Shrinkage Fitting of Circular Discs into Two Similar Circular Holes in Infinite Plate*

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In cases where two circular inserts with the same elastic constants are shrink-fitted into two similar circular holes in an infinite plate, the stress concentration has been investigated by using the dimensionless principal stress, on the basis of the uniform compressible stress acting on the interface of a single hole with the shrinkage insert, and the following conclusions can be made concerning the tendency of \( \bar{N}_m \) or \( \bar{N}_{tm} \) which shows the dimensionless maximum value of the maximum principal stress or of double the maximum shearing stress respectively:

1. \( \bar{N}_m \) and \( \bar{N}_{tm} \) on the plate and on the inserts are generated on the symmetrical axis connecting the centers of the two circular holes.

2. \( \bar{N}_m \) and \( \bar{N}_{tm} \) on the plate are mainly functions of the following two values: (1) the ratio of Young's moduli of the plate and the inserts \( (\alpha') \), and (2) the ratio of the distance between the two centers and the diameter of the circular holes \( (\cosh \xi \eta) \).

The values of \( \bar{N}_m \) and \( \bar{N}_{tm} \) become greater as the value of \( \xi \eta \) approaches zero, and the value of \( \bar{N}_m \) increases proportionally as the value of \( \alpha' \) is further apart from a particular value.

The values of \( \bar{N}_m \) and \( \bar{N}_{tm} \) generated on the inserts increase proportionally as the value of \( \alpha' \) or \( \xi \eta \) approaches zero, and the value of \( \bar{N}_{tm} \) on the inserts is fairly smaller than that on the plate.

1. Introduction

Author's previous paper (1) reported on the analysis for the case where two shrinkage inclusions were considerably harder than the exterior medium. In the example (1) treated therein two reinforced glass rods set at a short distance apart are shrunk during the curing of plastics materials.

In this paper, the analysis for a more general case where two shrinkage inclusions have any elastic constants is reported, such as two wire inclusions sealed in glass which are expanded by heating, or a sudden change of heat given to two adjacent points on a metallic plate.

When two discs with the same diameter and elastic constants are shrink-fitted into two similar circular holes in an infinite plate, the patterns of the stress concentration occurring on the discs or on the plate are shown by the dimensionless principal stress on the basis of the uniform stress acting on the interface of a single hole with the shrinkage insert.

2. Basic equations

The use of the stress function in the bipolar coordinates is helpful in solving two-dimensional problems of an infinite plate containing two circular holes or of an eccentric ring. If \( a^* \) is the distance between either of the centers of the two poles and the origin, and the relations \( h = (\cosh \xi - \cos \eta)/a^* \), \( x = h^{-1} \sin \eta \), \( y = h^{-1} \sinh \xi \) are assumed, then a family of the curves \( \eta = \text{constant} \) represents the arcs of the circles touching the poles, and a family of curves \( \xi = \text{constant} \) represents the circles which have centers on the \( y \)-axis and surround one pole.

Denoting by \( \chi \) the stress function and by \( \bar{a} \xi \), \( \bar{a} \eta \), the radial, circumferential, and shearing stresses on the circle \( \xi = \text{constant} \), respectively, we can get the following equations:

\[
\begin{align*}
\bar{a}^* \xi \eta &= \left( (\cosh \xi - \cos \eta) \frac{\partial^2}{\partial \eta^2} - \sinh \xi \frac{\partial}{\partial \xi} \right) (hX) \\
\bar{a}^* \eta \xi &= \left( (\cosh \xi - \cos \eta) \frac{\partial^2}{\partial \xi^2} - \sin \xi \frac{\partial}{\partial \xi} \right) (hX) \\
\bar{a}^* \xi \xi &= - (\cosh \xi - \cos \eta) \frac{\partial^2}{\partial \xi^2} (hX)
\end{align*}
\]

Next \( hX \) which is necessary to obtain the displacement must satisfy the following relations:

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\[ hQ = \int \int \left( \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} - 1 \right) (hX) d\xi d\eta \]

\[ \left( \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} - 1 \right) (hQ) + 4 \frac{\partial^2}{\partial \xi \partial \eta} (hX) = 0 \] ....(2)

In the case of mean plane stress, denoting by \( u_r \) and \( u_\theta \) the radial and the tangential displacement on the circle \( \xi = \text{constant} \), respectively, we find

\[ Eu_r = \frac{\partial}{\partial \xi} - h \frac{\partial Q}{\partial \eta} \]

\[ Eu_\theta = \frac{\partial}{\partial \eta} + h \frac{\partial}{\partial \xi} \]

where \( E \): Young's modulus, \( \sigma \): Poisson's ratio, \( \mu = 1 - \sigma \)

2.1 Case where any stress distribution, being symmetrical with respect to the \( x \) and \( y \)-axes and in equilibrium on either circular hole, acts on two similar circular holes bored in an infinite plate and moreover the stress on the plate is zero at infinity

The stress function \((1)\) which satisfies the condition of the above caption is even for the variables \( \xi, \eta \) (see Fig. 1).

\[ hX = K (\cosh \xi - \cos \eta) \log_e (\cosh \xi - \cos \eta) + \sum_{n=1}^\infty \phi_n(\xi) \cos n\eta \] .......(4)

where

\[ \phi_n(\xi) = A_n \cosh (n+1)\xi + B_n \cosh (n-1)\xi \]

