quad (23)$$

Putting

$$\bar{\eta}_1 = \{(a + \sqrt{a^2 - 4})t + (a - \sqrt{a^2 - 4})\} / 2(t + 1) \quad (24)$$

$$t(\bar{\eta}_1 = -1) = t_{-1}, \quad t(\bar{\eta}_1 = 1) = t_1, \quad \phi(t) = (\mu^2 - t^2)(\lambda^2 - t^2), \quad (\lambda^2 = \mu^{-2}),$$

we can obtain the following expression from Eq. (23),

$$I = -\sqrt{a^2 - 4} \underbrace{\int_{t_{-1}}^{t_1} \frac{dt}{\sqrt{\phi(t)}}}_{I_1} - \frac{\sqrt{a^2 - 4}}{2} (a^2 - 8 + a\sqrt{a^2 - 4}) \underbrace{\int_{t_{-1}}^{t_1} \frac{dt}{(1+t)\sqrt{\phi(t)}}}_{I_2} \\ + \frac{(a^2 - 4)}{2} (a + 3\sqrt{a^2 - 4}) \underbrace{\int_{t_{-1}}^{t_1} \frac{dt}{(1+t)^2\sqrt{\phi(t)}}}_{I_3} - (a^2 - 4)^{3/2} \underbrace{\int_{t_{-1}}^{t_1} \frac{dt}{(1+t)^3\sqrt{\phi(t)}}}_{I_4} \quad (25)$$

Let us evaluate integrals I_1 , I_2 , I_3 , and I_4 .

$$I_1 = (2/\mu)F(\pi/2, k) \quad (26)$$

$$I_2 = \int_{t_{-1}}^{t_1} \frac{dt}{(1-t^2)\sqrt{\phi(t)}} - \int_{t_{-1}}^{t_1} \frac{t dt}{(1-t^2)\sqrt{\phi(t)}} \\ = (2/\mu)\Pi(\pi/2; -\lambda^2, k) + (1/\sqrt{\mu^2 + \lambda^2 - 2}) \left[\tan^{-1} \left\{ \frac{\sqrt{t^4 - (\mu^2 + \lambda^2)t^2 + 1} + t^2 - 1}{\sqrt{\mu^2 + \lambda^2 - 2}} \right\} \right]_{t=t_{-1}}^{t=t_1} \quad (27)$$

where $\lambda^2 = \mu^{-2}$ and $\Pi(\pi/2; -\lambda^2, k)$ denotes the complete elliptic integral of the third kind. Using the recurrence formula for the elliptic integrals⁽¹²⁾, we obtain

$$I_3 = [2 - (\mu^2 + \lambda^2)]I_2 + (\mu^2 - 1)I_1 - 2\mu E(\pi/2, k) / [2 - (\mu^2 + \lambda^2)] \quad (28)$$

$$I_4 = [6 - 3(\mu^2 + \lambda^2)]I_3 - \{6 - (\mu^2 + \lambda^2)\}I_2 + 2I_1 / [2(2 - \mu^2 - \lambda^2)] \quad (29)$$

Substituting Eqs. (26) ~ (29) into Eq. (25), we can finally obtain

$$I = -a \{ (\mu^2 - 1)/\mu \} F(\pi/2, k) - \mu E(\pi/2, k) \quad (30)$$

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