534.13

On Impact-Damper for Concentrated-Mass-Continuum System*

By Hidekatsu TOKUMARU** and Tadashi KOTERA***

Forced vibrations of a concentrated-mass-continuum system with an impact-body are discussed. The impact-body moves in a groove of the concentrated mass and strikes the concentrated mass at opposite ends of the groove.

Periodic solutions of the system excited by harmonic motion of a foundation are presented, and their stability is discussed. This paper treats only such periodic motion that during n periods of harmonic motion of the foundation two impacts occur at equal time intervals and at opposite ends of the groove (n : odd).

The equation of motion is written in the form of a partial integro-differential equation by the use of an influence function.

As an example, numerical calculations are carried out for the simplest system. That is, the continuum of this system is a uniform coil spring. Stability regions are determined and maximum displacements of the concentrated mass are calculated. The results show that the damping effects of the impact-damper are sufficiently good.

1. Introduction

Impact-damper is a damper using impacts between a vibrating body and an impact-body. It has been applied to, for example, turbine blades (1931)⁽¹⁾, wings of airplane (1945)⁽¹⁾ and recently lighting poles along high-speed highways⁽⁸⁾. Studies of dynamics of impact-damper have been restricted to a mass-spring system with one degree-of-freedom^{(2)~(5)}, and dynamics of impact-damper with infinitely many degrees of freedom has never been studied.

In this paper, a study of motion of an impact-damper for a concentrated-mass-continuum system (with infinitely many degrees-of-freedom) is presented, and examples of numerical calculations are shown.

2. Nomenclature

t : time
 x : coordinate along the axis of the continuum
 $y_a(x, t)$: absolute displacement of the continuum
 $y_r(x, t)$: relative displacement of the continuum with respect to a foundation
 $m(x)$: mass per unit length of the continuum
 M_1 : concentrated mass at the tip of the

continuum [this is contained in $m(x)$]
 M_2 : mass of an impact-body moving in a groove of the concentrated mass M_1
 M_s : total mass of the continuum (without M_1)
 $\mu_0 = M_2/M_1$
 $2z$: free path of impact-body
 $y_2(t)$: displacement of the impact-body (the origin is the center of the groove at stationary state)
 $v(t) = a_0 \cos(\omega t + \theta_0)$: displacement given to the foundation
 θ_0 : phase displacement of $v(t)$ at a moment when an impact occurs at the left side end of the groove
 k : internal viscous damping coefficient
 ϵ : coefficient of rebound
 $G(x, \xi)$: influence function
 $f(x, t)$: force acting on the continuum
 ω_i : i th natural circular frequency
 $\varphi_i(x)$: normalized characteristic function of i th mode
 i : number of modes of natural vibration
 $\zeta_i = \frac{k}{2\omega_i}$: ratio of damping coefficient with respect to ω_i
 $\eta_i = \int_0^l m(x)\varphi_i(x)dx$: mode participation factor
 l : total length of the continuum

* Received 18th October, 1968.

** Professor, Faculty of Engineering, Kyoto University.

*** Graduate Student, Division for Research of Engineering, Graduate School, Kyoto University Sakyo-ku, Kyoto.

3. Basic equations

Figure 1 shows a model of an impact-damper studied in this paper. In analysis, the following are assumed:

- (1) When there is no impact, the vibration system is linear.
- (2) There is not any external resistance.
- (3) Internal viscous resistance is proportional to a strain rate.
- (4) The impact-body M_2 moves without friction in the groove of the concentrated mass M_1 .
- (5) The impact between M_2 and M_1 is instantaneous, and their deformations are negligible.
- (6) The relation between velocities before and after the impact is defined with a constant coefficient of rebound ε .

Now, the equation of motion with respect to an elastic body with one dimension is expressed by, in general, the partial integro-differential equation as follows⁽⁷⁾:

$$y_r(x, t) = \int_0^l G(x, \xi) \left\{ f(\xi, t) - m(\xi) \frac{\partial^2 y_a(\xi, t)}{\partial t^2} \right\} d\xi \quad (1)$$

Though Fig. 1 shows a model of transverse vibration, Eq. (1) can be applied to not only transverse vibration but also longitudinal or torsional vibration. In the case of torsional vibration, $m(\xi)$ is the moment of inertia of the cross section, $y(x, t)$ the angle of torsion and $f(x, t)$ the moment of torsion.

In Eq. (1), internal damping is not considered. Considering the internal viscous resistance, Eq. (1) is reduced to

$$y_r(x, t) + k \frac{\partial y_r(x, t)}{\partial t} = \int_0^l G(x, \xi) \left\{ f(\xi, t) - m(\xi) \frac{\partial^2 y_a(\xi, t)}{\partial t^2} \right\} d\xi \quad (2)$$

Fig. 1 Vibration system

or

$$y_r + k \frac{\partial y_r}{\partial t} = \int_0^l G(x, \xi) \left\{ f(\xi, t) - m(\xi) \ddot{v}(t) - m(\xi) \frac{\partial^2 y_r}{\partial t^2} \right\} d\xi \quad (3)^*$$

where

$$y_a(x, t) = y_r(x, t) + v(t) \quad (4)$$

Let a general solution of Eq. (3) be expanded into a series in terms of a normalized orthogonal system of characteristic functions, and put it

$$y_r(x, t) = \sum_{i=1}^{\infty} \varphi_i(x) \psi_i(t) \quad (5)$$

where $\psi_i(t)$ is an unknown function of time. Then Eq. (3) becomes

$$\sum_{i=1}^{\infty} \varphi_i(x) \{ \ddot{\psi}_i(t) + k \dot{\psi}_i(t) \} + \int_0^l G(x, \xi) m(\xi) \sum_{i=1}^{\infty} \varphi_i(\xi) \ddot{\psi}_i(t) d\xi = \int_0^l G(x, \xi) \{ f(\xi, t) - m(\xi) \ddot{v}(t) \} d\xi \quad (6)$$

Multiplying both sides of Eq. (6) by $m(x) \varphi_i(x)$, and integrating it from 0 to l with respect to x , the following equation is given:

$$\ddot{\psi}_i(t) + k \omega_i^2 \dot{\psi}_i(t) + \omega_i^2 \psi_i(t) = \int_0^l \{ f(\xi, t) - m(\xi) \ddot{v}(t) \} \varphi_i(\xi) d\xi \quad (7)$$

Substituting the solution $\psi_i(t)$ of Eq. (7) into Eq. (5), the general solution of Eq. (3) is given [see Appendix (1)].

