On Impact-Damper for Concentrated-Mass-Continuum System*

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Forced vibrations of a concentrated-mass-continuum system with an impact-body are discussed. The impact-body moves in a groove of the concentrated mass and strikes the concentrated mass at opposite ends of the groove.

Periodic solutions of the system excited by harmonic motion of a foundation are presented, and their stability is discussed. This paper treats only such periodic motion that during $n$ periods of harmonic motion of the foundation two impacts occur at equal time intervals and at opposite ends of the groove ($n$: odd).

The equation of motion is written in the form of a partial integro-differential equation by the use of an influence function.

As an example, numerical calculations are carried out for the simplest system. That is, the continuum of this system is a uniform coil spring. Stability regions are determined and maximum displacements of the concentrated mass are calculated. The results show that the damping effects of the impact-damper are sufficiently good.

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1. Introduction

Impact-damper is a damper using impacts between a vibrating body and an impact-body. It has been applied to, for example, turbine blades (1931)(11), wings of airplane (1945)(12) and recently lighting poles along high-speed highways(8). Studies of dynamics of impact-damper have been restricted to a mass-spring system with one degree-of-freedom(12)-(13), and dynamics of impact-damper with infinitely many degrees of freedom has never been studied.

In this paper, a study of motion of an impact-damper for a concentrated-mass-continuum system (with infinitely many degrees-of-freedom) is presented, and examples of numerical calculations are shown.

2. Nomenclature

$t$: time
$x$: coordinate along the axis of the continuum
$y_a(x, t)$: absolute displacement of the continuum
$y_r(x, t)$: relative displacement of the continuum with respect to a foundation
$m(x)$: mass per unit length of the continuum
$M_1$: concentrated mass at the tip of the continuum [this is contained in $m(x)$]
$M_2$: mass of an impact-body moving in a groove of the concentrated mass $M_1$
$M_0$: total mass of the continuum (without $M_2$)
$\mu_0=M_2/M_1$
$2z$: free path of impact-body
$y_e(t)$: displacement of the impact-body (the origin is the center of the groove at stationary state)
$v(t)=a_0 \cos(\omega t + \theta_0)$: displacement given to the foundation
$\theta_0$: phase displacement of $v(t)$ at a moment when an impact occurs at the left side end of the groove
$k$: internal viscous damping coefficient
$\xi$: coefficient of rebound
$G(x, \xi)$: influence function
$f(x, t)$: force acting on the continuum
$\omega_i$: $i$th natural circular frequency
$\varphi_i(x)$: normalized characteristic function of $i$th mode
$i$: number of modes of natural vibration
$\zeta_i=\frac{k}{2\omega_i}$: ratio of damping coefficient with respect to $\omega_i$
$\eta_i=\int_0^l m(x)\varphi_i(x)dx$: mode participation factor
$l$: total length of the continuum

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3. Basic equations

Figure 1 shows a model of an impact-damper studied in this paper. In analysis, the following are assumed:

(1) When there is no impact, the vibration system is linear.
(2) There is no external resistance.
(3) Internal viscous resistance is proportional to a strain rate.
(4) The impact-body \( M_2 \) moves without friction in the groove of the concentrated mass \( M_1 \).
(5) The impact between \( M_2 \) and \( M_1 \) is instantaneous, and their deformations are negligible.
(6) The relation between velocities before and after the impact is defined with a constant coefficient of rebound \( \varepsilon \).

Now, the equation of motion with respect to an elastic body with one dimension is expressed by, in general, the partial integro-differential equation as follows:\(^{(2)}\):

\[
y_{s}(x, t)\frac{\partial^{2}y_{s}(x, t)}{\partial t^{2}} + m(\xi)\frac{\partial y_{s}(x, t)}{\partial t} \left( f(\xi, t) - m(\xi)\omega^{2}y_{s}(x, t) \right) d\xi = y_{s}(x, t) \varepsilon \quad (1)
\]

Though Fig. 1 shows a model of transverse vibration, Eq. (1) can be applied to not only transverse vibration but also longitudinal or torsional vibration. In the case of torsional vibration, \( m(\xi) \) is the moment of inertia of the cross section, \( y(x, t) \) the angle of torsion and \( f(x, t) \) the moment of torsion.

In Eq. (1), internal damping is not considered.

