Anisotropic Yield Criterion under the Maximum Shear Stress Theory

By Hisashi Igaki**, Masakatsu Sugimoto***, and Koichi Saito****

Yield criterion of anisotropic materials is developed under the maximum shearing stress hypothesis when the elements of materials are acted upon by three normal stresses in the direction of principal axes of anisotropy and a shearing stress around one of them.

It is found for isotropic materials that the yield surface can be represented by two elliptical cone surfaces and a cylindroid one in the stress co-ordinate which is composed of the π-plane and its normal as a shearing stress. Under the assumption that these surfaces are independently distorted, translated and rotated, the numerical yield surface of anisotropic materials is found to be in good agreement with the experimental results of plates in any direction other than the direction of the degrees of angle, 60° to 70° from initial path of tensile prestrain.

1. Introduction

The yield criterion of anisotropic materials was previously extended by authors, under the maximum shearing stress hypothesis, in which the principal axes of stress are coincident with those of anisotropy of materials(1). The results were applied to some problems on plastic deformations and technologies of materials(2). When \(x_1\), \(x_2\) and \(x_3\) are taken as the principal axes of anisotropy, and \(\sigma_{11}\), \(\sigma_{22}\) and \(\sigma_{33}\) are the normal stresses in these directions, respectively, that is, both principal axes are coaxial, the yield condition can be written as follows.

\[
\begin{align*}
\sigma_{11} - \sigma_{11} &= 2\tau_{12}\,0 \\
\sigma_{22} - \sigma_{22} &= 2\tau_{23}\,1 \\
\sigma_{33} - \sigma_{22} &= 2\tau_{33}\,2
\end{align*}
\]

(1)

where \(\tau_{12}\) etc. are the parameters of anisotropy as detailed in the references (1) and (2), and will be called "the principal shearing stresses of yielding" in this paper. Equation (1) is, for example, represented by the hexagon ABCDEF on the π-plane as shown in Fig. 1.

When these two principal axes are not coaxial, a stress state is in general described by the three normal stresses in the directions of principal axes of anisotropy in material and the three shearing stresses around them. But, in this paper, assuming that the elements of materials are acted upon by the three normal stresses \(\sigma_{11}\), \(\sigma_{22}\) and \(\sigma_{33}\), and a shearing stress around the \(x_2\)-axis as shown in Fig. 2 (that is \(\tau_{12} = \tau_{33} = 0\)), the anisotropic yield criterion is developed in terms of these stress components being under the maximum shearing stress hypothesis.

2. Isotropic yield criterion under the maximum shearing stress hypothesis

Prior to derivation of the anisotropic yield criterion, the yield condition of isotropic materials obeying the maximum shearing stress hypothesis is

\[
\begin{align*}
AB & : \sigma_{21} - \sigma_{21} = 2\tau_{12} \\
BC & : \sigma_{22} - \sigma_{22} = 2\tau_{23} \\
CD & : \sigma_{33} - \sigma_{33} = 2\tau_{33} \\
DE & : \sigma_{21} - \sigma_{21} = 2\tau_{12} \\
EF & : \sigma_{22} - \sigma_{22} = 2\tau_{23} \\
FA & : \sigma_{33} - \sigma_{33} = 2\tau_{33}
\end{align*}
\]

Fig. 1 Yield surface of anisotropic materials when the principal axes of stress coincide with those of anisotropy.

* Received 23rd May, 1969.
** Associate Professor, Faculty of Engineering, University of Osaka Prefecture, Osaka.
*** Lecturer, Faculty of Engineering, University of Osaka Prefecture.
**** Professor, Faculty of Engineering, University of Osaka Prefecture.
first expressed in the stress as shown in Fig. 2, where \( \tau_{ix} \) is defined as positive in the direction of the figure.

### 2-1 Equation of yield criterion

The three principal stresses \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) for the stress state in Fig. 2 can be obtained as follows, where \( \sigma_{ni} \) becomes a principal stress and is denoted by \( \sigma_n \):

\[
\sigma_1 = \frac{(\sigma_n + \sigma_{n2})}{2} + \sqrt{\left(\frac{(\sigma_n - \sigma_{n2})}{2}\right)^2 + \tau_{ix}^2} \\
\sigma_2 = \frac{(\sigma_n + \sigma_{n3})}{2} - \sqrt{\left(\frac{(\sigma_n - \sigma_{n3})}{2}\right)^2 + \tau_{ix}^2} \\
\sigma_3 = \frac{(\sigma_n + \sigma_{n1})}{2} - \sqrt{\left(\frac{(\sigma_n - \sigma_{n1})}{2}\right)^2 + \tau_{ix}^2} \tag{2}
\]

These principal stresses are divided into eight cases in their dimensional relationship because the inequality \( \sigma_1 > \sigma_2 > \sigma_0 \) always is valid. Excepting the case \( \sigma_1 = \sigma_2 = \sigma_3 \), the seven cases are classified into the following three for the maximum shearing stress \( \tau_{max} \):

1. \( \sigma_1 = \sigma_2 > \sigma_3 \), \( \sigma_1 > \sigma_2 > \sigma_3 \), \( \sigma_1 > \sigma_3 > \sigma_2 \)
   \[\tau_{max} = \frac{1}{2} \sqrt{\left(\sigma_{n1} + \sigma_{n2} - 2\sigma_n\right)^2} \tag{3}\]

2. \( \sigma_1 = \sigma_3 > \sigma_2 \), \( \sigma_2 > \sigma_3 > \sigma_1 \)
   \[\tau_{max} = \sqrt{\left(\sigma_{n1} - \sigma_{n3}\right)^2 + \tau_{ix}^2} \tag{4}\]

3. \( \sigma_2 = \sigma_3 > \sigma_1 \), \( \sigma_1 > \sigma_3 > \sigma_2 \)
   \[\tau_{max} = \frac{1}{2} \sqrt{\left(2\sigma_n - \sigma_{n1} - \sigma_{n2}\right)^2} \tag{5}\]