By Eqs. (4) and (1), we find the stress components produced on the infinite plate as follows:

\[ a^* \bar{\sigma}_{\xi\xi} = \phi_0(\xi) - (K/2) \cosh 2\xi + K \cosh \xi \cos \eta - (K/2) \cos 2\eta \]

\[ + L \int_{-\infty}^{\infty} \left\{ (n-1)(n+2) \phi_{n-1}(\xi)/2 \right\} \cos n\eta \]

\[ + \sum_{n=1}^\infty \left\{ (n+1)(n-2) \phi_n(\xi)/2 \right\} \cos n\eta \]

\[ a^* \bar{\sigma}_{\eta\eta} = (K/2) \cosh 2\xi + \phi_0(\xi) - \phi_1''(\xi)/2 - K \cosh \xi \cos \eta + (K/2) \cos 2\eta \]

\[ + \sum_{n=1}^\infty \left\{ (n+1)(n-2) \phi_{n-1}(\xi)/2 \right\} \cos n\eta \]

\[ a^* \bar{\tau}_{\xi\eta} = -K \cosh \xi \sin \eta + (K/2) \cosh 2\xi + \phi_0(\xi) - \phi_1''(\xi)/2 + \phi_1''(\xi)/2 \]

\[ \sum_{n=1}^\infty \left\{ (n+1)(n-2) \phi_{n-1}(\xi)/2 \right\} \sin n\eta \]

where \( \phi_0(\xi) = \phi_1(\xi) = \phi_1''(\xi) = 0 \)

Supposing the circles \( \pm \xi_1 = \text{constant} \) as two circular holes, we can put the stress components on the circular hole as follows:

\[ a^* \bar{\sigma}_{\xi\xi} = \sum_{n=1}^\infty A_n \sin n\eta \]

\[ a^* \bar{\sigma}_{\eta\eta} = \sum_{n=1}^\infty B_n \cos n\eta \] .......(6)

Here the subscript 1 of the stress notations indicates a value on the circle \( \xi_1 = \text{constant} \).

If \( \xi_1 > 0 \), the condition \((1)\) that the stress distribution on the circular hole should be in equilibrium is given by

\[ 2\pi \int_{-\infty}^{\infty} (a_n + c_n) e^{-\xi_1} = 0 \] .......(7)

And if \( u_r \) and \( u_\theta \) are, as previously mentioned \((1)\), the radial and the tangential components of the displacement on the circular hole, we can express them as follows:

\[ Ehu_r = \sum_{n=1}^\infty F_n \cos n\eta \]

\[ Ehu_\theta = \sum_{n=1}^\infty L_n \sin n\eta \] .......(8)

![Fig. 1](image1.png) The correlation of the bipolar and Cartesian coordinates

![Fig. 2](image2.png) The circular disc or the eccentric ring represented by the bipolar coordinates
2.2 Case where the stress distribution in equilibrium and symmetrical about the axis acts on the periphery of the disc

2.2.1 The stress function of the disc as a limiting case of the eccentric ring

If the stress distribution acts symmetrically on the inner or outer boundary of the eccentric ring and is in equilibrium on each boundary, Jeffery's stress function\textsuperscript{11} using the bipolar coordinates will be even to \( \eta \) (see Fig. 2), and the stress function is:

\[
\hat{h} = \hat{B}_0 \xi (\cosh \xi - \cos \eta) + \sum_{n=1}^{\infty} \hat{D}_n \sinh (n+1) \xi \sin n \eta 
\]

\[
\hat{Q} = \sum_{n=1}^{\infty} \hat{D}_n \sinh (n-1) \xi \cos n \eta 
\]

where

\[
\hat{h} = (\cosh \xi - \cos \eta) / \alpha 
\]

\[
\hat{D}_n = \hat{A}_n \sinh (n+1) \xi + \hat{B}_n \sinh (n-1) \xi 
\]

\[
\hat{C}_n = \hat{D}_n \cosh (n+1) \xi + \hat{C}_n \cosh (n-1) \xi 
\]

\[
\hat{N}(\xi, \eta) = \hat{A} \cos \eta + \hat{B} \sin \eta + \hat{C} \sinh \xi + \hat{D} \cosh \xi 
\]

and \( \hat{D}_1 = 0 \)

Here, to distinguish the eccentric ring from the infinite plate, the hat sign \( \hat{\cdot} \) is added to each notation of the inclusions.

Substituting Eq. (9) into Eq. (2), we find

\[
\frac{1}{2} \hat{h} = \hat{B}(\xi \cos \xi - \cos \eta) + \sum_{n=1}^{\infty} \hat{D}_n \sin n \eta 
\]

\[
\hat{N}(\xi, \eta) = \hat{A} \cos \eta + \hat{B} \sin \eta + \hat{C} \sinh \xi + \hat{D} \cosh \xi 
\]

where

\[
\hat{D}_n = \hat{A}_n \sinh (n+1) \xi + \hat{B}_n \sinh (n-1) \xi 
\]

\[
\hat{C}_n = \hat{D}_n \cosh (n+1) \xi + \hat{C}_n \cosh (n-1) \xi 
\]

\[
\hat{N}(\xi, \eta) = \hat{A} \cos \eta + \hat{B} \sin \eta + \hat{C} \sinh \xi + \hat{D} \cosh \xi 
\]

If all points of the eccentric ring on the \( y \)-axis before deformation are still assumed to lie on the same axis after deformation, \( \hat{N}(\xi, \eta) \) can be derived by substituting Eq. (9) and Eq. (10) into Eqs. (3) as follows:

\[
\hat{N}(\xi, \eta) = \hat{A} \cos \xi - \cos \eta + \hat{B} \sin \eta 
\]

So the displacement components are expressed by Eqs. (11) and (12) as

\[
\hat{\varepsilon}_i = (\hat{D} - 2) \hat{B}_0 \xi (\cosh \xi - \cos \eta) 
\]

\[
+ \hat{D}_0 \sin \xi \cos \eta 
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \hat{D}_n \sin (n+1) \xi \sin n \eta 
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \hat{D}_n \sin (n-1) \xi \sin n \eta 
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \hat{D}_n \sin (n+1) \xi \sin n \eta 
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \hat{D}_n \sin (n-1) \xi \sin n \eta 
\]