Considering the internal viscous resistance, Eq. (1) is reduced to

\[
y_{s}(x, t)\frac{\partial^{2}y_{s}(x, t)}{\partial t^{2}} + m(\xi)\frac{\partial y_{s}(x, t)}{\partial t} \left( f(\xi, t) - m(\xi)\omega^{2}y_{s}(x, t) \right) d\xi = y_{s}(x, t) \varepsilon \quad (2)
\]

![Fig. 1 Vibration system](image)

or

\[
y_{s}(x, t)\frac{\partial^{2}y_{s}(x, t)}{\partial t^{2}} = \int_{0}^{l} G(x, \xi) \left( f(\xi, t) - m(\xi)\omega^{2}y_{s}(x, t) \right) d\xi \quad (3)
\]

where

\[
y_{s}(x, t) = y_{s}(x, t) + \varepsilon(t) \quad (4)
\]

Let a general solution of Eq. (3) be expanded into a series in terms of a normalized orthogonal system of characteristic functions, and put it

\[
y_{s}(x, t) = \sum_{i=1}^{\infty} \psi_{i}(x) \phi_{i}(t) \quad (5)
\]

where \( \phi_{i}(t) \) is an unknown function of time. Then Eq. (3) becomes

\[
\sum_{i=1}^{\infty} \int_{0}^{l} G(x, \xi) \left( f(\xi, t) - m(\xi)\omega^{2}\psi_{i}(x) \phi_{i}(t) \right) d\xi = \int_{0}^{l} \phi_{i}(t) \left( f(\xi, t) - m(\xi)\omega^{2}\phi_{i}(t) \right) d\xi \quad (6)
\]

Multiplying both sides of Eq. (6) by \( m(x)\phi_{i}(x) \), and integrating it from 0 to 1 with respect to \( x \), the following equation is given:

\[
\psi_{i}(t) + \varepsilon(t) + \omega^{2}\phi_{i}(t) \quad (7)
\]

Substituting the solution \( \phi_{i}(t) \) of Eq. (7) into Eq. (5), the general solution of Eq. (3) is given [see Appendix (1)].

Now, let the impact-body \( M_2 \) strike the concentrated mass \( M_1 \) at the left side end at a moment of \( t=0 \), and let the change in velocity of \( M_1 \) be \( v_0 \). Then the impact force acting on the continuum is expressed as

\[
f(x, t) = M_1 v_0 \delta(x-l, t) \quad (8)
\]

where \( \delta(x-l, t) \) is a two-dimensional delta function of Dirac. Substituting Eq. (8) and \( v(t) = a_0 \cos(\omega t + \theta_0) \) into Eq. (7), Eq. (7) is reduced to

\[
\phi_{i}(t) = a_0 (A_{1i} \cos(\omega t + \theta_0) + A_{2i} \sin(\omega t + \theta_0))
\]

A general solution of Eq. (9) is given by

\[
\psi_{i}(t) = M_1 v_0 \left( \sinh(\sqrt{\xi_i^2 - 1} \omega t) - e^{-\xi_i t} \right)
\]

where the first term of the right hand side is a stationary solution of forced vibrations, where

\[
A_{1i} = \frac{a_0 \omega^2(\omega t)^2 - \omega^2}{(\omega t)^2 + (2\xi_i \omega t)^2 + \omega^2}
\]

and \( A_{2i} = \frac{\eta_i \omega^2(\omega t)^2}{(\omega t)^2 + (2\xi_i \omega t)^2 + \omega^2} \)

the second is a transient solution by \( \delta(t) \) and the

\[\text{*} M_0, m(\xi) \text{ in Eq. (3) must be } m(\xi) + M_0 \delta(\xi - L).\]
third is a general solution of a homogeneous equation corresponding to Eq. (9). \( A_i \) and \( B_i \) are constants defined by initial conditions.

Substituting Eq. (10) into Eq. (5) and using Eq. (4), the general solution of Eq. (3) is expressed with respect to absolute displacement as

\[
y_\theta(x, t) = c_1(x) \cos(\omega t \theta_2) + c_2(x) \sin(\omega t \theta_2) + v_0 H(x, t) + F(x, t)
\]

\[
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (11)
\]

where

\[
c_1(x) = \sum_{i=1}^{M_i} \phi_i(x) A_i + 1, \quad c_2(x) = \sum_{i=1}^{M_i} \phi_i(x) A_i
\]

\[
H(x, t) = \sum_{i=1}^{M_i} \left( M_i \phi_i(x) \phi_i(l) \sin \sqrt{\xi l_i - 1} \omega_i t \right) e^{-\xi u_i t}
\]

\[
F(x, t) = \sum_{i=1}^{M_i} \left( \phi_i(x) \phi_i(l) \sinh \sqrt{\xi l_i - 1} \omega_i t \right) + B_i \cosh \sqrt{\xi l_i - 1} \omega_i t e^{-\xi u_i t}
\]