Substituting the shearing stress of yielding in isotropic material \( \tau_{Y} \) for \( \tau_{max} \) in Eqs. (3) to (5) leads to the following yield condition under the maximum shearing stress hypothesis:

\[
\sqrt{\left(\sigma_{n1} - \sigma_{n3}\right)^2 + \tau_{ix}^2} = 2\tau_Y + \left(3/2\right)S_n \tag{6}
\]

\[
\sqrt{\left(\sigma_{n1} - \sigma_{n2}\right)^2 + \tau_{ix}^2} = 2\tau_Y - \left(3/2\right)S_n \tag{7}
\]

\[
\sqrt{\left(\sigma_{n2} - \sigma_{n3}\right)^2 + \tau_{ix}^2} = 2\tau_Y - \left(3/2\right)S_n \tag{8}
\]

where \( S_n \) is the deviator of the stress \( \sigma_{ni} \).

### 2-2 Yield surface

The equations of the yield condition (6) to (8) give the hypersurfaces in the stress space of the four dimensions \( \sigma_1, \sigma_2, \sigma_3, \tau_{ix} \) geometrically. The stress state \( (\sigma_{ni}, \sigma_{n2}, \sigma_{n3}, \tau_{ix}) \) is equivalent to the one \( (S_n, S_{n2}, S_{n3}, \tau_{ix}) \) in the theory of plasticity, where \( S_n, S_{n2}, \) and \( S_{n3} \) are the deviators of the stresses \( \sigma_{ni}, \sigma_{n2}, \) and \( \sigma_{n3} \), respectively. A stress state can be geometrically represented by a particular point on the hyperplane \( \sigma_n + \sigma_{n2} + \sigma_{n3} = 0 \) in the stress space \( \sigma_n, \sigma_{n2}, \sigma_{n3}, \tau_{ix} \), because \( S_n + S_{n2} + S_{n3} = 0 \). On the other hand the \( \pi \)-plane is made up of the intersection of this hyperplane and the hyperplane \( \tau_{ix} = 0 \), that is, the principal stress space. Therefore the hyperplane \( \sigma_1 + \sigma_2 + \sigma_3 = 0 \) can be displayed in the three dimensional space composed of the \( \pi \)-plane and the \( \tau_{ix} \)-axis as its normal, as shown in Fig. 3. In this way the equations of the yield condition (6), (7) and (8) can be illustrated as the yield surface in such a three dimensional stress space. The stress axes \( \sigma_1, \sigma_2, \) and \( \sigma_3, \) on the \( \pi \)-plane in Fig. 3 are shortened by \( \sqrt{2/3} \) times in their length as they are the projections of the co-ordinate axes in the stress space \( \sigma_n, \sigma_{n2}, \sigma_{n3}, \tau_{ix} \) on the \( \pi \)-plane. This reduced scale is assumed to be the scale of the stresses themselves after isometric drawing in geometry. Together with this, the scale of the \( \tau_{ix} \)-axis is reduced by \( \sqrt{2/3} \). For convenience the \( \sigma_{n2} \)-axis etc. on the \( \pi \)-plane are called "the \( \sigma_{n2} \)-axis" etc. and this three dimensional stress space will be called "the \( \pi \tau \)-stress space."

Now, OA, AB, BC and CP in Fig. 3 are regarded as the directed segments of a line which are parallel and have the same directions with the co-ordinate axes in the \( \pi \tau \)-stress space, respectively. The point P indicates the stress state \( (\sigma_{n1}, \sigma_{n2}, \sigma_{n3}, \tau_{ix}) \) when \( OA = \sigma_{n1}, \; AB = \sigma_{n2}, \; BC = \sigma_{n3}, \) and \( CP = \tau_{ix} \). If DC and EC are considered to be the directed segments of a line parallel to the co-ordinate axes and GP and FC are the ones which are normal to the \( \sigma_{n1}, \sigma_{n3}, \tau_{ix} \)-plane and positive for \( EC > 0 \), then \( AB = DB = DA = (\sigma_{n2} - \sigma_{n3}) \) is valid. Therefore,

\[
(\sigma_{n1} - \sigma_{n3}) = (2/\sqrt{3})GP
\]

is obtained because \( GP = FC = EC \sin 60^\circ \) and \( EC = OD = OA - DA = (\sigma_{n1} - \sigma_{n3}) \) hold good.

The stress state, where \( S_n \) is equal to a constant value \( (2/3)S_{n2} \) is represented by the straight line on the \( \pi \)-plane which passes through the point on the hyperplane \( \sigma_n + \sigma_{n2} + \sigma_{n3} = 0 \).
"\(\sigma_{xx}\)" and is normal to the \(\sigma_{xx}\)-axis\(^{[9]}\). As \(S_{xx}\) is independent of \(\tau_{yy}\), \(S_{xx} = (2/3)\sigma_{xx}\) always holds good on the plane normal to the \(\sigma_{xx}\)-axis (for example, the PGFC plane in Fig. 3).

In due consideration of these, the feature of the yield surface satisfying Eqs. (6), (7) and (8) can be obtained in the \(\pi\tau\)-stress space as follows.

i) Equation (6) signifies the elliptical cone surface of which axis is the \(\sigma_{xx}\)-axis and apex is at the point \(\sigma_{xx} = -2\tau_{yy}\) on its axis. The major axis of the cross section (the ellipse) normal to the \(\sigma_{xx}\)-axis is on the \(\pi\)-plane, the minor axis of it is parallel with the \(\tau_{yy}\)-axis and the ratio of their lengths is \(\sqrt{3}:1\). This surface will be here called "the elliptical cone surface I."

ii) Equation (7) represents the cylindrical surface of which axis is the \(\sigma_{yy}\)-axis. There is the same relationship for the major and minor axes of the cross section (the ellipse) normal to the \(\sigma_{yy}\)-axis as in the elliptical cone surface I.

iii) Equation (8) represents the elliptical cone surface of which axis is the \(\sigma_{yy}\)-axis and apex is at the point \(\sigma_{yy} = 2\tau_{yy}\) on its axis. This surface is symmetrical to the elliptical cone surface I with the origin and will be called "the elliptical cone surface II."