If the disc is taken as a limiting case of the eccentric ring, the stress and the displacement at the pole (\( \xi = \infty \)) must be continuous and finite; so the relations between coefficients in Eq. (9) must satisfy the next ones:

\[
\hat{B}_0 = 0, \hat{A}_1 + \hat{C}_1 = 0 
\]

\[
\hat{A}_n + \hat{C}_n = 0, \hat{B}_n + \hat{D}_n = 0 \quad (n \geq 2) 
\]

Substituting these equations into Eqs. (9) and (10), we find the stress and displacement functions as follows:

\[
\hat{h} = \sum_{n=1}^{\infty} \hat{D}_n (\xi \cos \eta) \sin n \eta 
\]

\[
\frac{1}{2} \hat{Q} = \sum_{n=1}^{\infty} \hat{D}_n \sinh (n+1) \xi \cos n \eta 
\]

\[
\hat{N}(\xi, \eta) = \hat{A} \cos \eta + \hat{B} \sin \eta + \hat{C} \sinh \xi + \hat{D} \cosh \xi 
\]

\[
\hat{\varepsilon}_i = \hat{D}_0 \sin \xi \cos \eta 
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \hat{D}_n (n+1) \xi \sin n \eta 
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \hat{D}_n \xi \sin n \eta 
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \hat{D}_n \xi \sin n \eta 
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \hat{D}_n \xi \sin n \eta 
\]

\[
\text{where if} \ \xi > 0,
\]

\[
\hat{D}_n (\xi) = \hat{D}_n (\xi) / \xi 
\]

\[
\hat{D}_n (\xi) = \hat{D}_n (\xi) / \xi 
\]

\[
\hat{D}_n (\xi) = \hat{D}_n (\xi) / \xi 
\]

\[
\text{if} \ \xi \geq 1,
\]

\[
\hat{D}_n (\xi) = \hat{D}_n (\xi) / \xi 
\]

\[
\hat{D}_n (\xi) = \hat{D}_n (\xi) / \xi 
\]

2.2.2 Stress components acting on the disc

Supposing the periphery of the disc is a circle \( \xi_1 = \text{constant} \), and distinguishing by underlining the stress notations of the disc from those of the infinite plate, we can assume the following relations as those in Eq. (6):

\[
\hat{D}_n (\xi_1) = \hat{D}_n (\xi_1) / \xi_1 
\]

\[
\hat{D}_n (\xi_2) = \hat{D}_n (\xi_2) / \xi_2 
\]

If Eq. (4) is compared with Eqs. (13), the stress components produced on the disc can be derived from Eqs. (5) by putting \( K = 0 \), \( \alpha^* = \alpha^* \), \( \phi_\xi (\xi) \rightarrow \phi_\xi (\xi) \), etc. especially, if \( \xi_1 \) and \( \xi_2 \) are derived by putting \( \xi = \xi_1 \) in the above stress components are equated to Eqs. (15) and the terms of \( \sin n \eta \) and \( \cos n \eta \) in both relations are compared with each other, we can obtain the following relations:

\[
\phi_\xi (\xi_1) = \frac{1}{n} \left[ \frac{\sinh (n+1) \xi_1}{n+1} \right] \left[ \frac{\sinh (n-1) \xi_1}{n-1} \right] 
\]

\[
\times \sum_{p=1}^{\infty} \Delta_p e^{-p \xi_1} \sin^2 \xi_1 + 2 \Delta_0 a^* \quad (n \geq 2) 
\]

\[
\phi_\xi (\xi_2) = \frac{2}{n} \sum_{p=1}^{\infty} \Delta_p e^{-p \xi_2} \sin^2 \xi_2 
\]

\[
- \frac{1}{2} \sum_{n=1}^{\infty} \hat{D}_n (\xi_2) \sin n \eta 
\]

\[
\sin \xi_1 = \sin \xi_2 
\]

\[
\xi_1 = \xi_2 = 0 
\]

Substituting \( \xi = \xi_1 \) into Eqs. (14) and comparing this equation with Eqs. (16), we find

\[
\hat{A}_1 = -\sum_{p=1}^{\infty} \Delta_p e^{-p \xi_1}, \hat{B}_0 = \hat{B}_n = 0 
\]

\[
\hat{A}_n = \sum_{p=1}^{\infty} \Delta_p e^{-p \xi_1} \quad (n \geq 2) 
\]

\[
\hat{B}_n = \sum_{p=1}^{\infty} \Delta_p e^{-p \xi_1} \quad (n \geq 2) 
\]
Substituting $B_9=0$ and $\xi=\xi_1$ into Eqs. (11) and (12), and multiplying both sides by $\hat{h}$, and then using Eqs. (14) and (16), we can put the displacement components

$$\hat{E}_{h1} = \sum_{n=1}^{\infty} F_n \cos n\eta$$

$$\hat{E}_{h2} = \sum_{n=1}^{\infty} L_n \sin n\eta$$

as in the case of the displacement components [Eqs. (8)]. When $\xi_1>0$ and $n \geq 2$, the following relations should be satisfied:

$$\hat{F}_* = 2\hat{F}/\hat{d} = (2-\hat{w}) \sum_{n=1}^{\infty} \hat{d}_n e^{-n\eta}/\hat{d}$$

$$\hat{F}_n = \hat{w}(\hat{X}_n/2 - (\hat{\xi}_n/\hat{d}) \sinh \hat{\xi}_1) - \hat{\xi}_n$$

$$+ (\hat{w} \cosh \hat{\xi}_1 - 2\hat{e}_1) \sum_{n=1}^{\infty} \hat{d}_n e^{-n\eta}/\hat{d}$$

$$- 2(\hat{B}/\hat{d}) \cosh \hat{\xi}_1$$

$$\hat{F}_n = \hat{w}(\hat{X}_n - (\hat{\xi}_n/\hat{d}) \sinh \hat{\xi}_1) - \hat{\xi}_n$$