Now, let the impact-body M_2 strike the concentrated mass M_1 at the left side end at a moment of $t=0$, and let the change in velocity of M_1 be v_0 . Then the impact force acting on the continuum is expressed as

$$f(x, t) = M_1 v_0 \delta(x-l, t) \quad (8)$$

where $\delta(x-l, t)$ is a two-dimensional delta function of Dirac. Substituting Eq. (8) and $v(t) = a_0 \cos(\omega t + \theta_0)$ into Eq. (7), Eq. (7) is reduced to

$$\ddot{\psi}_i(t) + 2\zeta_i \omega_i \dot{\psi}_i(t) + \omega_i^2 \psi_i(t) = M_1 v_0 \varphi_i(l) \delta(t) + \eta_i a_0 \omega^2 \cos(\omega t + \theta_0) \quad (9)$$

A general solution of Eq. (9) is given by

$$\psi_i(t) = a_0 \{ A_{1i} \cos(\omega t + \theta_0) + A_{2i} \sin(\omega t + \theta_0) \} + M_1 v_0 \varphi_i(l) \frac{\sinh \sqrt{\zeta_i^2 - 1} \omega_i t}{\sqrt{\zeta_i^2 - 1} \omega_i} e^{-\zeta_i \omega_i t} + (A_i \sinh \sqrt{\zeta_i^2 - 1} \omega_i t + B_i \cosh \sqrt{\zeta_i^2 - 1} \omega_i t) e^{-\zeta_i \omega_i t} \quad (10)$$

where the first term of the right hand side is a stationary solution of forced vibrations, where

$$A_{1i} = \frac{\eta_i \omega^2 (\omega_i^2 - \omega^2)}{(\omega_i^2 - \omega^2)^2 + (2\zeta_i \omega_i \omega)^2}$$

$$\text{and } A_{2i} = \frac{\eta_i \omega^2 2\zeta_i \omega_i \omega}{(\omega_i^2 - \omega^2)^2 + (2\zeta_i \omega_i \omega)^2}$$

the second is a transient solution by $\delta(t)$ and the

* M_1 is contained in $m(x)$. If $m(x)$ does not contain M_1 , $m(\xi)$ in Eq. (3) must be $m(\xi) + M_1 \delta(\xi-l)$.

third is a general solution of a homogeneous equation corresponding to Eq. (9). A_i and B_i are constants defined by initial conditions.

Substituting Eq. (10) into Eq. (5) and using Eq. (4), the general solution of Eq. (3) is expressed with respect to absolute displacement as

$$y_a(x, t) = a_0 \{c_1(x) \cos(\omega t + \theta_0) + c_2(x) \sin(\omega t + \theta_0)\} + v_0 H(x, t) + F(x, t) \quad \dots\dots\dots(11)$$

where

$$c_1(x) = \sum_{i=1}^{\infty} \varphi_i(x) A_{1i} + 1, \quad c_2(x) = \sum_{i=1}^{\infty} \varphi_i(x) A_{2i},$$

$$H(x, t) = \sum_{i=1}^{\infty} \frac{M_1 \varphi_i(x) \varphi_i(l)}{\sqrt{\zeta_i^2 - 1} \omega_i} \sinh \sqrt{\zeta_i^2 - 1} \omega_i t \times e^{-\zeta_i \omega_i t}$$

$$F(x, t) = \sum_{i=1}^{\infty} \varphi_i(x) (A_i \sinh \sqrt{\zeta_i^2 - 1} \omega_i t + B_i \cosh \sqrt{\zeta_i^2 - 1} \omega_i t) e^{-\zeta_i \omega_i t}$$

On the other hand, motion of M_2 contained in M_1 becomes uniform under assumption (4) in interval from an impact to the following impact. Therefore, it is expressed as

$$y_a(x, t)_p^{(k)} = -y_a(x, t)_p^{(k-1)}; \quad y_2(t)_p^{(k)} = -y_2(t)_p^{(k-1)} \quad \dots\dots\dots(13)$$

On the other hand, conditions of connection of motion at a moment of $t=0$ in k th impact-period was written as

$$\left. \begin{aligned} \text{(i)} \quad & y_a(x, 0_+)_p^{(k)} = y_a(x, n\pi/\omega)_p^{(k-1)}, \\ \text{(ii)} \quad & y_2(0_+)_p^{(k)} = y_2(n\pi/\omega)_p^{(k-1)}, \\ \text{(iii)} \quad & y_a(l, 0_+)_p^{(k)} = y_2(0_+)_p^{(k)} - z, \\ \text{(iv)} \quad & \dot{y}_a(l, 0_+)_p^{(k)} = \frac{1-\varepsilon\mu_0}{1+\mu_0} \dot{y}_a\left(l, \frac{n\pi}{\omega}\right)_p^{(k-1)} + \mu_0 \frac{1+\varepsilon}{1+\mu_0} \dot{y}_2\left(\frac{n\pi}{\omega}\right)_p^{(k-1)}, \\ & \dot{y}_2(0_+)_p^{(k)} = \frac{1+\varepsilon}{1+\mu_0} \dot{y}_a\left(l, \frac{n\pi}{\omega}\right)_p^{(k-1)} + \frac{\mu_0-\varepsilon}{1+\mu_0} \dot{y}_2\left(\frac{n\pi}{\omega}\right)_p^{(k-1)} \end{aligned} \right\} \quad \dots\dots\dots(14)$$

and $\dot{y}_a(l, 0_+)_p^{(k)} - \dot{y}_a(l, n\pi/\omega)_p^{(k-1)} = v_0$.

