Fundamental Analysis of Lateral Vibration of a Leaf Spring*

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The purpose of this paper is to clarify the lateral vibrational characteristics of a leaf spring. The frequency equations are formulated under the assumption so-called "point pressure". For a number of leaf springs, the natural frequencies and their normal modes are calculated, and good agreement with experiments is obtained. Furthermore, for a multi-leaf spring, it is found that the analytical method based on the stepped beam hypothesis fails to predict some of the significant modes in the high-frequency region, while the presented method yields favourable numerical data.

1. Introduction

For the advantages of low cost and reliabilities on strength and performance, multi-leaf springs are usually applied to a rear suspension system for automobiles. But the leaf spring has a defect that it exhibits many resonant peaks versus driving torque and irregularity of roads, even in a frequency range of 200 Hz, and amplifies vibrational forces transmitted to the automobile body-shell. On the other hand, increased emphasis on quieter vehicles requires a more fundamental understanding of the vibrational characteristics of the leaf spring. However, very little work has been performed in this special field except for an attempt by S. Kobayashi⁽¹⁾. He estimates normal modes and their frequencies by assuming that the leaf spring is a stepped beam, Fig. 1 (b). But the influence of the shorter leaf appears not to be considered in detail.

In this paper the frequency equations for lateral vibration of the leaf spring are derived under the assumption of "point pressure" that the leaves touch each other only at the bearing points and the clamp. Furthermore comparison of the test data with the numerical data obtained through different methods based on the stepped beam and the point pressure theory, shows how to get reasonably an idealized model of the leaf spring covering the case of the single leaf spring.

2. Theory

Let us consider a multi-leaf spring system simply supported at the ends of the main leaf and connected by two massless springs of stiffness k_t and k_s , shown in Fig. 1 (a). The leaf spring may be

assumed to consist of uniform cross-sectional straight beams fixed by the clamp at the middle and hinged at the ends. The beams are numbered 1, 2, 3, ..., N , starting from the longest beam (i.e. main leaf).

First we consider the left part of the system and take the origin of the rectangular co-ordinates at the left end of the longest beam. Neglecting the effects of shearing force and rotatory inertia, the equation governing the lateral vibration of a uniform cross-sectional beam is given as follows⁽²⁾

$$EI \frac{\partial^4 y}{\partial x^4} = -\rho A_s \frac{\partial^2 y}{\partial t^2} \quad \dots\dots\dots (1)$$

where

x : co-ordinate along beam

y : co-ordinate normal to beam

t : time

E : modulus of elasticity

I : area moment of inertia of cross section

A_s : cross-sectional area

ρ : mass per unit volume

(a) Idealized model based on point pressure hypothesis

(b) Idealized model based on stepped beam hypothesis
Fig. 1

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For the i -th beam, the solution to Eq. (1) can be expressed in terms of a normal co-ordinate $\sin \omega t$ and normal functions X_i and Y_i in the form

$$y = X_i \sin \omega t, \quad l_{i-1} \leq x \leq l_i \quad (2)$$

$$y = Y_i \sin \omega t, \quad l_i \leq x \leq l_N \quad (3)$$

$$X_i = A_i \sin k_i x + B_i \cos k_i x + C_i \sinh k_i x + D_i \cosh k_i x \quad (4)$$

$$Y_i = \alpha_i \sin k_i x + \beta_i \cos k_i x + \gamma_i \sinh k_i x + \delta_i \cosh k_i x \quad (5)$$

where $k_i^4 = \rho \left(\frac{A_i}{EI_i} \right) \omega^2$, and ω = angular frequency