$$+ (\hat{w} \cosh \hat{\xi}_1 - 2\hat{e}_1) \sum_{n=1}^{\infty} \hat{d}_n e^{-n\eta}/\hat{d}$$

$$- 2(\hat{B}/\hat{d}) \cosh \hat{\xi}_1$$

$$\hat{L}_n = \hat{w}(\hat{X}_n/2 - (\hat{\xi}_n/\hat{d}) \sinh \hat{\xi}_1) - \hat{\xi}_n$$

$$+ (\hat{w} \cosh \hat{\xi}_1 - 2\hat{e}_1) \sum_{n=1}^{\infty} \hat{d}_n e^{-n\eta}/\hat{d}$$

$$\hat{L}_n = \hat{w}(\hat{X}_n - (\hat{\xi}_n/\hat{d}) \sinh \hat{\xi}_1) - \hat{\xi}_n$$

$$+ (\hat{w} \cosh \hat{\xi}_1 - 2\hat{e}_1) \sum_{n=1}^{\infty} \hat{d}_n e^{-n\eta}/\hat{d}$$

$$\hat{L}_n = \hat{w}(\hat{X}_n/2 - (\hat{\xi}_n/\hat{d}) \sinh \hat{\xi}_1) - \hat{\xi}_n$$

$$+ (\hat{w} \cosh \hat{\xi}_1 - 2\hat{e}_1) \sum_{n=1}^{\infty} \hat{d}_n e^{-n\eta}/\hat{d}$$

$$\hat{L}_n = \hat{w}(\hat{X}_n - (\hat{\xi}_n/\hat{d}) \sinh \hat{\xi}_1) - \hat{\xi}_n$$

where $\hat{Z}_n$ and $\hat{X}_n$ are in the forms with the hat signs removed off the notations of the proviso in Eqs. (16) [see Eqs. (27)].

In Fig. 3, if the circle $\xi_1$ constant with the center $O$ or $O$, shows the circular hole before expanding or the periphery of the disc before shrinking respectively, any two points $A$ and $B$ on each circle which have a clockwise angle $\theta$ from the $y$-axis passing through each center, must be fixed as above mentioned. Then Eqs. (6) and (15) can be rearranged by the relation

$$\cos \theta = (\cosh \hat{\xi}_1 \cos \eta - 1)(\cosh \hat{\xi}_1 - \cos \eta)^{-1}$$

as follows:

$$\frac{\hat{X}_1}{\hat{X}_1} = \sum_{n=1}^{\infty} \hat{a}_n \cos n\eta = \sum_{n=1}^{\infty} \hat{a}_n \hat{d} \cosh n\eta = \hat{X}_1$$

$$\frac{\hat{X}_1}{\hat{X}_1} = \sum_{n=1}^{\infty} \hat{a}_n \sin n\eta = \sum_{n=1}^{\infty} \hat{a}_n \hat{d} \sinh n\eta = \hat{X}_1$$

$$\frac{\hat{X}_1}{\hat{X}_1} = \sum_{n=1}^{\infty} \hat{a}_n \hat{d} \cosh n\eta = \hat{X}_1$$

$$\frac{\hat{X}_1}{\hat{X}_1} = \sum_{n=1}^{\infty} \hat{a}_n \hat{d} \sinh n\eta = \hat{X}_1$$

where $X_n$ and $Z_n$ are in the forms with the hat signs removed off the notations of the proviso in Eqs. (16) [see Eqs. (27)].

2.3 Shrinkage fitting perfectly fixed on the boundary ($X_n=0$)

When two discs (radius $R$) are to be shrink-fitted into similar circular holes (radius $r$) in the plate, we expand uniformly the whole plate by heating. As soon as the radius of the hole expands to $R$, we insert the two discs smoothly in the circular holes, and then fix the boundaries. At that instant, if this fixed boundary of the circular hole and disc is treated with a circle $\xi_1$ constant, the size of this plate is expanded $R/r$ times to the initial size; so between $d^*$, $a^*$ of the plate before expanding and $\hat{d}^*$, $\hat{a}^*$ of the disc before shrinking there must hold the next relation:

$$\hat{d}^*/\hat{a}^* = a^*/d^* = R/r$$

where $d^* = r \cosh \xi_1$, $a^* = r \sinh \xi_1$.

In Fig. 3, if the circle $\xi_1$ constant with the center $O$ or $O$, shows the circular hole before expanding or the periphery of the disc before shrinking respectively, any two points $A$ and $B$ on each circle which have a clockwise angle $\theta$ from the $y$-axis passing through each center, must be fixed as above mentioned. Then Eqs. (6) and (15) can be rearranged by the relation

$$\cos \theta = (\cosh \hat{\xi}_1 \cos \eta - 1)(\cosh \hat{\xi}_1 - \cos \eta)^{-1}$$

as follows:

$$\frac{\hat{X}_1}{\hat{X}_1} = \sum_{n=1}^{\infty} \hat{a}_n \cos n\eta = \sum_{n=1}^{\infty} \hat{a}_n \hat{d} \cosh n\eta = \hat{X}_1$$

$$\frac{\hat{X}_1}{\hat{X}_1} = \sum_{n=1}^{\infty} \hat{a}_n \sin n\eta = \sum_{n=1}^{\infty} \hat{a}_n \hat{d} \sinh n\eta = \hat{X}_1$$

$$\frac{\hat{X}_1}{\hat{X}_1} = \sum_{n=1}^{\infty} \hat{a}_n \hat{d} \cosh n\eta = \hat{X}_1$$

$$\frac{\hat{X}_1}{\hat{X}_1} = \sum_{n=1}^{\infty} \hat{a}_n \hat{d} \sinh n\eta = \hat{X}_1$$

where $X_n$ and $Z_n$ are in the forms with the hat signs removed off the notations of the proviso in Eqs. (16) [see Eqs. (27)].