From Eqs. (13) and (14) the following equations are obtained:

$$\text{(i)} \quad y_a(x, 0_+)_p^{(k)} = -y_a(x, n\pi/\omega)_p^{(k)} \quad \dots\dots\dots(15)$$

$$\text{(ii)} \quad y_2(0_+)_p^{(k)} = -y_2(n\pi/\omega)_p^{(k)} \quad \dots\dots\dots(16)$$

$$\text{(iii)} \quad y_a(l, 0_+)_p^{(k)} = y_2(0_+)_p^{(k)} - z \quad \dots\dots\dots(17)$$

$$\text{(iv)} \quad \dot{y}_a(l, 0_+)_p^{(k)} = -\frac{1-2\varepsilon\mu_0-\varepsilon}{1+2\mu_0-\varepsilon} \dot{y}_a(l, n\pi/\omega)_p^{(k)} \quad \dots\dots\dots(18)$$

$$\dot{y}_2(0_+)_p^{(k)} = -\frac{1+\varepsilon}{1+2\mu_0-\varepsilon} \dot{y}_a\left(l, \frac{n\pi}{\omega}\right)_p^{(k)} \quad \dots\dots\dots(19)$$

$$\text{(v)} \quad \dot{y}_a(l, 0_+)_p^{(k)} + \dot{y}_a(l, n\pi/\omega)_p^{(k)} = v_0 \quad \dots\dots\dots(20)$$

Substituting Eqs. (11) and (12) into $y_a(x, t)_p^{(k)}$, $y_2(t)_p^{(k)}$, the coefficients of periodic solutions can be determined. First, the following equations are obtained:

$$\left. \begin{aligned} F(x, 0_+) + F(x, n\pi/\omega) + v_0 H(x, n\pi/\omega) &= 0 \\ \dot{F}(x, 0_+) + \dot{F}(x, n\pi/\omega) + v_0 \dot{H}(x, n\pi/\omega) &= 0 \end{aligned} \right\} \quad \dots\dots\dots(21)$$

$$\left. \begin{aligned} v_0 &= \frac{4\mu_0\omega}{n\pi} [a_0 \{c_1(l) \cos \theta_0 + c_2(l) \sin \theta_0\} + F(l, 0) + z] \\ v_0 &= -\frac{2\mu_0(1+\varepsilon)}{1+2\mu_0-\varepsilon} [a_0 \omega \{-c_1(l) \sin \theta_0 + c_2(l) \cos \theta_0\} + \dot{F}(l, 0)] \end{aligned} \right\} \quad \dots\dots\dots(22)$$

$$\text{and} \quad \dot{y}_2(0_+) = -\frac{v_0}{2\mu_0}, \quad y_a(l, 0_+) + z = \frac{v_0 n\pi}{4\mu_0\omega} \quad \dots\dots\dots(23)$$

From Eq. (21)

$$y_2(t) = y_a(l, 0) + z + \dot{y}_2(0_+)t \quad \dots\dots\dots(12)$$

where $y_a(l, 0)$ and $y_2(0_+)$ are unknown constants.

In the above, basic equations of motion of the system treated in the paper were determined. Next, let us determine periodic solutions of impact-vibration of the system.

4. Periodic solutions

This paper treats only such periodic motion that during n (odd) periods of harmonic motion of the foundation 2 impacts occur at equal time intervals and at opposite side ends of the groove.

Now, let certain $n/2$ periods (from the moment of an impact at the left hand side end to the moment of the following impact at the right hand side end) be named the k th impact-period, and the previous impact-period the $k-1$ th, and put the displacements at respective impact-period as

$$y_a(x, t)_p^{(k)}, y_2(t)_p^{(k)}; y_a(x, t)_p^{(k-1)}, y_2(t)_p^{(k-1)}$$

where a subscript p means periodic state and it is assumed that at a moment of every impact $t=0$. Then, conditions for periodic motion are written as follows:

$$A_i = -v_0 \frac{M_1 \varphi_i(l)}{\sqrt{\zeta_i^2 - 1} \omega_i} \frac{(D_2 + D_3) D_3}{(1 + D_2 D_3)^2 - (D_1 D_3)^2}, \quad B_i = -v_0 \frac{M_1 \varphi_i(l)}{\sqrt{\zeta_i^2 - 1} \omega_i} \frac{D_1 D_3}{(1 + D_2 D_3)^2 - (D_1 D_3)^2}$$

and from Eq. (22)

$$c_1(l) \cos \theta_0 + c_2(l) \sin \theta_0 = \frac{v_0}{a_0 \omega} C_1 - \frac{z}{a_0}, \quad c_2(l) \cos \theta_0 - c_1(l) \sin \theta_0 = -\frac{v_0}{a_0 \omega} C_2 \quad (24)$$

where

$$D_1 = \sinh \sqrt{\zeta_i^2 - 1} n \pi \frac{\omega_i}{\omega}, \quad D_2 = \cosh \sqrt{\zeta_i^2 - 1} n \pi \frac{\omega_i}{\omega}, \quad D_3 = \exp(-\zeta_i n \pi \omega_i / \omega)$$

$$C_1 = \frac{n \pi}{4 \mu_0} + \sum_{i=1}^{\infty} \frac{M_1 \varphi_i^2(l)}{\sqrt{\zeta_i^2 - 1}} \frac{\omega}{\omega_i} \frac{D_1 D_3}{(1 + D_2 D_3)^2 - (D_1 D_3)^2}$$

$$C_2 = \frac{1 + 2 \mu_0 - \varepsilon}{2 \mu_0 (1 + \varepsilon)} - \sum_{i=1}^{\infty} M_1 \varphi_i^2(l) \frac{(D_2 + D_3 - D_1 \zeta_i / \sqrt{\zeta_i^2 - 1}) D_3}{(1 + D_2 D_3)^2 - (D_1 D_3)^2}$$

Here, if there is no viscous damping ($\zeta_i = 0$), then

$$C_1 = \frac{n \pi}{4 \mu_0} + \sum_{i=1}^{\infty} M_1 \varphi_i^2(l) \frac{\omega}{\omega_i} \tan \frac{n \pi \omega_i}{2 \omega}, \quad C_2 = \frac{1 + 2 \mu_0 - \varepsilon}{2 \mu_0 (1 + \varepsilon)} - \frac{1}{2} \sum_{i=1}^{\infty} M_1 \varphi_i^2(l)$$

and C_1 does not uniformly converge over the whole region of ω_i / ω .

Now, eliminating θ_0 from Eq. (24), a quadratic equation with respect to v_0

$$(C_1^2 + C_2^2) \left(\frac{v_0}{a_0 \omega} \right)^2 - 2 C_1 \frac{z}{a_0} \frac{v_0}{a_0 \omega} + \left[\left(\frac{z}{a_0} \right)^2 - \{c_1^2(l) + c_2^2(l)\} \right] = 0 \quad (25)$$

is obtained, and from this equation

$$\frac{v_0(\pm)}{a_0 \omega} = \frac{1}{C_1^2 + C_2^2} \left[C_1 \frac{z}{a_0} \pm \sqrt{(C_1^2 + C_2^2) \{c_1^2(l) + c_2^2(l)\} - C_2^2 \left(\frac{z}{a_0} \right)^2} \right] \quad (26)$$

is obtained. Here $v_0(\pm)$ corresponds with the sign $\pm$ preceding the radical sign.