On the other hand, motion of \( M_1 \) contained in \( M_2 \) becomes uniform under assumption (4) in interval from an impact to the following impact. Therefore, it is expressed as

\[
y_\theta(x, t_p) = y_\theta(x, t_{p-1}) \left( x^{(p)} \right); \quad y_\theta(t_p) = y_\theta(t_{p-1}) \left( x^{(p)} \right)
\]

\[
(13)
\]

On the other hand, conditions of connection of motion at a moment of \( t = 0 \) in \( k \) th impact-period was written as

\[
( i ) \quad y_\theta(x, 0) = y_\theta(x, \tau_1) \left( x^{(1)} \right),
\]

\[
(ii) \quad y_\theta(0, 0) = y_\theta(0, \tau_1) \left( x^{(1)} \right),
\]

\[
(iii) \quad y_\theta(l, 0) = y_\theta(l, \tau_1) \left( x^{(1)} \right),
\]

\[
(iv) \quad \dot{y}_\theta(l, 0) = \frac{1 - \epsilon \mu_0}{1 + \mu_0} \dot{y}_\theta(l, \tau_1) \left( x^{(1)} \right) + \frac{1 - \epsilon}{1 + \mu_0} \dot{y}_\theta(l, \tau_1) \left( x^{(1)} \right)
\]

\[
(14)
\]

and \( \dot{y}_\theta(l, 0) = \dot{y}_\theta(l, \tau_1) \left( x^{(1)} \right) = y_\theta(0, 0) \).

From Eqs. (13) and (14) the following equations are obtained:

\[
(i) \quad y_\theta(x, 0) = y_\theta(x, \tau_1) \left( x^{(1)} \right)
\]

\[
(ii) \quad y_\theta(0, 0) = y_\theta(0, \tau_1) \left( x^{(1)} \right)
\]

\[
(iii) \quad y_\theta(l, 0) = y_\theta(l, \tau_1) \left( x^{(1)} \right)
\]

\[
(iv) \quad \dot{y}_\theta(l, 0) = \frac{1 - 2 \epsilon \mu_0 - \epsilon}{1 + 2 \mu_0 - \epsilon} \dot{y}_\theta(l, \tau_1) \left( x^{(1)} \right) \left( x^{(1)} \right)
\]

\[
(15)
\]

\[
(i) \quad y_\theta(l, 0) = y_\theta(l, \tau_1) \left( x^{(1)} \right)
\]

\[
(16)
\]

\[
(i) \quad y_\theta(l, 0) = y_\theta(l, \tau_1) \left( x^{(1)} \right)
\]

\[
(17)
\]

\[
(i) \quad y_\theta(l, 0) = y_\theta(l, \tau_1) \left( x^{(1)} \right)
\]

\[
(18)
\]

\[
(i) \quad y_\theta(l, 0) = y_\theta(l, \tau_1) \left( x^{(1)} \right)
\]

\[
(19)
\]

Substituting Eqs. (11) and (12) into \( y_\theta(x, t_p) \), \( y_\theta(t_p) \), the coefficients of periodic solutions can be determined. First, the following equations are obtained:

\[
F(x, 0) + F(x, \tau_1) + v_0 H(x, \tau_1) = 0
\]

\[
F(x, 0) + F(x, \tau_1) + v_0 H(x, \tau_1) = 0
\]

\[
v_\theta = 4 \mu_0 \frac{\omega}{\mu_0} \left[ c_1(l) \sin \theta_3 + c_2(l) \sin \theta_3 \right] + F(l, 0) + z
\]

\[
v_\theta = - \frac{2 \mu_0 (1 + \epsilon) \omega}{1 + 2 \mu_0 - \epsilon} \left[ c_1(l) \sin \theta_3 + c_2(l) \sin \theta_3 \right] + F(l, 0)
\]

\[
(20)
\]

and \( \dot{y}_\theta(0) = \frac{v_\theta}{2 \mu_0}, \quad y_\theta(l, 0) + z = \frac{v_\theta \mu_0}{4 \mu_0 \omega} \)

\[
(21)
\]