In this paper, the displacement on the plate is determined to be symmetrical about the $x$ and $y$ axes, or that on the disc is determined to be symmetrical only about the $y$ axis (see Fig. 3). So let us consider on the $y$ axis the middle point $O'$ of the deformed hole or of the deformed periphery of the disc, and draw a circle with center $O'$ which has the same radius as the initial one, as shown by the dot-dash circle (see Fig. 4). For shrinkage calculation, it is convenient to use the displacement in relation to circle $O'$. The difference $\Delta r_1$ between center $O$ and $O'$ for the hole is found by using $u_1$ in Eqs. (8) as follows:

$$\Delta r_1 = (u_1 + u_{1}^*)/2$$

Similarly, for the disc the difference $\Delta r_1$ is.

Fig. 3 The correlation of the circular hole and the periphery of the circular disc before shrinkage.

Fig. 4 The relation between the displacement for O-circle and the relative displacement for O' circle which has on the y-axis the middlepoint O' of the deformed hole.
found by using $\hat{u}_t$, in Eqs. (18) as follows:

$$\ddot{r}_t := (\hat{r}_t(t), s_t + \hat{u}_t(t)) / 2$$

Let us add the dash signs to the notations of the displacement components to show the relative displacement to the circle $O$.

Then from Fig. 4 we find:

$$u_{t_1} = u_{t_2} = a_{t_1} \cos \theta, \quad u_{t_2} = a_{t_1} \sin \theta$$

Multiplying both sides of the above equations by $Eh$, we can get the rearranged equations as follows:

$$Eh u_{t_1} = (Eh / \hat{E}h) \hat{E} h u_{t_1} = Eh(r - R)$$

Putting $\alpha = 0$ in Eqs. (25), we can get the equations in which the starred capitals can be loaned from author's previous paper (9).

Let us show collectively the unknown factors and notations included in those equations.

$$Y a^*_x = K - \sum_{p=1}^{\infty} a_p e^{-n-1} / \sin h\xi_1$$

$$X a^*_x = \sum_{p=1}^{\infty} a_p \sin h(n-1) / \sin h\xi_1$$

$$Z a^*_x = \sum_{p=1}^{\infty} \frac{a_p}{n(n-)} / \sin h\xi_1$$

$$G(a_x) = 1 / 2 + \sin h\xi_1 / \cos h\xi_1$$

$$k_p = \sin h n_1 \{n_1(2 \sin h n_1 + n_1 \sin 2 \xi_1)\}^{-1}$$

$$f_1 = \sin h n_1 \{n_1(2 \sin h n_1 + n_1 \sin 2 \xi_1)\}^{-1}$$

$$g_n = \cos h n_1 - \cos h 2 \xi_1$$

$$\Theta = 2 \cos h \xi_1$$

In Eqs. (25) we put Eqs. (27) in the starred capitals without hat signs and put Eqs. (26) in the starred capitals with hat signs. And we introduce new factors $w^*$, $g^*$, combining with $\alpha$, in place of $w$, $g$ as follows:

$$w^* = w - \alpha \hat{w}, \quad g^* = g_n + \alpha$$

$$b_m = \lim_{m \to \infty} \frac{a_m}{n_m} e^{-n_m / 2} (2 - w^*)^{-1}$$

where $b_m$ represents the uniform compressible stress acting on the outer boundary of a disc shrink-fitted into a single circular hole in an infinite plate. Also, as shown in author's previous paper (9), for a certain integer $m$, we find

$$\sum_{p=1}^{\infty} a_p e^{-n_p} / a_p = \sum_{p=1}^{m} a_p e^{-n_p} / a_p = e^{-n_1} (X_{n+1} - e^{-n_1} X_m)$$

And therefore, the first equation in Eqs. (25) is given by

$$Y (1 - 2 \sin h \xi_1) \cos h \xi_1 - f_x + w^* G(\xi_1) \sin h \xi_1) + (w^* / 2 - 6k_2) X_1 + 6g_2 Z_2 + 2 \alpha \sin h \xi_1 \{X_{n+1} - e^{-n_1} X_n\} e^{-n_1} + 2w' \sin h \xi_1 \sum_{m=1}^{n} (2b_m X_m \sin h \xi_1 + f_n Z_n)$$

$$= (2 - w') b_m \sin h \xi_1$$
Similarly, from the 2nd equation in Eqs. (25), we find
\[(g_2 + 4k_2 \cosh Z_2) \sinh Z_2 = (w' - 2 - 6k_2) \cosh Z_2 X_2 + 3(w' - 4k_2) \sinh Z_2 X_2 = 0 \tag{31}\]

From the 3rd equation in Eqs. (25), for \(m \geq n \geq 2\), we find
\[
\begin{align*}
\Theta_2 Y (n - 2) g_{n+1} Z_{n-1} &= \left( w' \sinh Z_n + 2g_n \cosh Z_n \right) Z_n + (n + 2) g_{n+1} Z_{n+1} \\
+ \left[ (w' / 2) (n - 1) Y - (n - 2) k_n \cosh Z_n \left( \sinh Z_n \right)^{n+1} X_n \\
+ \cosh Z_n \left( 2k_n \sinh Z_n \left( \sinh Z_n \right)^{n+1} - w' X_n \right) X_n \\
+ \left[ (w' / 2) (n + 1) Y + (n + 2) k_n \cosh Z_n \left( \sinh Z_n \right)^{n+1} X_n \right] X_{n+1} = 0
\end{align*}
\tag{32}\]