The whole yield surface is a closed surface which is composed of these elliptical cone and cylindrical surfaces, as shown in Fig. 4. The orthogonal projection of this yield surface is shown in Fig. 5. Figure 5 (a) shows the projection of this to the \(\pi\)-plane. Figs. 5 (b) and (c) are the projections of this to the direction of the arrows \(\mathbf{2}\) and \(\mathbf{3}\), respectively.

3. Yield criterion of anisotropic materials

3.1 Development of anisotropic yield criterion

In the process of extending the isotropic yield condition described above in terms of \(\sigma_{xx}, \ldots, \tau_{yy}\) to the anisotropic one, we presuppose that the anisotropic yield condition is restricted with the following provisions.

i) The anisotropic yield condition is represented by Eq. (1) when the three stresses \(\sigma_{xx}, \sigma_{yy}, \) and \(\sigma_{zz}\) act only at any point of the cross section of materials.

ii) When only the shearing stress \(\tau_{zy}\) acts and \(\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0\), yielding of materials depends merely upon the value of \(\tau_{zy}\) in yielding irrespective of any yield criterion adopted. Such a value of \(\tau_{zy}\) is the parameter of anisotropy and will be called "the shearing stress of yielding on the co-ordinate plane". In anisotropic materials this value is, in general different according to the direction of \(\tau_{zy}\). So, \(\tau_{zy}\) is denoted by \(\tau_{zy} (>0)\) when \(\tau_{zy} > 0\) (its direction is shown in Fig. 2), by \(\tau_{zy} (>0)\) when \(\tau_{zy} < 0\). Then the anisotropic yield condition, when \(\tau_{zy}\) acts only, can be given as follows.

\[
\begin{align*}
\tau_{zy} &= \tau_{zy} (>0) \quad (9) \\
\tau_{zy} &= -\tau_{zy} (<0) \quad (10)
\end{align*}
\]

iii) When the anisotropic yield condition will be applied to isotropic materials, it is represented by Eqs. (6) to (8).

The subsequent yield surface, when an initially isotropic material is deformed plastically, is often assumed to be composed of combination of translating and expanding the initial yield one\(^{[9]}-^{[1]}\). In addition to this, however, it will be necessary in general to take into account of the distortion and rotation of yield surface for the purpose of better agreement with experimental results. Backhaus\(^{[a]}\) introduced these distortion and rotation into the yield surface, using the affine transformation for stress deviators.

Such a procedure of introducing the anisotropic yield surface as explained above will be used here. Namely, the cylindrical and two elliptical cone surfaces which compose the yield surface for the
maximum shearing stress hypothesis are translated or distorted separately so that above three provisions (i) to (iii) for yielding may be fulfilled.

3.2 Translation and distortion of the elliptical cone surfaces

The elliptical cone surfaces I and II are translated in parallel with their axes remaining on the \( \pi \)-plane. Then the lines of intersection of these surfaces and the \( \pi \)-plane \( A_B C_0 \) and \( D_E F_0 \) are translated to the lines \( A_B C_0 \) and \( D_E F_0 \), respectively, as shown in Fig. 6, and these lines become the yield surface corresponding to the equations of the yield condition for \( \tau_{xy} = 0 \). According to the features of anisotropy, it may be necessary to expand or contract the yield surface individually to the positive or negative direction of the \( \tau_{xy} \)-axis, being bounded by the \( \pi \)-plane.

Such translated and distorted “elliptical cone surfaces” as explained above are formulated as follows. \( m_x, m_y, n_x \) and \( n_y \) in the following equations are the positive numbers and mean expanding of the yield surface when they are larger than unity, contracting of the surface when they are smaller than unity.

“The elliptical cone surface I” can be written as,

\[
\sqrt{\frac{(\sigma_{xx}-\sigma_{yy})+2(\tau_{xy}-\tau_{xy})}{2}\left(\frac{\tau_{xy}}{m_x}\right)^2 + \left(\frac{\tau_{xy}}{m_y}\right)^2}
= (\tau_{xy} + \tau_{xy}) + (3/2)S_{xy}
\]

when \( \tau_{xy} \geq 0 \) (11-a)

\[
\sqrt{\frac{(\sigma_{xx}-\sigma_{yy})+2(\tau_{xy}-\tau_{xy})}{2}\left(\frac{\tau_{xy}}{m_x}\right)^2 + \left(\frac{\tau_{xy}}{m_y}\right)^2}
= (\tau_{xy} + \tau_{xy}) + (3/2)S_{xy}
\]

when \( \tau_{xy} \leq 0 \) (11-b)

“The elliptical cone surface II” can be written as,

\[
\sqrt{\frac{(\sigma_{xx}-\sigma_{yy})+2(\tau_{xy}-\tau_{xy})}{2}\left(\frac{\tau_{xy}}{n_x}\right)^2 + \left(\frac{\tau_{xy}}{n_y}\right)^2}
= (\tau_{xx} + \tau_{xx}) - (3/2)S_{xy}
\]

when \( \tau_{xy} \geq 0 \) (12-a)

\[
\sqrt{\frac{(\sigma_{xx}-\sigma_{yy})+2(\tau_{xy}-\tau_{xy})}{2}\left(\frac{\tau_{xy}}{n_x}\right)^2 + \left(\frac{\tau_{xy}}{n_y}\right)^2}
= (\tau_{xx} + \tau_{xx}) - (3/2)S_{xy}
\]

when \( \tau_{xy} \leq 0 \) (12-b)

Fig. 6 Translation of elliptical cone surfaces

* When the elliptical cone surface is distorted in addition to being translated, it is no longer an elliptical cone one in general, but for the purpose of simplicity it will be called “the elliptical cone surface.” This convention is similarly applied to the cylindrical surface.