From Eq. (21)

\[
(22)
\]

\[
(23)
\]
\[ A_i = -v_0 \frac{M_i^j \phi^i_j(t)}{\sqrt{\zeta_o^2 \omega_o^2 - 1}} \frac{D_1 + D_2 \zeta_o^2 + (D_2 \zeta_o^2)^2}{(1 + D_2 \zeta_o^2)^2 - (D_2 \zeta_o^2)^2}, \]
\[ B_i = -v_0 \frac{M_i^j \phi^i_j(t)}{\sqrt{\zeta_o^2 \omega_o^2 - 1}} \frac{D_1 D_2 \zeta_o^2 + (D_1 D_2 \zeta_o^2)^2}{(1 + D_2 \zeta_o^2)^2 - (D_2 \zeta_o^2)^2} \]

and from Eq. (22)

\[ c_1(t) \cos \theta_1 + c_2(t) \sin \theta_1 = \frac{v_0}{a_0} C_1 - \frac{z}{a_0}, \quad c_2(t) \cos \theta_2 - c_1(t) \sin \theta_2 = -\frac{v_0}{a_0} C_2 \]

\[ \text{where} \]
\[ D_1 = \sinh(\sqrt{\zeta_o^2 \omega_o^2 - 1} \omega_o t), \quad D_2 = \cosh(\sqrt{\zeta_o^2 \omega_o^2 - 1} \omega_o t), \quad D_3 = \exp(-\sqrt{\zeta_o^2 \omega_o^2 - 1} \omega_o t) \]
\[ C_1 = \frac{\omega_o}{4 \mu_o} + \sum_{i=1}^{\infty} \frac{M_i^j \phi^i_j(t)}{\omega_o} \frac{D_1 D_2}{(1 + D_2 \zeta_o^2)^2 - (D_2 \zeta_o^2)^2} \]
\[ C_2 = \frac{1 + 2 \mu_o - \epsilon}{2 \mu_o (1 + \epsilon)} - \sum_{i=1}^{\infty} M_i^j \phi^i_j(t) \left( \frac{D_1 + D_2 \zeta_o^2}{(1 + D_2 \zeta_o^2)^2} \right) \]

Here, if there is no viscous damping ($\zeta_o = 0$), then
\[ C_1 = \frac{\omega_o}{4 \mu_o} + \sum_{i=1}^{\infty} M_i^j \phi^i_j(t) \frac{\omega_o}{2 \omega_o} \tan^{-1} \frac{\omega_o}{\omega_o}, \quad C_2 = \frac{1 + 2 \mu_o - \epsilon}{2 \mu_o (1 + \epsilon)} - \frac{1}{2} \sum_{i=1}^{\infty} M_i^j \phi^i_j(t) \]
and $C_1$ does not uniformly converge over the whole region of $\omega_o / \omega_o$.

Now, eliminating $\theta_2$ from Eq. (24), a quadratic equation with respect to $v_0$
\[ (C_i + C_j) \left( \frac{v_0}{a_0} \right)^2 - 2C_i \frac{z}{a_0} \frac{v_0}{a_0} + \left( \frac{z}{a_0} \right)^2 \left( c_i^2(t) + c_j^2(t) \right) = 0 \]

is obtained, and from this equation
\[ \frac{v_0(\pm)}{a_0} = \frac{1}{C_i + C_j} \left[ C_1 \frac{z}{a_0} \pm \sqrt{(C_i + C_j) \left( c_i^2(t) + c_j^2(t) \right) - C_i^2 \left( \frac{z}{a_0} \right)^2} \right] \]

is obtained. Here $v_0(\pm)$ corresponds with the sign + preceding the radical sign.

In this manner, if $v_0$ is obtained, $A_i$ and $B_i$ can be determined, $\theta_1$ from Eq. (24), $y_0(t)$ and $y_1(t, 0)$ from Eq. (23). Thus the periodic solutions can be determined.

But in order that the formally obtained periodic solutions may be realized, the following conditions must be satisfied:

1. The velocity jump $v_0$ by impact is positive, and
2. at the interval $0 \leq t \leq \pi / \omega_o$
\[ y_0(t, t) - z \equiv y_0(t) \leq y_0(t, 0) + z \quad \text{ lacking to (27)} \]
First, the condition for $v_0$ to be real is
\[ \frac{z}{a_0} \leq \left[ 1 + \left( \frac{C_1}{C_2} \right)^2 \right] \left( c_i^2(t) + c_j^2(t) \right) \equiv \frac{z}{a_0} \quad \text{(28)} \]
and then the range of the free paths for positive $v_0$ is expressed as follows:

(1) Concerning $v_0(\pm)$

(a) $0 < z / a_0 < \sqrt{c_i^2(t) + c_j^2(t)}$ for $C_i < 0$
(b) $0 < z / a_0 < z / a_0$ for $C_i > 0$

(2) Concerning $v_0(\pm)$

(a) positive $z$ does not exist for $C_i < 0$
(b) $\sqrt{c_i^2(t) + c_j^2(t)} < z / a_0 < z / a_0$ for $C_i > 0$.