In this manner, if v_0 is obtained, A_i and B_i can be determined, θ_0 from Eq. (24), $\dot{y}_2(0_+)$ and $y_a(l, 0)$ from Eq. (23). Thus the periodic solutions can be determined.

But in order that the formally obtained periodic solutions may be realized, the following conditions must be satisfied:

(i) The velocity jump v_0 by impact is positive, and

(ii) at the interval $0 \leq t \leq n \pi / \omega$

$$y_a(l, t) - z \leq y_2(t) \leq y_a(l, t) + z \quad (27)$$

First, the condition for v_0 to be real is

$$\left(\frac{z}{a_0} \right)^2 \leq \left\{ 1 + \left(\frac{C_1}{C_2} \right)^2 \right\} \{c_1^2(l) + c_2^2(l)\} \equiv \left(\frac{z_{gr}}{a_0} \right)^2 \quad (28)$$

and then the range of the free paths for positive v_0 is expressed as follows:

(1) Concerning $v_0(+)$

(a) $0 < z/a_0 < \sqrt{c_1^2(l) + c_2^2(l)}$ for $C_1 < 0$

(b) $0 < z/a_0 < z_{gr}/a_0$ for $C_1 > 0$

(2) Concerning $v_0(-)$

(a) positive z does not exist for $C_1 < 0$

(b) $\sqrt{c_1^2(l) + c_2^2(l)} < z/a_0 < z_{gr}/a_0$ for $C_1 > 0$.

Next, as the second condition (27) can not be examined except by numerical calculations, only the results of numerical calculations are shown in this paper (see Chap. 6).

5. Stability of periodic solutions

In the above, periodic solutions are found. But

in order that the periodic solutions may be realized, they must further be stable dynamically. Let us define here the stability of periodic solutions as follows: "Now, let a periodic state be realized in some manner. And let the state be disturbed slightly by some cause. Then if disturbances tend to converge to 0 with impacts, the original periodic state is defined as being stable."

Referring to expressions in Chap. 4, disturbed motion is written as follows:

$$y_a(x, t)^{(k)} = a_0 \{c_1(x) \cos(\omega t + \theta_0^{(k)}) + c_2(x) \sin(\omega t + \theta_0^{(k)})\} + v_0^{(k)} H(x, t) + \sum_{i=1}^{\infty} \varphi_i(x) (A_i^{(k)} \sinh \sqrt{\zeta_i^2 - 1} \omega_i t + B_i^{(k)} \cosh \sqrt{\zeta_i^2 - 1} \omega_i t) e^{-\zeta_i \omega_i t} \quad (29)$$

$$y_2(t)^{(k)} = \dot{y}_2(0_+)^{(k)} t + y_a(l, 0)^{(k)} + z \quad (30)$$

and

$$y_a(x, t)^{(k-1)} = -a_0 \{c_1(x) \cos(\omega t + \theta_0^{(k-1)}) + c_2(x) \sin(\omega t + \theta_0^{(k-1)})\} - v_0^{(k-1)} H(x, t) - \sum_{i=1}^{\infty} \varphi_i(x) (A_i^{(k-1)} \sinh \sqrt{\zeta_i^2 - 1} \omega_i t + B_i^{(k-1)} \cosh \sqrt{\zeta_i^2 - 1} \omega_i t) e^{-\zeta_i \omega_i t} \quad (31)$$

$$y_2(t)^{(k-1)} = -\dot{y}_2(0_+)^{(k-1)} t - y_a(l, 0)^{(k-1)} - z \quad (32)$$

Conditions of connection corresponding to Eq. (14) become

$$(i) \quad y_a(x, 0_+)^{(k)} = y_a \left(x, \frac{n \pi + \theta_0^{(k)} - \theta_0^{(k-1)}}{\omega} \right)^{(k-1)} \quad (33)$$

$$(ii) \quad y_2(0_+)^{(k)} = y_2 \left(\frac{n \pi + \theta_0^{(k)} - \theta_0^{(k-1)}}{\omega} \right)^{(k-1)} \quad (34)$$

$$(iii) \quad y_a(l, 0_+)^{(k)} = y_2(0)^{(k)} - z \quad (35)$$

$$(iv) \quad \dot{y}_a(l, 0_+)^{(k)} = \frac{1-\varepsilon\mu_0}{1+\mu_0} \dot{y}_a\left(l, \frac{n\pi + \theta_0^{(k)} - \theta_0^{(k-1)}}{\omega}\right)^{(k-1)} + \mu_0 \frac{1+\varepsilon}{1+\mu_0} \dot{y}_2\left(\frac{n\pi + \theta_0^{(k)} - \theta_0^{(k-1)}}{\omega}\right)^{(k-1)} \dots\dots\dots(36)$$

$$\dot{y}_2(0_+)^{(k)} = \frac{1+\varepsilon}{1+\mu_0} \dot{y}_a\left(l, \frac{n\pi + \theta_0^{(k)} - \theta_0^{(k-1)}}{\omega}\right)^{(k-1)} + \frac{\mu_0 - \varepsilon}{1+\mu_0} \dot{y}_2\left(\frac{n\pi + \theta_0^{(k)} - \theta_0^{(k-1)}}{\omega}\right)^{(k-1)} \dots\dots\dots(37)$$

$$(v) \quad \dot{y}_a(l, 0_+)^{(k)} - \dot{y}_a\left(l, \frac{n\pi + \theta_0^{(k)} - \theta_0^{(k-1)}}{\omega}\right)^{(k-1)} = v_0^{(k)} \dots\dots\dots(38)$$