As contrasted to the case of isotropic materials, “the elliptical cone surfaces” for anisotropic materials may happen to intersect the \( \tau_{xy} \)-axis after they are translated or distorted. This condition can be obtained as follows by using Eqs. (9) to (12-b).

When “the elliptical cone surface I” intersects \( \tau_{xy} \), which is positive:

\[
\tau_{xy} = 2m_x \sqrt{\frac{\tau_{xy}}{m_x}}
\]

When “the elliptical cone surface I” intersects \( \tau_{xy} \), which is negative:

\[
\tau_{xy} = 2m_x \sqrt{\frac{\tau_{xy}}{m_x}}
\]

When “the elliptical cone surface II” intersects \( \tau_{xy} > 0 \):

\[
\tau_{xy} = 2n_x \sqrt{\frac{\tau_{xy}}{n_x}}
\]

When “the elliptical cone surface II” intersects \( \tau_{xy} < 0 \):

\[
\tau_{xy} = 2n_x \sqrt{\frac{\tau_{xy}}{n_x}}
\]

3.3 Distortion of the cylindrical surface

If the cylindrical surface is translated, distorted and rotated adequately, keeping its axis parallel to the \( \sigma_{xx} \)-axis, its intersection with the \( \pi \)-plane gives the yield surface being represented by the equations \( \sigma_{xx} = \sigma_{xx} = 2\tau_{xy} \) and \( \sigma_{yy} = \sigma_{yy} = 2\tau_{xy} \). Various methods can be considered for translating, distorting and rotating of the cylindrical surface, but in this paper only the following distortion of the surface will be considered by dividing it into four parts.

\[
\sqrt{\frac{(\sigma_{xx}-\sigma_{yy})^{2} + (\tau_{xy}/q)^{2}}{2} = 1}
\]

(for \( \sigma_{xx} \geq \sigma_{yy}, \tau_{xy} \geq 0 \) ).........(13-a)

\[
\sqrt{\frac{(\sigma_{xx}-\sigma_{yy})^{2} + (\tau_{xy}/q)^{2}}{2} = 1}
\]

(for \( \sigma_{xx} \leq \sigma_{yy}, \tau_{xy} \geq 0 \) ).........(13-b)

\[
\sqrt{\frac{(\sigma_{xx}-\sigma_{yy})^{2} + (\tau_{xy}/q)^{2}}{2} = 1}
\]

(for \( \sigma_{xx} \geq \sigma_{yy}, \tau_{xy} \leq 0 \) ).........(13-c)

\[
\sqrt{\frac{(\sigma_{xx}-\sigma_{yy})^{2} + (\tau_{xy}/q)^{2}}{2} = 1}
\]

(for \( \sigma_{xx} \leq \sigma_{yy}, \tau_{xy} \leq 0 \) ).........(13-d)

where \( q_x \) is equal to \( q_x \) if “the cylindrical surface” after distortion intersects the positive \( \tau_{xy} \)-axis, \( q_x = q_x \) if it does the negative one. However the inequality \( q_x > q_x \) is valid when “the elliptical cone
surface" after translation and distortion intersects the positive $\tau_{xx}$-axis, $q_x > \tau_{xx}$, when it does the negative one. Figure 7 shows the features of the yield surface represented by Eqs. (13-a) to (13-d), where the arrow head (1) indicates the yield surface for the cases $q_4 = \tau_{33}$ and $q_x = \tau_{xx}$ as viewed from the arrow head direction.

3-4 An example of anisotropic yield surfaces

From the equations explained above, an anisotropic yield surface can be represented in the $\pi$-stress space, as exemplified in Fig. 8. In this figure (a), (b) and (c) have the same meaning as those in Fig. 5. The values of the principal shearing stress of yielding $\tau_{10}$ etc., the shearing stress of yielding on the co-ordinate plane $\tau_{10}$ etc. and other parameters of anisotropy are entered in the figure. In this example "the cylindrical surface" intersects the positive $\tau_{xx}$-axis because of $q_4 = \tau_{33} = 34.8$ kg/mm$^2$, and "the elliptical cone surface II" intersects the negative $\tau_{xx}$-axis as the inequality $q_x > \tau_{xx}$ holds good, because of

$$2 \eta \sqrt{\pi_{10} \pi_{10}} = 2 \times 0.583 \sqrt{20.0 \times 10.0} = 16.5 = \tau_{xx}.$$

4. Comparison between theoretical and experimental results

As shown in Eq. (1) or Eqs. (11-a) to (13-d), the anisotropic yield condition based on the maximum shearing stress hypothesis conveniently includes Bauschinger effect in terms of the linear function of stresses. Consequently these equations are considered to be useful for analyzing the plastic problems of anisotropy. And, for this purpose, it should be verified that these equations can actually express the yield surface of existing anisotropic materials with good approximation. In this paper, for an example, this anisotropic yield condition is compared with the experimental results for anisotropic yielding of the materials pretrained in simple tension.

4-1 The case when both principal axes are coincident

4-1-1 Experimental procedure

The experimental work used brass tubings of which chemical compositions are shown in Table 1. These cylindrical specimens had an outer diameter of 20.5 mm and an inner diameter of 18.5 mm, and were annealed at $600^\circ$C for 45 minutes. Yield stresses were observed when various kinds of proportional loadings of the combination of internal pressure and axial tension or compression were applied to the cylindrical specimens which had been pretrained by 1.8% or 20% tensile plastic strain (logarithmic strain) $\varepsilon_p$ in the axial direction of specimens. The

<p>| Table 1 Chemical compositions (%) of test specimens (brass tubing) |</p>
<table>
<thead>
<tr>
<th>Cu</th>
<th>Zn</th>
<th>Sn</th>
<th>Pb</th>
<th>Fe</th>
</tr>
</thead>
<tbody>
<tr>
<td>65.01</td>
<td>34.88</td>
<td>0.00</td>
<td>0.00</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Logarithmic strain %