Next, as the second condition (27) cannot be examined except by numerical calculations, only the results of numerical calculations are shown in this paper (see Chap. 6).

5. Stability of periodic solutions

In the above, periodic solutions are found. But in order that the periodic solutions may be realized, they must further be stable dynamically. Let us define here the stability of periodic solutions as follows: "Now, let a periodic state be realized in some manner. And let the state be disturbed slightly by some cause. Then if disturbances tend to converge to 0 with impacts, the original periodic state is defined as being stable."

Referring to expressions in Chap. 4, disturbed motion is written as follows:
\[ y_0(x, t) = -y_0(x, t) + y_0(x, 0) + z \quad \text{(29)} \]
and
\[ y_0(x, t) = -y_0(x, t) + y_0(x, 0) + z \quad \text{(30)} \]

Conditions of connection corresponding to Eq. (14) become

1. $y_0(x, 0) = y_0(x, 0) + \frac{\pi + \theta_0}{\omega_o} \theta_0(\pi - \theta_0) \quad \text{(31)}$
2. $y_0(x, 0) = y_0(x, 0) + \frac{\pi + \theta_0}{\omega_o} \theta_0(\pi - \theta_0) \quad \text{(32)}$
3. $y_0(x, 0) = y_0(x, 0) + \frac{\pi + \theta_0}{\omega_o} \theta_0(\pi - \theta_0) \quad \text{(33)}$
Substituting Eqs. (29) ~ (32) into these equations, replacing

\[ \theta_0^{(k)} = \theta_0 + \sum_{j=0}^{k-1} \delta \theta_s^{(j)}, \quad v_0^{(k)} = v_0 + \delta v_0^{(k)}, \quad A_1^{(k)} = A_1 + \delta A_1^{(k)} \]

\[ B_1^{(k)} = B_1 + \delta B_1^{(k)}, \quad \hat{y}_0^{(k)} = \hat{y}_0 + \delta \hat{y}_0^{(k)} \]

\[ y_0^{(k)} = y_0 + \delta y_0^{(k)} \]

\[ \delta F(\xi, \omega) = \sum_{i=1}^{\infty} \varphi_i(x) (\delta A_i^{(k)}) \sinh \chi_{i}^{(k)} - 1 \omega_i t + \delta B_i^{(k)} \cosh \chi_{i}^{(k)} - 1 \omega_i t e^{-\xi_i \omega_i t} \]

and putting them in order under the assumption \( \omega_i/2 \chi_i \omega_i < 1 \), the following equations are obtained:

\[ \delta F(x, 0)^{(k)} = \delta F(x, \omega)^{(k-1)} + \delta \hat{F}(x, \frac{\omega \pi}{\omega}) + v_0 \delta H(x, \frac{\omega \pi}{\omega}) \]

\[ \delta \hat{y}_0(l, 0, \omega)^{(k)} = \delta y_0(l, 0, \omega)^{(k-1)} + \frac{\omega \pi}{\omega} \delta \hat{y}_0(l, 0, \omega)^{(k-1)} + \frac{\omega \pi}{\omega} \delta \hat{y}_0(l, 0, \omega)^{(k-1)} = 0 \]

\[ \frac{\Lambda_1}{1 + \mu_0} \left[ a_0 \omega c_i(l) \cos \theta_0 + c_i(l) \sin \theta_0 \sum_{j=0}^{k-1} \delta \theta_j^{(j)} - \delta F(l, 0)^{(k)} \right] \]

\[ + \left( \frac{\Lambda_1}{1 + \mu_0} + \frac{\Lambda_2}{1 + \mu_0} \right) \delta \hat{y}_0(l, 0, \omega)^{(k-1)} = 0 \]

\[ \frac{1 + \varepsilon}{1 + \mu_0} \left[ a_0 \omega c_i(l) \cos \theta_0 + c_i(l) \sin \theta_0 \sum_{j=0}^{k-1} \delta \theta_j^{(j)} - \delta F(l, 0)^{(k)} \right] \]

\[ + \left( \frac{\Lambda_1}{1 + \mu_0} + \frac{\Lambda_2}{1 + \mu_0} \right) \delta \hat{y}_0(l, 0, \omega)^{(k-1)} = 0 \]

The relation (33) is satisfied automatically.