Substituting Eqs. (29) ~ (32) into these equations, replacing

$$\begin{aligned} \theta_0^{(k)} &= \theta_0 + \sum_{j=0}^{k-1} \delta\theta_0^{(j)}, & v_0^{(k)} &= v_0 + \delta v_0^{(k)}, & A_i^{(k)} &= A_i + \delta A_i^{(k)} \\ B_i^{(k)} &= B_i + \delta B_i^{(k)}, & \dot{y}_2(0_+)^{(k)} &= \dot{y}_2(0_+) + \delta \dot{y}_2(0_+)^{(k)} \\ y_a(l, 0_+)^{(k)} &= y_a(l, 0_+) + \delta y_a(l, 0_+)^{(k)} \\ \delta F(x, t)^{(k)} &= \sum_{i=1}^{\infty} \varphi_i(x) (\delta A_i^{(k)} \sinh \sqrt{\zeta_i^2 - 1} \omega_i t + \delta B_i^{(k)} \cosh \sqrt{\zeta_i^2 - 1} \omega_i t) e^{-\zeta_i \omega_i t} \dots\dots\dots(39) \end{aligned}$$

and putting them in order under the assumption $(\omega_1/2\zeta_1\omega)\delta\theta_0^{(k)} \ll 1$, the following equations are obtained:

$$\left. \begin{aligned} &\delta F(x, 0)^{(k)} + \delta F\left(x, \frac{n\pi}{\omega}\right)^{(k-1)} + \delta v_0^{(k-1)} H\left(x, \frac{n\pi}{\omega}\right) \\ &+ \left\{ \dot{F}\left(x, \frac{n\pi}{\omega}\right) + v_0 \dot{H}\left(x, \frac{n\pi}{\omega}\right) \right\} \frac{\delta\theta_0^{(k-1)}}{\omega} = 0 \\ &\delta y_a(l, 0_+)^{(k)} + \delta y_a(l, 0_+)^{(k-1)} + \frac{n\pi}{\omega} \delta \dot{y}_2(0_+)^{(k-1)} + \dot{y}_2(0_+) \frac{\delta\theta_0^{(k-1)}}{\omega} = 0 \\ &\frac{\mu_0(1+\varepsilon)}{1+\mu_0} \left[a_0 \omega \{c_1(l) \cos \theta_0 + c_2(l) \sin \theta_0\} \sum_{j=0}^{k-1} \delta\theta_0^{(j)} - \delta \dot{F}(l, 0)^{(k)} \right] \\ &- \delta v_0^{(k)} \dot{H}(l, 0_+) - \frac{\mu_0(1+\varepsilon)}{1+\mu_0} \delta \dot{y}_2(0_+)^{(k-1)} = 0 \\ &\frac{1+\varepsilon}{1+\mu_0} \left[a_0 \omega \{c_1(l) \cos \theta_0 + c_2(l) \sin \theta_0\} \sum_{j=0}^{k-1} \delta\theta_0^{(j)} - \delta \dot{F}(l, 0)^{(k)} \right] \\ &+ \delta \dot{y}_2(0_+)^{(k)} + \frac{\mu_0 - \varepsilon}{1+\mu_0} \delta \dot{y}_2(0_+)^{(k-1)} = 0 \\ &\delta \dot{F}(l, 0)^{(k)} + \delta \dot{F}\left(l, \frac{n\pi}{\omega}\right)^{(k-1)} + \dot{H}\left(l, \frac{n\pi}{\omega}\right) \delta v_0^{(k-1)} \\ &+ \left\{ \dot{F}\left(l, \frac{n\pi}{\omega}\right) + v_0 \dot{H}\left(l, \frac{n\pi}{\omega}\right) \right\} \frac{\delta\theta_0^{(k-1)}}{\omega} = 0 \end{aligned} \right\} \dots\dots\dots(40)$$

The relation (35) is automatically satisfied.

Substituting Eq. (39) into Eq. (40) and eliminating $\delta\theta_0^{(k)}$ and $\delta \dot{y}_2(0_+)^{(k)}$, the following linear simultaneous difference equations with constant coefficients are obtained:

$$\left. \begin{aligned} &\alpha_1^{(i)} \delta \bar{A}_i^{(k)} + \alpha_2^{(i)} \delta \bar{A}_i^{(k-1)} + \beta_1^{(i)} \delta B_i^{(k+1)} + \beta_2^{(i)} \delta B_i^{(k)} + \beta_3^{(i)} \delta B_i^{(k-1)} = 0 \\ &\alpha_3^{(i)} \delta \bar{A}_i^{(k)} + \alpha_4^{(i)} \delta \bar{A}_i^{(k-1)} + \alpha_5^{(i)} \delta \bar{A}_i^{(k-2)} + \beta_4^{(i)} \delta B_i^{(k+1)} + \beta_5^{(i)} \delta B_i^{(k)} + \beta_6^{(i)} \delta B_i^{(k-1)} + \beta_7^{(i)} \delta B_i^{(k-2)} = 0 \end{aligned} \right\} \dots\dots\dots(41)$$

where

$$\begin{aligned} \delta \bar{A}_i^{(k)} &= \delta A_i^{(k)} + \delta v_0^{(k)} \frac{M_1 \varphi_i(l)}{\sqrt{\zeta_i^2 - 1} \omega_i} \\ \alpha_1^{(i)} &= \sqrt{\zeta_i^2 - 1} n\pi \frac{\omega_i}{\omega} + \mu_0 \frac{\lambda_0}{\lambda_1} D_1 D_3 \\ \alpha_2^{(i)} &= \sqrt{\zeta_i^2 - 1} n\pi \frac{\omega_i}{\omega} \left(D_2 - \frac{\zeta_i}{\sqrt{\zeta_i^2 - 1}} D_1 \right) D_3 + \left(\mu_0 \frac{\lambda_0}{\lambda_1} - \frac{n\pi}{\omega} \frac{\lambda_2}{\lambda_1} \right) D_1 D_3 \\ \alpha_3^{(i)} &= -\sqrt{\zeta_i^2 - 1} n\pi \frac{\omega_i}{\omega} + \frac{1-\varepsilon\mu_0}{1+\mu_0} \frac{\lambda_0}{\lambda_1} D_1 D_3 \\ \alpha_4^{(i)} &= \sqrt{\zeta_i^2 - 1} n\pi \frac{\omega_i}{\omega} - \left(\frac{\lambda_0}{\lambda_1} + \omega \frac{\lambda_3}{\lambda_1} \right) D_1 D_3, & \alpha_5^{(i)} &= \frac{\lambda_0}{\lambda_1} \frac{\mu_0(1+\varepsilon)}{1+\mu_0} D_1 D_3 \\ \beta_1^{(i)} &= \mu_0 \left(\frac{\lambda_0}{\lambda_1} - 1 \right) \\ \beta_2^{(i)} &= -\zeta_i n\pi \frac{\omega_i}{\omega} + \mu_0 \frac{\lambda_0}{\lambda_1} (1 + D_2 D_3) - \frac{n\pi}{\omega} \frac{\lambda_2}{\lambda_1} \\ \beta_3^{(i)} &= \sqrt{\zeta_i^2 - 1} n\pi \frac{\omega_i}{\omega} \left(D_1 - \frac{\zeta_i}{\sqrt{\zeta_i^2 - 1}} D_2 \right) D_3 + \mu_0 + \left(\mu_0 \frac{\lambda_0}{\lambda_1} - \frac{n\pi}{\omega} \frac{\lambda_2}{\lambda_1} \right) D_2 D_3 \end{aligned}$$