<table>
<thead>
<tr>
<th>Axial tension</th>
<th>$\varepsilon_p$ %</th>
<th>$\varepsilon_2$ %</th>
<th>$\varepsilon_p/\varepsilon_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.20</td>
<td>-0.558</td>
<td>0.464</td>
<td></td>
</tr>
<tr>
<td>1.88</td>
<td>-0.832</td>
<td>0.443</td>
<td></td>
</tr>
<tr>
<td>10.4</td>
<td>-5.05</td>
<td>0.486</td>
<td></td>
</tr>
<tr>
<td>19.9</td>
<td>-9.97</td>
<td>0.501</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Internal pressure of the open end type</th>
<th>$\varepsilon_p$ %</th>
<th>$\varepsilon_2$ %</th>
<th>$\varepsilon_p/\varepsilon_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.12</td>
<td>-0.452</td>
<td>0.404</td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td>-0.725</td>
<td>0.414</td>
<td></td>
</tr>
<tr>
<td>10.9</td>
<td>-4.92</td>
<td>0.451</td>
<td></td>
</tr>
<tr>
<td>19.2</td>
<td>-7.69</td>
<td>0.401</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Axial compression</th>
<th>$\varepsilon_p$ %</th>
<th>$\varepsilon_2$ %</th>
<th>$\varepsilon_p/\varepsilon_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.00</td>
<td>0.466</td>
<td>0.466</td>
<td></td>
</tr>
<tr>
<td>-1.97</td>
<td>0.908</td>
<td>0.461</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 9 Verification for initial isotropy of test specimens (brass tubing)
axial strain was measured with Martens' mirror type extensometer of a gauge length 50 mm, the circumferential strain at the position of the mean diameter of tubing was obtained by assuming the incompressibility of materials and measuring the mean value of the outer diameter at four points of tubing with the micrometer having a vernier 0.001 mm. The stresses when the largest plastic strain in absolute value among the axial, circumferential and radial strains reaches 0.1% and 0.2% are called the 0.1% and 0.2% offset stress, respectively. Two yield surfaces, when these two stresses are defined as the yield stress, were observed.

4.1.2 Verification for initial isotropy of test specimens

Figure 9 shows the experimental results when virgin specimens were loaded with the axial tension and compression and the internal pressure of the open end type. It gives the true stress-logarithmic strain curves, and the table in it indicates the ratio of the logarithmic circumferential strain $e_\phi$ to the axial one $e_x$. It may be considered from Fig. 9 that the material of specimens will be isotropic at least in the plane of cylindrical surface of specimens. Therefore the initial tension in the axial direction will make the test specimen anisotropic. Then reloading of the combination of internal pressure and axial load gives the stress state where the directions of the axial, circumferential and radial principal stresses $\sigma_x$, $\sigma_\phi$, and $\sigma_r$ coincide with those of principal axes of anisotropy.

4.1.3 Experimental results

The points with marks ○ in Figs. 10 and 11 represent the observed yield stresses for the pre-strain $e_\phi^p$=1.8%, which are plotted on the $\pi$-plane, and the points with marks ● do those for the pre-strain $e_\phi^p$=20%. Lines connecting these points gives the yield curves of materials which have been pre-strained by $e_\phi^p$. If the specimens are assumed to be initially isotropic it may be predicted that the yield curve prestrained in axial tension will be symmetric for the $\sigma_x$-axis. So, a whole yield curve will be the closed curve composed of the solid line and the dashed one which is obtained from doubling that along the $\sigma_x$-axis.

4.1.4 Approximation with the anisotropic yield condition based on the maximum shearing stress hypothesis

The experimental subsequent yield curve can not be completely expressed with the yield condition (1). However, if it may be possible to substitute the observed points for the appropriate hexagon, it will be useful for studying the plastic problems of anisotropy. The hexagon $A_0B_1C_0D_1E_0F_0$ in Fig. 10 or $A_0B_2C_0D_2E_0F_2$ in Fig. 11 can be obtained from tracing through the measuring point on the positive $\sigma_x$-axis and the midway among other measuring points.

As shown in both figures, these hexagons tolerably agree with the subsequent yield curves connecting the observed points in a row, though this is not so partially. It is presumed from this that experimental curves can not be manifested with the simple combination of expanding and translating the initial yield curve, and formulation of the experimental yield curve may become seriously complex. In a sense of this the yield condition of Eq. (1) may be considered to be highly useful.

4.2 The case when both principal axes are not coincident

4.2.1 Outline of experiments

We can make use of the following experimental results which have been already published. Namely the one is the results for the specimens of brass, copper and aluminium which were carried out by one of authors(1), the other is those of aluminium alloy by Szczepiński(2). As explained later these are obtained in such manner that the plate which is isotropic...
at least in the tension of the plate plane is prestrained in simple tension, may become anisotropic and then tension test pieces are cut from its plate in several directions different from the initial tension one, and reloaded with tension.

As shown in Fig. 12, if the axes $x_1$, $x_2$ and $x_3$ are taken to coincide with the directions of the initial tension, plate width and plate thickness, respectively, and the virgin material is isotropic in plane, these axes become coincident with the principal directions of anisotropy in the specimen after it is loaded with the tension load $W$ in the $x_1$ direction and the plastic strain is residual in it. The stress $\sigma_{\alpha}$ is defined as the tension stress when the second tension is applied to the specimen in the direction of the angle of $\alpha$ from the $x_1$-axis. Then the stress components for the co-ordinate of the principal axes of anisotropy can be written as follows.

$$
\begin{align*}
\sigma_{\alpha} &= \sigma_{\alpha\alpha}\cos^2\alpha, \quad \sigma_{\beta\beta} = \sigma_{\alpha\alpha}\sin^2\alpha, \quad \sigma_{\gamma\gamma} = 0 \\
\tau_{\alpha\beta} &= 0, \quad \tau_{\beta\gamma} = \sigma_{\alpha\alpha}\sin\alpha\cos\alpha
\end{align*}
$$

These experiments, that is, correspond to those which are carried out by applying the proportional loading of the stress ratio

$$
\sigma_{\alpha} : \sigma_{\beta} : \sigma_{\gamma} = \cos^2\alpha : \sin^2\alpha : \sin\alpha\cos\alpha
$$
to the anisotropic material.