Substituting Eq. (39) into Eq. (40) and eliminating \( \delta \theta_s^{(k)} \) and \( \delta \hat{y}_0^{(k)} \), the following linear simultaneous difference equations with constant coefficients are obtained:

\[ \alpha_1^{(i)} \delta A_1^{(i)} + \alpha_2^{(i)} \delta A_2^{(i-1)} + \beta_1^{(i)} \delta B_1^{(i+1)} + \beta_2^{(i)} \delta B_2^{(i)} + \beta_3^{(i)} \delta B_3^{(i)} + \beta_4^{(i)} \delta B_4^{(i)} = 0 \]

\[ \alpha_5^{(i)} \delta A_5^{(i)} + \alpha_6^{(i)} \delta A_6^{(i-1)} + \alpha_7^{(i)} \delta A_7^{(i-2)} + \beta_5^{(i)} \delta B_5^{(i+1)} + \beta_6^{(i)} \delta B_6^{(i)} + \beta_7^{(i)} \delta B_7^{(i)} + \beta_8^{(i)} \delta B_8^{(i-1)} + \beta_9^{(i)} \delta B_9^{(i-2)} = 0 \]
\[ \beta_i^{(1)} = \frac{1 - \varepsilon \mu_0}{1 + \mu_0} \left( \frac{\lambda_0}{\lambda_1} - 1 \right) \]

\[ \beta_i^{(2)} = \zeta \mu_0 \omega_i + \frac{1 - \varepsilon \mu_0}{1 + \mu_0} \left( \frac{\lambda_0}{\lambda_1} - 1 \right) - \omega \frac{\lambda_0}{\lambda_1} \]

\[ \beta_i^{(3)} = -\zeta \mu_0 \omega_i + \mu_0 (1 + \varepsilon) \left( \frac{\lambda_0}{\lambda_1} - 1 \right) - \left( \frac{\lambda_0}{\lambda_1} + \omega \frac{\lambda_0}{\lambda_1} \right) D_0 \]

\[ \lambda_i^{(o)} = \left( C + \frac{1}{2 \mu_0} \right) \omega_i \]

\[ \lambda_i^{(o)} = \frac{1 + 2 \mu_0 - \varepsilon}{2 \mu_0 (1 + \varepsilon)} \omega_i \]

\[ \lambda_i = \frac{\omega_i}{\omega} + z \]

General solutions of Eq. (41) are given by

\[ \delta A_i^{(1)}(z) = a_i z_i^{(1)} \quad \delta B_i^{(1)}(z) = b_i z_i^{(1)} \]

where \(a_i\) and \(b_i\) are constants and \(z_i\) is unknown. Substituting Eq. (42) into Eq. (41),

\[ \begin{bmatrix} \alpha_i^{(1)} z_i^{(1)} + \alpha_i^{(2)} \beta_i^{(1)} z_i^{(2)} + \beta_i^{(2)} z_i^{(3)} + \beta_i^{(3)} z_i^{(4)} \\ \alpha_i^{(2)} z_i^{(2)} + \alpha_i^{(3)} \beta_i^{(2)} z_i^{(2)} + \beta_i^{(3)} z_i^{(3)} + \beta_i^{(4)} z_i^{(4)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

is obtained. Therefore, the characteristic equation is written as

\[ \gamma_i^{(1)} z_i^{(1)} + \gamma_i^{(2)} z_i^{(2)} + \gamma_i^{(3)} z_i^{(3)} + \gamma_i^{(4)} z_i^{(4)} + \gamma_i^{(5)} = 0 \]  

where

\[ \gamma_i^{(1)} = \alpha_i^{(1)} \beta_i^{(1)} - \alpha_i^{(2)} \beta_i^{(2)} - \alpha_i^{(3)} \beta_i^{(3)} - \alpha_i^{(4)} \beta_i^{(4)} - \alpha_i^{(5)} \]

\[ \gamma_i^{(2)} = \alpha_i^{(2)} \beta_i^{(1)} + \alpha_i^{(3)} \beta_i^{(2)} + \alpha_i^{(4)} \beta_i^{(3)} + \alpha_i^{(5)} \beta_i^{(4)} - \alpha_i^{(6)} \beta_i^{(5)} - \alpha_i^{(7)} \]

\[ \gamma_i^{(3)} = \alpha_i^{(3)} \beta_i^{(1)} + \alpha_i^{(4)} \beta_i^{(2)} + \alpha_i^{(5)} \beta_i^{(3)} + \alpha_i^{(6)} \beta_i^{(4)} + \alpha_i^{(7)} \]

\[ \gamma_i^{(4)} = \alpha_i^{(4)} \beta_i^{(1)} + \alpha_i^{(5)} \beta_i^{(2)} + \alpha_i^{(6)} \beta_i^{(3)} + \alpha_i^{(7)} \beta_i^{(4)} + \alpha_i^{(8)} \]

is obtained. Therefore, the solution of Eq. (41) becomes

\[ \delta A_i^{(1)}(z) = \sum_{j=1}^{3} a_j z_j^{(1)} \delta B_i^{(1)}(z) = \sum_{j=1}^{3} b_j z_j^{(1)} \]