From the 4th equation in Eqs. (25), for \(m \geq n \geq 2\), we find
\[
\begin{align*}
2Y \left[ 2k_n \cosh Z_n + h_n + f_n \sinh (n - 1) \right] \sinh Z_n \\
+ \left( n - 2 \right) \left( w' / 2 - n \left( n - 1 \right) k_n \cosh Z_n \left( \sinh Z_n \right)^{-1} \right) Z_n \\
- \left[ n \left( n - 1 \right) m \cosh Z_n + 2g_n \sinh Z_n \cosh Z_n \left( \sinh Z_n \right)^{n+1} \right] Z_n \\
+ \left( n + 2 \right) \left( w' / 2 - n \left( n + 1 \right) k_n \cosh Z_n \left( \sinh Z_n \right)^{-1} \right) Z_{n+1} \\
+ \left[ (n - 1) \left( n - 1 \right) + 2k_n \sinh Z_n \left( \sinh Z_n \right)^{-1} \right] Z_{n-1} \\
- 2 \cosh Z_n \left( 2k_n \cosh Z_n \left( \sinh Z_n \right)^{n+1} \right) Z_n \\
+ \left[ \left( n + 1 \right) \left( n + 1 \right) - 2k_n \sinh Z_n \left( \sinh Z_n \right)^{n+1} \right] Z_{n+1} = 0
\end{align*}
\tag{33}\]

Denoting by \(X_0, Z_0\) etc. the conditions of Eqs. (7) in equilibrium on the fitting boundary, we find
\[
e^{w' X_0} = \left( m(n - 2) Z_{n-1} - m(n - 1) e^{-w' X_0} Z_n + mX_{n+1} + e^{-w' X_0} \right) e^{-w' X_0} = 0
\tag{34}\]

Dividing both sides of Eqs. (30) \(\sim\) (34) by \(b_n\) and giving \(n\) in Eqs. (32) and (33) \(\sim n\), we can get \((2m + 1)\) equations. From them, therefore, the approximation of the unknown \(Y b_n^{-1}, X_0 b_n^{-1} \cdots X_{n+1} b_n^{-1}, Z_{n-1}, Z_n\) can be obtained.

And the relations between these unknown factors and the coefficients of Fourier's series in Eqs. (6) or \(K\) in Eqs. (27) are expressed in author's previous paper \(\text{(*)}\) as follows:

\[
\begin{align*}
c_0 / a^* &= YG \left( \xi_1 \right) + 2 \sum_{n=1}^{\infty} \left( 2k_n X_n \sinh \xi_1 + f_n Z_n \right) \\
a_0 / a^* &= X_0 - 2X_n \cosh \xi_1 + X_{n+1} \\
c_0 / a^* &= 2X_n \cosh \xi_1 \left( \sinh \left( n - 2 \right) - n \left( n - 1 \right) \right) \cosh \xi_1 + 2(n - 1) \cosh \xi_1 \left( \sinh \left( n - 1 \right) \right) \cosh \xi_1 + (n + 1) (n + 2) \cosh \xi_1 \\
K / a^* &= \left( \sinh \left( n - 1 \right) \right) \cosh \xi_1 \left( \sinh \left( n - 1 \right) \right) \cosh \xi_1 + (n + 1) (n + 2) \cosh \xi_1 \\
&\text{where } X_0 = 0
\end{align*}
\tag{35}\]

2.3-1 Stress field of the infinite plate

As mentioned in the previous paper \(\text{(*)}\), the coefficients \(A_n, B_n\) which are necessary to derive the stress field of the plate, are given by:

\[
\begin{align*}
A_n / a^* &= (K / a^* - Y \tanh \xi_1) / 2 \\
B_n / a^* &= (Y / Z) \cosh 2 \xi_1 \tanh \xi_1 + c_0 / a^* \\
A_n / a^* &= \left( K / a^* \right) / \left( n(n + 1) - n(n + 1) \sinh \xi_1 \cosh \xi_1 \right) \\
&\text{where } X_n = 0 \\
B_n / a^* &= \left( K / a^* \right) / \left( n(n + 1) - n(n + 1) \sinh \xi_1 \cosh \xi_1 \right) \\
&\text{where } X_n = 0
\end{align*}
\tag{36}\]

Substituting the unknown \(Y b_n^{-1}, X_0 b_n^{-1}, Z_{n-1} b_n^{-1}\) and \(c_0(a^* b_n)^{-1}\), \(K(a^* b_n)^{-1}\) of Eqs. (35) into Eqs. (36), we can obtain \(A_n(a^* b_n)^{-1}, B_n(a^* b_n)^{-1}\) etc.

Consequently, by using Eqs. (5) the stress field of the infinite plate is shown as the dimensionless form based on \(b_n\).

2.3-2 Stress field of the disc

Similarly, as for the stress field of the plate, we substitute Eqs. (16), (29), (21) and (35) into (17) in order to obtain \(\phi_h(\xi_1)\) etc.

Then we find
\[
\begin{align*}
\hat{A}_1 / a^* &= -e^{-w' X_0} (X_{n+1} - e^{-w' X_0} X_n) \\
\hat{B}_1 / a^* &= -YG(\xi_1) + e^{-w' X_0} (X_{n+1} - e^{-w' X_0} X_n) - 2 \sum_{n=1}^{\infty} \left( 2k_n X_n \sinh \xi_1 + f_n Z_n \right) \\
\hat{A}_n / a^* &= -e^{w' X_0} \left( (n - 1) - n(n + 1) \sinh \xi_1 \cosh \xi_1 \right) e^{-w' X_0} (X_{n+1} - e^{-w' X_0} X_n) \\
\hat{B}_n / a^* &= e^{w' X_0} \left( (n + 1) - n(n + 1) \sinh \xi_1 \cosh \xi_1 \right) e^{-w' X_0} (X_{n+1} - e^{-w' X_0} X_n)
\end{align*}
\tag{37}\]
Especially, for the circumferential stress component \( \tau_{\theta\theta} \), \( \phi_1''(\xi_1) \) etc. are given by Eqs. (17) as follows:

\[
\phi_1''(\xi_1) = -4 \sum_{n=1}^{\infty} \alpha_n e^{-n \xi_1},
\]

\[
\phi_2''(\xi_1) = -(\xi_1^2 - 1) \phi_1'(\xi_1) - 2n \phi_1(\xi_1) \]