The chemical compositions and dimensions of test specimens are shown in Table 2.

**4.2.2 Verification for isotropy of the test specimens**

In regard to the specimens of brass and copper, the tension test was carried out on five test pieces which had been cut out of the virgin plate in the direction of the angle of $\beta$ (22.5° interval) from the latest rolling direction. True stress-logarithmic strain curves of them are all on the thick solid lines in Fig. 13. In respect of the specimens of aluminium, the tension test was performed on the test pieces of $\beta=0^\circ$, 45° and 90° and their results are shown in Fig. 13. The table in Fig. 13 indicates the ratio of the logarithmic strains $\varepsilon_{p}^*$ and $\varepsilon_{p}^*$ in the direction of plate thickness and plate width, respectively, to the logarithmic strain $\varepsilon_{p}^*$ (=$-\varepsilon_{\alpha p}^*$) in the direction of plate length, where these are the residual strains after the tension test have been performed. It may be presumed from Fig. 13 that the material of these plates will be initially isotropic only with regard to tension in the plane of plate.

![Graph showing stress-strain relationship](image)

Table 2 Chemical compositions (%) and dimensions (mm) of test specimens (plates)

<table>
<thead>
<tr>
<th>Material</th>
<th>Chemical compositions</th>
<th>Test piece for initial tension</th>
<th>Test piece for second tension</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cu</td>
<td>Al</td>
<td>Zn</td>
</tr>
<tr>
<td>Brass</td>
<td>Residuals</td>
<td>29.37</td>
<td>29.37</td>
</tr>
<tr>
<td>Copper</td>
<td>99.91</td>
<td>0.23</td>
<td>0.00</td>
</tr>
<tr>
<td>Aluminium</td>
<td>0.16</td>
<td>58.89</td>
<td>0.23</td>
</tr>
<tr>
<td>Aluminium</td>
<td>Residuals</td>
<td>0.5</td>
<td>4.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Material</th>
<th>Investigator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brass</td>
<td>K. Saito</td>
</tr>
<tr>
<td>Copper</td>
<td></td>
</tr>
<tr>
<td>Aluminium</td>
<td></td>
</tr>
</tbody>
</table>

* Only in this case the tension test was resumed after tension was unloaded during the tension test and the strains $\varepsilon_{\alpha p}^*$ and $\varepsilon_{\beta p}^*$ were measured.

Fig. 13 Verification for isotropy of test specimens (plates).
For the specimens of aluminium alloy, the tension test was carried out on each test piece which had been cut from four virgin plates in the direction of angles of $\alpha=0^\circ$, 30$^\circ$, 60$^\circ$ and 90$^\circ$ to the plane length direction. These tension stress-strain curves are indicated in Fig. 14, which shows good agreement with each other, but the strains in the directions of plate thickness and width are not recorded. It is, however, assumed here that these plate materials are initially isotropic for the tension in the plane of plate.

4.2.3 Experimental results

The second subsequent yielding in tension is defined by the offset stress $\sigma_{p(a)}$ and some discussions are given here for the two cases when the yield stress is taken as 0.1% offset stress $\sigma_{0.1(a)}$ and 0.2% offset stress $\sigma_{0.2(a)}$, respectively. These values obtained from experiments are indicated by the marks O in Figs. 17 to 20 which represent the relationship between $\sigma_{p(a)}$ and $\alpha$ in the polar co-ordinate. The values of the strain $\epsilon_{p(a)}$ in them are the logarithmic plastic prestrain given by the first tension in the direction of the angle of $\alpha=0^\circ$.

4.2.4 Approximation with the anisotropic yield condition based on the maximum shearing stress hypothesis

As $\sigma_{x}$, $\sigma_{y}$, and $\tau_{xy}$ may all be positive in these experiments, measuring values can be approximated by using the equations of the yield condition (11-a), (13-a) and (13-b).

i) We assume the yield surfaces on the $\pi$-plane to be as shown in Fig. 15, referring to Figs. 10 and 11, when the initially isotropic materials of a plate are subjected to the tension for the first time. These yield surfaces are for the specimens of brass, Fig. 15 (a) being the one when $\epsilon_{p(a)}$ is small and Fig. 15 (b) the one when $\epsilon_{p(a)}$ is large. For simplicity of the following treatment, the yield surface is assumed to be as shown in Fig. 15 (a) in spite of the kind of materials and the value of the strain $\epsilon_{p(a)}$. Then the principal shearing stresses of yielding are decided as follows from measuring of $\sigma_{p(\pi)}$ and $\sigma_{p(\pi/2)}$, referring to the table in Fig. 1.

$$
\tau_{10} = \tau_{12} = \sigma_{p(\pi)} / \sqrt{2}, \quad \tau_{20} = \tau_{22} = \sigma_{p(\pi/2)} / \sqrt{2} \quad \text{......(15)}
$$

ii) It is assumed that the $\tau_{12}$-axis in the $\pi$-stress space will remain in intersection with "the cylinderoid surface" after pretrained with the strain $\epsilon_{p(a)}$. The shape of "the cylinderoid surface" can be determined because $\tau_{20}$ and $\tau_{22}$ are equal to $\tau_{30}$ and $\tau_{33}$, respectively, when the values of $\tau_{30}$ and $\tau_{33}$ are observed (if the specimen is regarded as an isotropic material, the equality $\tau_{30} = \tau_{33}$ is valid after the plastic deformation under the $\tau_{12}$ direction tension). From the experiments of plates, however, $\tau_{30}$ can not be determined. So, the value of $\tau_{30}$ is presumed from the results of experiments which were carried out on the specimens of brass and copper by the authors(9), copper by Mair and Pugh(10) and nickel by Iago and Shishmarev(11). These are the results of the torsion test of thin-walled tubings which had been initially isotropic and were pretrained in tension.