Considering the definition of stability,

\[ \lim_{k \to \infty} \delta A_i^{(1)}(k) = 0, \quad \lim_{k \to \infty} \delta B_i^{(1)}(k) = 0 \]

must be satisfied, and therefore for four radicals \(z_i\) of Eq. (44) \(|z_i|<1\) must be satisfied. The necessary and sufficient conditions for \(|z_i|<1\) are given by Schur's theorem, in this paper the other stability criteria is used [see Appendix (2)]. The latter stability conditions are

\[ d_i^{(1)} > 0, \quad d_i^{(2)} > 0, \quad d_i^{(3)} > 0, \quad d_i^{(4)} > 0 \]

\[ d_i^{(5)} d_i^{(6)} d_i^{(7)} d_i^{(8)} > 0 \]

(\(a_i\) expressing the mode number of natural vibration (omitted))

Consequently, stability of the periodic solution can be discriminated by Eq. (46) for all positive integers \(i\). For large \(i\), Eq. (46) can be approximated as follows. Considering that the terms \(\sqrt{\xi_i^2 - 1} \mu_0 \omega_i / \omega\) and \(\zeta \mu_0 \omega_i / \omega\) are sufficiently larger

6. Numerical calculations

Next, the results of numerical calculations for a concrete model of the impact-damper are shown. Figure 2 shows a model of which the continuum is

![Vibration system](image)
a uniform coil spring. Natural circular frequency and normalized characteristic function of this vibration system are given by
\[
\omega_1 \tan \frac{\omega_1 l}{a} = M_1 \quad \text{(49)}
\]
\[
\varphi_1(x) = \frac{2}{\sqrt{M_1 + M_2 \sin^2 \frac{\omega_1 l}{a}}} \sin \frac{\omega_1 l}{a} \quad \text{(50)}
\]
where \(a = l \sqrt{k/M_1} \) is the velocity of wave propagation in the spring and \(k \) is the spring-constant for the whole spring.

In this paper, stability regions of the periodic solutions and the amplitude of the concentrated mass \(M_1 \) are calculated. To simplify calculation, let all parameters be nondimensional. In the determination of the stability region, the independent parameters are \(M_2/M_1, \omega_1/\omega_0, z/\omega_0 \), \(\xi_1, \eta_1, \xi \), and \(\epsilon \).

Now, if the value of \(M_2/M_1 \) is given, then all \(\omega_1/\omega_0 \) are determined by Eq. (49) and all ratios of the \(i\)th natural frequencies to the first \(\omega_i/\omega_1 \) are determined. The values of \(\omega_i/\omega_0 \) are, therefore, determined if \(\omega_1/\omega_0 \) is given. And all values of \(\xi_i \) are determined if the value of \(\xi_i = \omega_i/\omega_0 \) is given, as \(\xi_i = (\omega_i/\omega_0) \xi_1 \).

The stability regions are expressed in \(\omega_1/\omega_0 - z/\omega_0 \) plane for five values provided of \(M_2/M_1, \eta_1, \xi_1, \) and \(\epsilon \). In the paper, \(M_2/M_1 = 0.1, \xi_1 = 0.1 \) are adopted. Such a vibration system behaves as if it were a system with one degree-of-freedom in free vibration, and the infinite series \(C_1 \) and \(C_2 \) and the displacement \(y_a(l, t) \) can be sufficiently approximated by only their first term. And then \(n = 1 \) is adopted because this case is the most fundamental periodic state.

Figure 3 shows the stability region in the case of \(M_2/M_1 = 0.1, \xi_1 = 0.5 \). Boundary lines of the stability region for \(i = 3 \) lie over one for \(i = 2 \). This figure concerns \(v_a(+) \), and the stability region concerning \(v_a(-) \) is omitted because it is too narrow. The boundary line II is determined by Eq. (27).
Figure 4 shows the distribution of the amplitude in the stability region described in Fig. 3. Dotted curves are used only conveniently not to complicate the figure.