$$\begin{aligned}\beta_4^{(i)} &= \frac{1-\varepsilon\mu_0}{1+\mu_0} \left(\frac{\lambda_0}{\lambda_1} - 1 \right) \\ \beta_5^{(i)} &= \zeta_i n\pi \frac{\omega_i}{\omega} + \frac{1-\varepsilon\mu_0}{1+\mu_0} \left(1 + \frac{\lambda_0}{\lambda_1} D_2 D_3 \right) - \left(\frac{\lambda_0}{\lambda_1} - 1 \right) - \omega \frac{\lambda_3}{\lambda_1} \\ \beta_6^{(i)} &= -\zeta_i n\pi \frac{\omega_i}{\omega} + \frac{\mu_0(1+\varepsilon)}{1+\mu_0} \left(\frac{\lambda_0}{\lambda_1} - 1 \right) - 1 - \left(\frac{\lambda_0}{\lambda_1} + \omega \frac{\lambda_3}{\lambda_1} \right) D_2 D_3, \quad \beta_7^{(i)} = \frac{\mu_0(1+\varepsilon)}{1+\mu_0} \left(1 + \frac{\lambda_0}{\lambda_1} D_2 D_3 \right), \\ \lambda_0 &= -\left(C_2 + \frac{1}{2\mu_0} \right) v_0, \quad \lambda_1 = \left\{ C_2 + \frac{1+2\mu_0-\varepsilon}{2\mu_0(1+\varepsilon)} \right\} v_0 \\ \lambda_2 &= \ddot{F}(l, n\pi/\omega) + v_0 \ddot{H}(l, n\pi/\omega) = v_0 \sum_{i=1}^{\infty} \frac{M_1 \varphi_i^2(l)}{\sqrt{\zeta_i^2 - 1}} \omega_i \frac{\{(2\zeta_i^2 - 1)D_1 - 2\zeta_i \sqrt{\zeta_i^2 - 1}(D_2 + D_3)\} D_3}{(1 + D_2 D_3)^2 - (D_1 D_3)^2} \\ \lambda_3 &= C_1 v_0 / \omega - z\end{aligned}$$

General solutions of Eq. (41) are given by

$$\partial A_i^{(k)} = a_i' z_i^k, \quad \partial B_i^{(k)} = b_i z_i^k \dots\dots\dots (42)$$

where a_i' and b_i are constants and z_i is unknown. Substituting Eq. (42) into Eq. (41),

$$\begin{bmatrix} \alpha_1^{(i)} z_i + \alpha_2^{(i)} & \beta_1^{(i)} z_i^2 + \beta_2^{(i)} z_i + \beta_3^{(i)} \\ \alpha_3^{(i)} z_i^2 + \alpha_4^{(i)} z_i + \alpha_5^{(i)} & \beta_4^{(i)} z_i^3 + \beta_5^{(i)} z_i^2 + \beta_6^{(i)} z_i + \beta_7^{(i)} \end{bmatrix} \begin{bmatrix} a_i' \\ b_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \dots\dots\dots (43)$$

is obtained. Therefore, the characteristic equation is written as

$$\gamma_0^{(i)} z_i^4 + \gamma_1^{(i)} z_i^3 + \gamma_2^{(i)} z_i^2 + \gamma_3^{(i)} z_i + \gamma_4^{(i)} = 0 \dots (44)$$

where

$$\begin{aligned}\gamma_0^{(i)} &= \alpha_1^{(i)} \beta_4^{(i)} - \alpha_3^{(i)} \beta_1^{(i)} \\ \gamma_1^{(i)} &= \alpha_1^{(i)} \beta_5^{(i)} + \alpha_2^{(i)} \beta_4^{(i)} - \alpha_3^{(i)} \beta_2^{(i)} - \alpha_4^{(i)} \beta_1^{(i)} \\ \gamma_2^{(i)} &= \alpha_1^{(i)} \beta_6^{(i)} + \alpha_2^{(i)} \beta_5^{(i)} - \alpha_3^{(i)} \beta_3^{(i)} \\ &\quad - \alpha_4^{(i)} \beta_2^{(i)} - \alpha_5^{(i)} \beta_1^{(i)} \\ \gamma_3^{(i)} &= \alpha_1^{(i)} \beta_7^{(i)} + \alpha_2^{(i)} \beta_6^{(i)} - \alpha_4^{(i)} \beta_3^{(i)} - \alpha_5^{(i)} \beta_2^{(i)} \\ \gamma_4^{(i)} &= \alpha_2^{(i)} \beta_7^{(i)} - \alpha_5^{(i)} \beta_3^{(i)}\end{aligned}$$

Putting the radicals of Eq. (44) as z_{ij} ($j=1, 2, 3, 4$), the solutions of Eq. (41) become

$$\partial A_i^{(k)} = \sum_{j=1}^4 a_{ij}' z_{ij}^k, \quad \partial B_i^{(k)} = \sum_{j=1}^4 b_{ij} z_{ij}^k \dots\dots\dots (45)$$

Considering the definition of stability,

$$\lim_{k \rightarrow \infty} \partial \bar{A}_i^{(k)} = 0, \quad \lim_{k \rightarrow \infty} \partial B_i^{(k)} = 0$$

must be satisfied, and therefore for four radicals z_{ij} of Eq. (44) $|z_{ij}| < 1$ must be satisfied. The necessary and sufficient conditions for $|z_{ij}| < 1$ are given by Schur's theorem, but in this paper the other stability criterion is used [see Appendix (2)]. The latter stability conditions are

$$\left. \begin{aligned} d_1 > 0, \quad d_2 > 0, \quad d_3 > 0, \quad d_4 > 0, \\ d_1 d_2 d_3 - d_3^2 - d_1^2 d_4 > 0 \end{aligned} \right\} \dots\dots\dots (46)$$

where

$$\begin{aligned}d_1 &= 2(2\gamma_0 + \gamma_1 - \gamma_3 - 2\gamma_4)/d_0 \\ d_2 &= 2(3\gamma_0 - \gamma_2 + 3\gamma_4)/d_0 \\ d_3 &= 2(2\gamma_0 - \gamma_1 + \gamma_3 - 2\gamma_4)/d_0 \\ d_4 &= (\gamma_0 - \gamma_1 + \gamma_2 - \gamma_3 + \gamma_4)/d_0 \\ d_0 &= \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\end{aligned}$$