Figure 16 shows the relationship between the plastic shearing stress in torsion $\tau_{p(a)}$ and the ratio of the shearing stress in torsion $\tau_{12}$ to the stress $\sigma_{p(a)}$ at the starting point of unloading in the initial tension, of which experiments were performed by
the authors, where the axes $x_1$, $x_2$ and $x_3$ refer to the axial, circumferential and radial directions, respectively. The values of $\tau_{50}$ obtained by Mair and Pugh through extrapolation, are plotted on the ordinate in Fig. 16. If such a method is applied to the results observed here, the value of $\tau_{50}/\tau_{\text{max}}$ is decided from the intersection of the dashed line and the ordinate. As Iagn and Shishmarev adopt 0.018% offset stress as the yield stress, their results are indicated at the point where the equivalent plastic strain

$$\varepsilon_P^p = (2/3)\tau_{\text{max}}^p = (2/3)\tau_{50}^p$$

is equal to 0.018%, which is calculated under the maximum plastic shearing strain.

From Fig. 16 the values of $\tau_{50}/\sigma_{91u}$ will be considered not to be greatly influenced by the kind of materials and the value of $\varepsilon_P^p$. So, in the experiment of plates, when the offset stress $\sigma_{91u}(a)$ is used as the yield stress the value of $\tau_{50}/\sigma_{91u}$ is taken to be equal to $0.54 \sim 0.60$ at the point of $\varepsilon_P^p = 0.1%$, and when the offset stress $\sigma_{91u}(a)$ is used it is $0.57 \sim 0.64$ at $\varepsilon_P^p = 0.2%$. There is little difference in those values when either the equivalent plastic strain $\varepsilon_P^p = (1/\sqrt{3})\tau_{50}^p$ derived from the second invariant of the plastic strain tensor or the maximum principal plastic strain in absolute value

$$\varepsilon_P^p_{\text{max}} = (1/2)\tau_{50}^p$$

used in the paragraph 4.1 will be adopted. As $\sigma_{91u}$ in the experiment of plates can be considered to be approximately equal to $\sigma_{91u,10}$ or $\sigma_{91u,20}$, the following equations are obtained and will be used hereafter.

$$\tau_{50} = (0.54 \sim 0.60)\sigma_{91u,10}$$

Fig. 16 Shearing stress—shearing strain diagram and shearing stress of yielding in torsion after plastic deformation in tension

**Table**

<table>
<thead>
<tr>
<th>Number of test piece</th>
<th>Material</th>
<th>$\varepsilon_P^p$</th>
<th>$\sigma_{91u}/\varepsilon_P$</th>
<th>Investigator</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Brass</td>
<td>1.70</td>
<td>1.23</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>2.50</td>
<td>1.32</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>4.06</td>
<td>1.50</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>Copper</td>
<td>0.75</td>
<td>1.32</td>
<td>K. Saito and H. Igaki</td>
</tr>
<tr>
<td>II</td>
<td></td>
<td>1.24</td>
<td>1.63</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td></td>
<td>2.10</td>
<td>1.86</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td>3.01</td>
<td>2.10</td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>Copper</td>
<td>1.3</td>
<td></td>
<td>Mair and Pugh</td>
</tr>
<tr>
<td>b</td>
<td></td>
<td>4.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td></td>
<td>8.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>Nickel</td>
<td>2.3</td>
<td></td>
<td>Iagn and Shishmarev</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>4.1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\varepsilon_P^p$: plastic prestrain with initial tension

$\sigma_{91u}$: stress at the starting point of unloading in initial tension

$\varepsilon_P^p$: initial yield stress in tension

Fig. 17 Subsequent yielding of the specimen of brass prestrained by $\varepsilon_P^p$.
\[ \tau_{30} = (0.57 \sim 0.64) \sigma_{\theta_{0} \theta_{0}} \]  \hspace{1cm} (16\text{-b})

These can be written as follows.

\[ \tau_{30} = \zeta \sigma_{\theta_{1} \theta_{1}} \]  \hspace{1cm} (17)

where \( \zeta \) is the constant in the parentheses of Eqs. (16\text{-a}) and (16\text{-b}).

iii) The values of \( \tau_{30}, \tau_{31}, \) \( \tau_{32} \) and \( \tau_{30} \) (= \( \tau_{31} \)) obtained above give the feature of “the cylindroid surface” passing through the two points \( \sigma_{\theta_{1} \theta_{1}} \) and \( \sigma_{\theta_{1} \theta_{1}} \). The relation between \( \sigma_{\theta_{1}} \) and \( \alpha \) obtained from substituting Eqs. (14), (15) and (17) for Eqs. (13\text{-a}) and (13\text{-b}) signifies the yield condition under the consideration that yielding is prescribed with “the cylindroid surface”, all over the range of \( \alpha \). Such relations required from the upper and lower limits of \( \zeta \) give the curves which are designated as “cylindroid surface” in Figs. 17 to 20. The experimental results are located within the curves, especially in the range of \( \alpha < 60^\circ \), excepting the observations of \( \alpha = 0^\circ \) and \( 90^\circ \). Therefore it is necessary to approximate the experimental yield curves with “the elliptical cone surface 1” at least in the range of \( \alpha < 60^\circ \).

iv) Substituting Eqs. (14) and (15) for Eq. (11\text{-a}) and assuming \( \sigma_{\theta_{1} \theta_{1}} \) to be prescribed with “the elliptical cone surface 1”, the following equation is obtained.

\[ m_{3} = \sigma_{\theta_{1} \theta_{1}} / (2 \sigma_{\theta_{1} \theta_{1}} - \sigma_{\theta_{1} \theta_{1}}) \]  \hspace{1cm} (18)

So the yield condition, when yielding is assumed to be prescribed only with “the elliptical cone surface 1”, is obtained. These curves are shown as “elliptical cone surface 1” in Figs. 17 to 20.

v) The solid lines illustrated in these figures are the yield loci when the observations for yielding of materials prestrained by \( \varepsilon_{p} \) can be assumed to be approximated with the anisotropic yield criterion based on the maximum shearing stress hypothesis.