(\( n \geq 2 \))

Let us rearrange the 2nd equation in Eqs. (5) at \( K=0 \), \( \xi=\xi_1 \), and add hat signs to \( \alpha^* \), \( \phi^* \) \( (\xi_1) \) and others; especially, let the above \( \phi_1''(\xi_1) \) take the place of \( \phi_1''(\xi_1) \), then the term of \( \cos n \eta \) of \( \tau_{\theta\theta} \) in Eqs. (5) can be expressed as:

\[
(n-1)(n-2)\phi_1''(\xi_1)/2 - (n^2-1)\phi_3''(\xi_1) \cos \xi_1
\]

\[
+ (n+1)(n+2)\phi_1''(\xi_1)/2 - \phi_3''(\xi_1) \sinh \xi_1
\]

\[-2(\tilde{X}_{n+1} - 2\tilde{X}_n \cosh \xi_1 + \tilde{X}_{n-1}) \delta^*
\]

in which, by using Eqs. (21) and (35) we have

\[
\tilde{X}_{n+1} - 2\tilde{X}_n \cosh \xi_1 + \tilde{X}_{n-1} = \delta_n/\delta^* = a_n/a^*
\]

Also, referring to the 1st in Eqs. (5), consequently we find

\[
\frac{\tau_{\theta\theta}}{\tau_{\theta\theta}^*} = 2\sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha^*} \cos n \eta
\]

\[-2 \sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha^*} \cos n \eta
\]

(38)

If \( \tilde{A}_n(\delta^* b_\alpha)^{-1} \) and \( \tilde{B}_n(\delta^* b_\alpha)^{-1} \) are derived from Eqs. (37) whose both sides are divided by \( b_\alpha \), the dimensionless stress field of the disc can be shown by Eqs. (5) on the basis of \( b_\alpha \).

3. Numerical results

Typical inclusion-matrix combinations calculated in this paper on the assumption of Poisson's ratio 0.3 for both materials, are three kinds, each of which has two cases of \( X_\alpha=0 \) and \( X_\alpha=0 \) as the boundary condition (see Table 1).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( w=1-\sigma )</th>
<th>( \tilde{w} )</th>
<th>( \bar{w} )</th>
</tr>
</thead>
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<tr>
<td>1/3</td>
<td>2/3</td>
<td>2/3</td>
<td>(2/3)²</td>
</tr>
<tr>
<td>1</td>
<td>0.7 (0.5)</td>
<td>0.7 (0.5)</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.7</td>
<td>0.7</td>
<td>-1.4</td>
</tr>
</tbody>
</table>

Fig. 5 The stress distribution on the shrinkage boundary

As to values of \( \frac{\tau_{\theta\theta}}{b_\alpha} \) or \( \frac{\tau_{\theta\theta}}{b_\alpha} \) on the fitting boundary, the solid curves in Fig. 5 are all alike independently of \( \alpha \) or \( \xi_1 \) value; moreover, according as the value of \( \alpha \) or \( \xi_1 \) increases, the curves of \( \frac{\tau_{\theta\theta}}{b_\alpha} \) or \( \frac{\tau_{\theta\theta}}{b_\alpha} \) generally tend to become even and to approach 1 or 0 respectively, and whether \( X_\alpha=0 \) or \( X_\alpha=0 \), the maximum value of \( \frac{\tau_{\theta\theta}}{b_\alpha} \) curves, which are concave with respect to the angle \( \theta \) (V-shape), occurs at \( \theta=\pi \). We express this value by \( \frac{\tau_{\theta\theta}}{b_\alpha} \).

If \( X_\alpha=0 \) and \( \alpha=0 \), the value of \( \frac{\tau_{\theta\theta}}{b_\alpha} \) at \( \theta=\pi \) on the fitting boundary varies from positive to negative numbers, and finally approaches -1 according as the value of \( \xi_1 \) increases from around 0.4, but still the shape of the stress distribution retains a V-shape similar to \( \frac{\tau_{\theta\theta}}{b_\alpha} \) curve in Fig. 5.

However, regardless of the value of \( \xi_1 \) or \( \eta_1 \), if \( X_\alpha=0 \) at around \( \alpha=1/3 \), \( \frac{\tau_{\theta\theta}}{b_\alpha} \) curves tend to become even, and at around \( \alpha=1 \), they turn into convex ones (A-shape), namely

\[
\frac{\tau_{\theta\theta}}{b_\alpha} = -\frac{\tau_{\theta\theta}}{b_\alpha}
\]

(see Fig. 5).

Consequently, if \( X_\alpha=0 \), we find \( \frac{\tau_{\theta\theta}}{b_\alpha} \).
\[ \pm \frac{1}{3} \alpha^{2/3} \leq \frac{\eta H_1}{b_w} < \frac{1}{3} \alpha^{1/3} \quad \text{at } \alpha = 0; \]
\[ \pm \frac{1}{4} \alpha \leq \frac{\eta H_1}{b_w} < \frac{1}{4} \alpha \quad \text{at } \alpha \approx 1/4. \]

Next if \( X_a = 0 \), regardless of the value of \( \xi \) or \( \alpha \), \( \eta H_1 \) curves form \( \Delta \)-shape and we find
\[ -\frac{\eta H_1}{b_w} \leq \frac{1}{3} \alpha^{1/3} \quad \text{at } \alpha = 0; \]
\[ \frac{1}{4} \alpha \leq \frac{1}{4} \alpha \quad \text{at } \alpha = 0. \]

In order to show the above relations directly, the values of \( \frac{\eta H_1}{b_w} \), \( \frac{\eta H_1}{b_w} \), were indicated as parameters of \( \alpha \)-value with respect to \( d^*/r \), at the top in Figs. 6 and 7.