(the subscripts i expressing the mode number of natural vibration are omitted)

Consequently, stability of the periodic solution can be discriminated by Eq. (46) for all positive integers i . For large i , Eq. (46) can be approximated as follows. Considering that the terms $\sqrt{\zeta_i^2 - 1} n\pi \omega_i / \omega$ and $\zeta_i n\pi \omega_i / \omega$ are sufficiently larger

than other terms in $\alpha_1^{(i)} \sim \alpha_5^{(i)}$ and $\beta_1^{(i)} \sim \beta_7^{(i)}$, Eq. (44) becomes

$$\gamma_0' z^3 + \gamma_1' z + \gamma_2' z + \gamma_3' = 0 \dots\dots\dots (47)$$

where

$$\begin{aligned}\gamma_0' &= \left(\frac{\lambda_0}{\lambda_1} - 1 \right) \left(\mu_0 + \frac{1-\varepsilon\mu_0}{1+\mu_0} \right) \\ \gamma_1' &= \left(1 + \frac{\lambda_0}{\lambda_1} e^{-(n\pi/2\zeta_1)(\omega_1/\omega)} \right) \left(\mu_0 + \frac{1-\varepsilon\mu_0}{1+\mu_0} \right) \\ &\quad - \left(\frac{\lambda_0}{\lambda_1} - 1 \right) - \omega \frac{\lambda_3}{\lambda_1} - \frac{n\pi}{\omega} \frac{\lambda_2}{\lambda_1} \\ \gamma_2' &= \mu_0 \frac{\varepsilon - \mu_0}{1 + \mu_0} \left(\frac{\lambda_0}{\lambda_1} - 1 \right) - 1 + \frac{n\pi}{\omega} \frac{\lambda_2}{\lambda_1} \\ &\quad - e^{-(n\pi/2\zeta_1)(\omega_1/\omega)} \left(\frac{\lambda_0}{\lambda_1} + \omega \frac{\lambda_3}{\lambda_1} + \frac{n\pi}{\omega} \frac{\omega_1}{2\zeta_1} + \frac{n\pi}{\omega} \frac{\lambda_2}{\lambda_1} \right) \\ \gamma_3' &= \mu_0 \frac{\varepsilon - \mu_0}{1 + \mu_0} + e^{-(n\pi/2\zeta_1)(\omega_1/\omega)} \left(\mu_0 \frac{\lambda_0}{\lambda_1} \frac{\varepsilon - \mu_0}{1 + \mu_0} \right. \\ &\quad \left. + \frac{n\pi}{\omega} \frac{\omega_1}{2\zeta_1} + \frac{n\pi}{\omega} \frac{\lambda_2}{\lambda_1} \right)\end{aligned}$$

In this case the stability conditions are

$$d_1' > 0, \quad d_3' > 0, \quad d_1' d_2' - d_3' > 0 \dots\dots\dots (48)$$

where

$$\begin{aligned}d_1' &= (3\gamma_0' + \gamma_1' - \gamma_2' - 3\gamma_3')/d_0' \\ d_2' &= (3\gamma_0' - \gamma_1' - \gamma_2' + 3\gamma_3')/d_0' \\ d_3' &= (\gamma_0' - \gamma_1' + \gamma_2' - \gamma_3')/d_0' \\ d_0' &= \gamma_0' + \gamma_1' + \gamma_2' + \gamma_3'\end{aligned}$$

6. Numerical calculations

Next, the results of numerical calculations for a concrete model of the impact-damper are shown. Figure 2 shows a model of which the continuum is

Fig. 2 Vibration system

a uniform coil spring. Natural circular frequency and normalized characteristic function of this vibration system are given by

$$\frac{\omega_i l}{a} \tan \frac{\omega_i l}{a} = \frac{M_s}{M_1} \quad \dots\dots\dots (49)$$

$$\varphi_i(x) = \sqrt{\frac{2}{M_s + M_1 \sin^2 \frac{\omega_i l}{a}}} \sin \frac{\omega_i l}{a} \frac{x}{l} \quad \dots\dots\dots (50)$$

where $a = l\sqrt{k/M_s}$ is the velocity of wave propagation in the spring and k is the spring-constant for the whole spring.

In this paper, stability regions of the periodic solutions and the amplitude of the concentrated mass M_1 are calculated. To simplify calculation, let all parameters be nondimensional. In the determination of the stability region, the independent parameters are M_s/M_1 , M_2/M_1 , ω_1/ω , z/a_0 , ζ_1 , n and ε .

Now, if the value of M_s/M_1 is given, then all $\omega_i l/a$ are determined by Eq. (49) and all ratios of the i th natural frequencies to the first $\alpha_i = \omega_i/\omega_1$ are determined. The values of ω_i/ω are, therefore, determined if ω_1/ω is given. And all values of ζ_i are determined if the value of $\zeta_1 = k\omega_1/2$ is given, as $\zeta_i = (k\omega_1/2)(\omega_i/\omega_1)$.

The stability regions are expressed in $\omega_1/\omega - z/a_0$ plane for five values provided of M_s/M_1 , M_2/M_1 , ζ_1 , n and ε . In the paper, $M_s/M_1 = 0.1$, $\zeta_1 = 0.1$ are adopted. Such a vibration system behaves as if it were a system with one degree-of-freedom in free vibration, and the infinite series C_1 and C_2 and the displacement $y_a(l, t)$ can be sufficiently approximated

by only their first term. And then $n=1$ is adopted because this case is the most fundamental periodic state.

Figure 3 shows the stability region in the case of $M_2/M_1 = 0.1$, $\varepsilon = 0.5$. Boundary lines of the stability region for $i \geq 3$ lie over one for $i=2$. This figure concerns $v_0(+)$, and the stability region concerning $v_0(-)$ is omitted because it is too narrow. The boundary line II is determined by Eq. (27).