4-2-5 Some considerations

i) “The elliptical cone surface 1” has a good approximation to the experimental yield locus in the range of \( \alpha = 0^\circ \) to \( \alpha = 45^\circ \) regardless of the kind of materials, the values of \( \varepsilon_{p} \) and \( \varepsilon_{h,1} \) or \( \varepsilon_{h,2} \) used.

ii) Excepting the case of \( \varepsilon_{h,1} \) for aluminium, “the elliptical cone surface 1” and “the cylindroid surface” intersect each other at a point between \( \alpha \)

---

* In Figs. 17 to 19 the cylindroid surface and the value of \( \zeta \) are designated only for the larger strain \( \varepsilon_{p} \), and this is the same as the smaller \( \varepsilon_{p} \). This applies correspondingly to the elliptical cone surface 1 described later.
NII-Electronic Library Service

Vol. 13, No. 61, 1970  Anisotropic Yield Criterion under the Maximum Shear Stress Theory  835

\[ = 60^\circ \text{ and } \alpha = 70^\circ. \] There is little influence of \( \zeta \) on the "cylindroid surface" in the range of this intersecting point to \( \alpha = 90^\circ. \) Increasing \( \epsilon_{\infty}, \) tends to give a more inaccurate approximation of the yield locus in the neighbourhood of the intersecting point of both curves.

iii) When \( \delta_{0.11(a)} \) is considered to be a definition of the yield stress for aluminium, "the elliptical cone surface I" has a better approximation to the experimental yield locus over the whole range of \( \alpha. \) This is the same for the case of \( \epsilon_{\infty}, \) when \( \delta_{0.11(a)} \) and \( \delta_{0.21(a)} \) are adopted for aluminium and copper, respectively.

iv) There is generally no fundamental difference in the degree of agreement between the experimental and theoretical anisotropic yield criteria when either \( \delta_{0.11(a)} \) or \( \delta_{0.21(a)} \) is applied as the yield stress.

v) The value of \( \tau_{\infty} \) at the intersecting point of the \( \tau_{\infty} \)-axis and "the elliptical cone surface I" (or its extension) is obtained from using the value of \( m_2 \) in Eq. (18). Then the ratio of \( \tau_{\infty} \) to \( \sigma_{\infty} \) in the experiment of plates becomes 0.69 (for brass, \( \epsilon_{\infty} = 18.9\% \), \( \sigma_{0.11(a)} \sim 1.10 \) (for copper, \( \epsilon_{\infty} = 7.7\% \), \( \sigma_{0.21(a)} \)). These values are larger than the observations in Fig. 16. Therefore this justifies the assumption, explained in the paragraph 4.2.4 ii), that "the cylindroid surface" will remain in interaction with the \( \tau_{\infty} \)-axis after prestrained by \( \epsilon_{\infty}. \)

vi) Hill's anisotropic yield criterion which is given by the following equation (19) was compared with the experimental results of the plate specimens of brass, copper and aluminium, previously published by authors\(^{(12)}\)

\[
F'(\sigma_{01} - \sigma_{02})^2 + G(\sigma_{02} - \sigma_{12})^2 + H(\sigma_{11} - \sigma_{22})^2 \\
+ 2L\tau_{\infty}^2 + 2M\tau_{\infty}^2 + 2N\tau_{\infty}^2 = 1
\]  

(19) Namely, the parameters of anisotropy being decided with the observation, the comparison was carried out with the same method as the one in Figs. 17 to 20. For example the one dotted-dashed line in Fig. 17 (a) shows the yield locus when \( \delta_{0.11(a)} \) is adopted as the yield stress for the material of brass subject to the prestrain \( \epsilon_{\infty} = 18.9\%. \) According to this figure Hill's theory has a good approximation for the experimental results in the range of \( \alpha = 45^\circ \) to \( \alpha = 90^\circ, \) but does not so in the range of \( \alpha = 0^\circ \) to \( \alpha = 45^\circ, \) on the contrary to the case of the anisotropic yield criterion developed in this paper.

5. Conclusions

The anisotropic yield criterion based on the maximum shear stress hypothesis is developed for the anisotropic material which is acted upon by the three normal stresses in the directions of principal axes of anisotropy and a shearing stress around one of them, the states being an example of the case when the principal axes of anisotropy are not coincident with those of stress.

It is first found for isotropic materials that the yield surface under the maximum shear stress hypothesis is composed of two elliptical cone surfaces and a cylindroid surface in the \( \pi, \tau \)-stress space where the \( \pi \)-axis is normal to the \( \tau \)-plane.

The anisotropic yield criterion is derived by assuming that the appropriate translation and distortion of these elliptical cone and cylindroid surfaces give the yield surface of anisotropic materials.

It is certified with the experiment that the anisotropic yield criterion under the maximum shear stress hypothesis when both principal axes are coincident with each other gives a good approximation to the yield surface of the anisotropic materials of brass which are prestrained in simple tension.

The anisotropic yield criterion developed in this

NII-Electronic Library Service
paper when both principal axes are not coaxial can well approximate the experimental results of yielding, except in the range of $\alpha=60^\circ$ to $\alpha=70^\circ$, when tension tests are carried out in the direction of $\alpha$ from the first tension direction for the plate materials pretrained in tension.

Comparing the same results of plates with Hill's anisotropic yield criterion, there are merits and demerits for the anisotropic yield criterion derived here and the Hill's one. As Bauschinger effect can be easily introduced into the anisotropic yield criterion developed here, this criterion has better advantages in precise expressing of the yielding of actual materials having anisotropy.

Acknowledgements

Authors wish to thank Dr. M. Hamada for his valuable suggestions and kind instruction throughout the work.

References