On the boundary of the disc, \( \frac{\eta H_1}{b_w} \) and \( \frac{\eta H_1}{b_w} \) represent both compressive stresses and when \( X_a = 0 \), there is no difference between the two principal stresses because \( \frac{\eta H_1}{b_w} = \frac{\eta H_1}{b_w} \) when \( X_a = 0 \). \( \frac{\eta H_1}{b_w} \) shows a convex curve (\( \Delta \)-shape) with respect to \( \eta \) and has the maximum value which is smaller than \( \frac{\eta H_1}{b_w} \) (see dotted curve in Fig. 5).

The greatest principal stress is made dimensionless on the basis of \( b_w \) and also the difference between the two principal stresses is treated likewise. Now, the former and the latter are represented by \( \frac{N}{N}, \frac{N_m}{N_m} \), respectively, and the maximum value of \( N \), \( N_m \), by \( \frac{N}{N}, \frac{N_m}{N_m} \) respectively.

In order to see where \( \frac{N}{N} \) and \( \frac{N_m}{N_m} \) on the infinite plate or on the disc result from shrinkage, the relations between \( \eta \) and a dimensionless pair of principal stresses on the circle \( \xi = \text{constant} \) were illustrated by various figures, according to many combinations of \( \xi \) with inclusion and matrix. Figure 8 is one of these figures.

From these figures the values of \( \frac{N}{N} \) and \( \frac{N_m}{N_m} \) were obtained, as shown in Table 2. Therefore, we can conclude that \( \frac{\eta H_1}{b_w} \), \( \frac{\eta H_1}{b_w} \), and \( \frac{\eta H_1}{b_w} \), \( \frac{\eta H_1}{b_w} \), are most important. But if \( X_a = 0 \) and \( \alpha \leq 1/3 \) we need to derive \( \frac{N}{N} \) and \( \frac{N_m}{N_m} \) from the solid curves in Fig. 9, because \( \frac{N}{N} \) increases remarkably from the shrinkage boundary on the \( y \)-axis to the origin, according as the value of \( \xi \) decreases from a special value. And the values of \( \frac{N_m}{N_m} \) on the disc at \( X_a = 0 \) are shown at the bottom in Fig. 6 and at \( X_a = 0 \) in Fig. 7.

In either case, the value of \( \frac{N}{N} \) which is double the maximum shearing stress is considerably smaller than on the infinite plate.

4. Conclusions

When two similar discs are shrink-fitted into
two similar circular holes in an infinite plate, the principal stresses are expressed in dimensionless forms on the basis of the uniform compressible stress acting on the interface of a single hole in the infinite plate where the same inclusion is shrunk. By means of these dimensionless forms, the stress concentration has been investigated mainly into three kind of inclusion and matrix (Poisson’s ratio for each kind is about 0.3). As to \( \hat{N}_m \) and \( \hat{N}_{rm} \); i.e. the maximum dimensionless value of the greatest principal stress and of double the maximum shearing stress, we can conclude as follows:

1. \( \hat{N}_m \) and \( \hat{N}_{rm} \) appearing on the infinite plate or on the disc take their position on the symmetrical axis which connects the centers of the two circular holes.

2. (i) \( \hat{N}_m \) and \( \hat{N}_{rm} \) on the infinite plate

   The above values are mainly functions of \( E/\hat{E} \) (or \( \alpha \)), which shows the ratio of Young’s modulus of the plate to that of the disc, and they are also functions of \( \cosh \xi_1 \); which shows the ratio of the distance between the centers of the two circular holes to the diameter of the circular hole. The values of \( \hat{N}_m \) and \( \hat{N}_{rm} \) increase according as the value of \( \xi_1 \) approaches 0; especially, the value of \( \hat{N}_m \) becomes larger according as the value of \( \alpha \) deviates from around 1/4 at \( X_0 = 0 \), or from around 1 at \( X_0 = \pi/2 \).

2. (ii) \( \hat{N}_m \) and \( \hat{N}_{rm} \) on the disc

   The above values increase according as the value of \( \alpha \) or \( \xi_1 \) approaches 0, and the value of \( \hat{N}_{rm} \) on the disc is fairly smaller than that on the plate.

Table 2 shows the formulas of \( \hat{N}_{rm} \) and \( \hat{N}_m \) with respect to \( \alpha \) and the shrinkage boundary conditions; and Figs. 6, 7, 9 show the values with

<table>
<thead>
<tr>
<th>Classification</th>
<th>Shrinkage boundary conditions</th>
<th>( \hat{N}_m )</th>
<th>( \hat{N}_{rm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infinite plate</td>
<td>Fixed ((X_0=0))</td>
<td>( \frac{\hat{E}/\hat{E}_m}{\xi_1} ) ( \xi_1 )</td>
<td>See Fig. 9 (Solid curve)</td>
</tr>
<tr>
<td></td>
<td>Slidable ((X_0=0))</td>
<td>( \frac{\hat{E}/\hat{E}_m}{\xi_1} ) ( \xi_1 )</td>
<td>Fig. 6 ( \frac{\hat{E}/\hat{E}_m}{\xi_1} - \frac{\gamma_1/\eta_1}{\xi_1} )</td>
</tr>
<tr>
<td>Circular disc</td>
<td>Fixed ((X_0=0))</td>
<td>( \frac{\hat{E}/\hat{E}_m}{\xi_1} ) ( \xi_1 )</td>
<td>Fig. 6 ( \frac{\hat{E}/\hat{E}_m}{\xi_1} - \frac{\gamma_1/\eta_1}{\xi_1} )</td>
</tr>
<tr>
<td></td>
<td>Slidable ((X_0=0))</td>
<td>( \frac{\hat{E}/\hat{E}_m}{\xi_1} ) ( \xi_1 )</td>
<td>See Fig. 7</td>
</tr>
</tbody>
</table>
respect to \cosh \xi.

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References