Percentages of these amplitudes to the resonance amplitude of the system without an impact-body are shown in Fig. 5. The relation between $z/a_0$ and the amplitude at $\omega = \omega_0$ is shown in Fig. 6.

Figure 7 shows the relation between $\varepsilon$ and the stability region when $M_2/M_1 = 0.2$. On the other hand, in Fig. 8 the relation between $M_2/M_1$ and the stability region in the case of $\varepsilon = 0.1$ is shown.

Figure 9 shows distribution of the amplitude...
corresponding to the value of $M_2/M_1$ and $z/a_o$.

Figure 10 shows the change in the amplitude corresponding to $M_2/M_1$ and $z/a_o$. On the other hand, the change in the amplitude corresponding to $\varepsilon$ is shown in Fig. 11, where parameters are $M_2/M_1$ and $z/a_o$.

7. Conclusions

The results of numerical calculations show that the stability regions of the periodic solutions exist and that the effects of damping are sufficiently good.

With respect to the parameters adopted in this paper ($M_2/M_1 = 0.1$, $\zeta_1 = 0.1$, $n = 1$), the following conclusions are obtained.

1. The stability region is wider as the value of $M_2/M_1$ is smaller or $\varepsilon$ is nearer to 1.
2. For a fixed value of $z/a_o$, the damping effect in the case of $\omega = \omega_1$ is good as $M_2/M_1$ is larger. When $M_2/M_1$ is fixed too, the change of the damping effect with respect to $\varepsilon$ is small.
3. For fixed values of $\varepsilon$ and $M_2/M_1$, the damping effect at $\omega = \omega_1$ is good as $z/a_o$ is larger. But if $z/a_o$ is larger than the resonance amplitude of the system without an impact-body, impact may not occur in some cases of the initial conditions.
4. As seen from Figs. 10 and 11, however the parameters may be selected, the damping effect may not be better than a certain extent.
5. As seen from Fig. 11 the damping effect is best in the case of $\varepsilon=1$. But the amplitude can not be decreased under 0.7 (about 7.7% in percentage of the amplitude to the resonance amplitude). Also in the case of the system with one degree-of-freedom, the damping effect is best when $\varepsilon=1$.
6. As seen from Figs. 4 and 5, the amplitude can be larger than in the case without an impact if $\omega_2/\omega_1$ is larger than a certain extent. In that case the system can not be used as the damper.

Numerical calculations in this paper were carried out by KDC-II (HITAC 5020) at the Computation Center of Kyoto University.
Appendix

(1) Corresponding to Eq. (3), natural vibration is expressed by
\[ y_i(x, t) = - \int_0^l G(x, \xi) m(\xi) y_i(\xi, t) d\xi \]  \hspace{1cm} (51)
Putting the solution of Eq. (51)
\[ y_i(x, t) = \varphi_i(x) \cos (\sqrt{\lambda} t + \alpha) \]  \hspace{1cm} (52)
the following Fredholm type homogeneous integral equation is obtained:
\[ \varphi_i(x) = \lambda \int_0^l G(x, \xi) m(\xi) \varphi_i(\xi) d\xi \]  \hspace{1cm} (53)
Here putting the eigenvalues and the eigenfunctions \( \lambda_i = \omega_i^2 \) and \( \varphi_i(x) \) respectively, Eq. (53) is reduced to
\[ \varphi_i(x) = \omega_i^2 \int_0^l G(x, \xi) m(\xi) \varphi_i(\xi) d\xi \]  \hspace{1cm} (54)
\( i = 1, 2, 3, \ldots \)
These \( \varphi_i(x) \) are orthogonal functions with respect to weight \( m(x) \):
\[ \int_0^l m(x) \varphi_i(x) \varphi_j(x) dx = \delta_{ij} \]  \hspace{1cm} (55)
where \( \varphi_i(x) \) are normalized.

(2) Replacing \( z_i \) in Eq. (44) by \( (s_i + 1)/(s_i - 1) \), the inside of the unit circle in \( z_i \)-plane is mapped to the left half-plane in \( s_i \)-plane \((s)\). Therefore, applying Hurwitz's stability criterion to the polynomial with respect to \( s_i \), stability can be discriminated. By this transformation, Eq. (44) yields to
\[ s_1^4 + d_1^{(1)} s_1^3 + d_2^{(1)} s_1^2 + d_3^{(1)} s_1 + d_4^{(1)} = 0 \]  \hspace{1cm} (56)
of which the stability discriminant is Eq. (47).

References

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