Fig. 4 Distribution of amplitude for $v_0(+)$

I: Boundary-line of stability region

II: Boundary-line determined from

$y_a(l, t) + z \geq y_2(t) \geq y_a(l, t) - z$

Fig. 3 Stability region for $v_0(+)$

Fig. 5 Damping effect

Fig. 6 Amplitude at $\omega_1/\omega=1$
 $M_2/M_1=0.1$, $M_2/M_1=0.2$, $\zeta_1=0.1$, $i=1$, $n=1$
 Fig. 7 Stability region for $v_0(+)$

 $M_2/M_1=0.1$, $\zeta_1=0.1$, $n=1$, $\varepsilon=1.0$, $i=1$
 Fig. 8 Stability region for $v_0(+)$
(a) $M_2/M_1=0.2$ (b) $M_2/M_1=0.4$ (c) $M_2/M_1=0.6$ (d) $M_2/M_1=0.8$ (e) $M_2/M_1=1.0$ Fig. 9 Distribution of amplitude for $v_0(+)$ ($M_2/M_1=0.1$, $\zeta_1=0.1$, $\varepsilon=1.0$, $n=1$)

Figure 4 shows the distribution of the amplitude in the stability region described in Fig. 3. Dotted curves are used only conveniently not to complicate the figure.

Percentages of these amplitudes to the resonance amplitude of the system without an impact-body are shown in Fig. 5. The relation between z/a_0 and the amplitude at $\omega=\omega_1$ is shown in Fig. 6.

Figure 7 shows the relation between ε and the stability region when $M_2/M_1=0.2$. On the other hand, in Fig. 8 the relation between M_2/M_1 and the stability region in the case of $\varepsilon=1$ is shown.

Figure 9 shows distribution of the amplitude

corresponding to the value of M_2/M_1 and z/a_0 .

Figure 10 shows the change in the amplitude corresponding to M_2/M_1 and z/a_0 . On the other hand, the change in the amplitude corresponding to ϵ is shown in Fig. 11, where parameters are M_2/M_1 and z/a_0 .

7. Conclusions

The results of numerical calculations show that the stability regions of the periodic solutions exist and that the effects of damping are sufficiently good.

With respect to the parameters adopted in this paper ($M_2/M_1=0.1$, $\zeta_1=0.1$, $n=1$), the following

Fig. 10 Distribution of amplitude for $v_0(+)$

(a) $M_2/M_1=0.2$

(b) $M_2/M_1=0.4$

conclusions are obtained.

(1) The stability region is wider as the value of M_2/M_1 is smaller or ϵ is nearer to 1.

(2) For a fixed value of z/a_0 , the damping effect in the case of $\omega=\omega_1$ is good as M_2/M_1 is larger. When M_2/M_1 is fixed too, the change of the damping effect with respect to ϵ is small.

(3) For fixed values of ϵ and M_2/M_1 , the damping effect at $\omega=\omega_1$ is good as z/a_0 is larger. But if z/a_0 is larger than the resonance amplitude of the system without an impact-body, impact may not occur in some cases of the initial conditions.

(4) As seen from Figs. 10 and 11, however the parameters may be selected, the damping effect may not be better than a certain extent.

(5) As seen from Fig. 11 the damping effect is best in the case of $\epsilon=1$. But the amplitude can not be decreased under 0.7 (about 7.7% in percentage of the amplitude to the resonance amplitude). Also in the case of the system with one degree-of-freedom, the damping effect is best when $\epsilon=1$ ⁽³⁾.

(6) As seen from Figs. 4 and 5, the amplitude can be larger than in the case without an impact if ω_1/ω is larger than a certain extent. In that case the system can not be used as the damper.

Numerical calculations in this paper were carried out by KDC-II (HITAC 5020) at the Computation Center of Kyoto University.

(c) $M_2/M_1=0.6$

(d) $M_2/M_1=0.8$

(e) $M_2/M_1=1.0$

Fig. 11 Distribution of amplitude for $v_0(+)$

Appendix

(1) Corresponding to Eq. (3), natural vibration is expressed by

$$y_r(x, t) = - \int_0^l G(x, \xi) m(\xi) \ddot{y}_r(\xi, t) d\xi \quad \dots\dots(51)$$

Putting the solution of Eq. (51)

$$y_r(x, t) = \varphi(x) \cos(\sqrt{\lambda} t + \alpha) \quad \dots\dots\dots(52)$$

the following Fredholm type homogeneous integral equation is obtained:

$$\varphi(x) = \lambda \int_0^l G(x, \xi) m(\xi) \varphi(\xi) d\xi \quad \dots\dots\dots(53)$$

Here putting the eigenvalues and the eigenfunctions $\lambda_i = \omega_i^2$ and $\varphi_i(x)$ respectively, Eq. (53) is reduced to

$$\varphi_i(x) = \omega_i^2 \int_0^l G(x, \xi) m(\xi) \varphi_i(\xi) d\xi \quad \dots\dots\dots(54)$$

$$i = 1, 2, 3, \dots$$

These $\varphi_i(x)$ are orthogonal functions with respect to weight $m(x)$:

$$\int_0^l m(x) \varphi_i(x) \varphi_j(x) dx = \delta_{ij} \quad \dots\dots\dots(55)$$

where $\varphi_i(x)$ are normalized.

(2) Replacing z_i in Eq. (44) by $(s_i+1)/(s_i-1)$, the inside of the unit circle in z_i -plane is mapped to the left half-plane in s_i -plane⁽⁶⁾. Therefore, applying Hurwitz's stability criterion to the polynomial with respect to s_i , stability can be discriminated. By this transformation, Eq. (44) yields to

$$s_i^4 + d_1^{(i)} s_i^3 + d_2^{(i)} s_i^2 + d_3^{(i)} s_i + d_4^{(i)} = 0 \quad \dots\dots\dots(56)$$

of which the stability discriminant is Eq. (47).

References

- (1) A. E. Kobrinskii: *Mechanizmi s Uprugimi Svyazyami*, (1964), Nauka.
- (2) S. F. Masri: *Analytical and Experimental Studies of Impact Dampers*, (1965), CIT.
- (3) K. Kuphal: *Z. AMM*, Bd. 45-2/3 (1965), S. 73.
- (4) M. I. Feigin: *Prikl. Math. Mech.*, Vol. 30, No. 5 (1966), p. 942.
- (5) D. M. Egle: *ASME Paper*, 67-Vibr. -10 (1967).
- (6) F. Peterka: *Strojn. Chas.*, Vol. 16, No. 3 (1965), p. 314.
- (7) J. Kondo: *Integral equation and its application*, (1963), p. 166, Corona Publishing Co.
- (8) K. Nishioka et al.: *Sumitomo Kinzoku*, Vol. 16, No. 1 (1964), p. 44.