The coefficients $A_i, B_i, C_i, D_i, \alpha_i, \beta_i, \gamma_i, \delta_i$ ($i=1, 2, 3, \dots, N$) must be evaluated to satisfy the boundary conditions for each section. For the first beam, the boundary conditions at the left end and the shear, moment, slope, and deflection at the contact point with the second beam are given as

$$x=0, \quad Y_1=0 \quad (6)$$

$$\frac{d^2 X_1}{dx^2} = 0 \quad (7)$$

$$x=l_1, \quad X_1=Y_1 \quad (8)$$

$$\frac{dX_1}{dx} = \frac{dY_1}{dx} \quad (9)$$

$$\frac{d^2 X_1}{dx^2} = \frac{d^2 Y_1}{dx^2} \quad (10)$$

$$\frac{d^3 X_1}{dx^3} + \frac{F_1}{EI_1} = \frac{d^3 Y_1}{dx^3} \quad (11)$$

where F_1 : shearing force transmitted at the tip of leaf. Making use of Eqs. (6) and (7), we obtain

$$B_i = D_i = 0 \quad (12)$$

Then the coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i$ can be expressed as functions of A_i, C_i , and F_1 by substituting Eq. (12) into Eqs. (8), (9), (10), and (11).

$$\begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \delta_i \end{bmatrix} = \begin{bmatrix} 1, 0, -\sigma_i \cos z_i \\ 0, 0, \sigma_i \sin z_i \\ 0, 1, \sigma_i \cosh z_i \\ 0, 0, -\sigma_i \sinh z_i \end{bmatrix} \begin{bmatrix} A_i \\ C_i \\ F_1 \end{bmatrix} \quad (13)$$

where $z_i = k_i l_i$, $\sigma_i = 1/(2EI_i k_i^3)$

$$\begin{bmatrix} A_i \\ B_i \\ C_i \\ D_i \\ F_i \end{bmatrix} = \frac{1}{\nu_i} \begin{bmatrix} \sinh(u_i - z_i) - \cos u_i \sin z_i & -\cos u_i \cos z_i & \sinh u_i \cos z_i & \cosh u_i \cos z_i \\ \sin u_i \sin z_i & \sin u_i \cos z_i + \sinh(u_i - z_i) & -\sinh u_i \sin z_i & -\cosh u_i \sin z_i \\ \sin u_i \cosh z_i & \cos u_i \cosh z_i & -\cosh u_i \sinh z_i + \sin(u_i - z_i) & -\cosh u_i \cosh z_i \\ -\sin u_i \sinh z_i & -\cos u_i \sinh z_i & \sinh u_i \sinh z_i & \sinh u_i \cosh z_i + \sin(u_i - z_i) \\ -\mu_i \sin u_i & -\mu_i \cos u_i & \mu_i \sinh u_i & \mu_i \cosh u_i \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \delta_i \end{bmatrix} \quad (23)$$

where $z_i = k_i l_i$, $u_i = k_i l_{i-1}$, $\nu_i = \sin(u_i - z_i) + \sinh(u_i - z_i)$, $\mu_i = 2EI_i k_i^3$.

If the coefficients $\alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}, \delta_{i-1}$ for the $(i-1)$ th beam can be expressed as linear homogeneous first-order equations in A_i, C_i , and F_i , F_{i-1} can be also written as similar linear function of the three variables A_i, C_i, F_i from Eq. (23).

Let

$$\frac{\nu_i}{2} Y_{i-1} = p_1 A_i + q_1 C_i + r_1 F_i \quad (24)$$

$$-\frac{F_{i-1} \nu_i}{EI_i k_i^3} = p_2 A_i + q_2 C_i + r_2 F_i \quad (25)$$

Fig. 2

Next we consider the case in which there is one leaf or more (this may be called "intermediate beam") between the first beam and the N th one, Fig. 2. The boundary conditions for the left end of the intermediate beam are given as

$$x=l_{i-1}, \quad X_i=Y_{i-1} \quad (14)$$

$$\frac{d^2 X_i}{dx^2} = 0 \quad (15)$$

$$\frac{d^3 X_i}{dx^3} = -\frac{F_{i-1}}{EI_i} \quad (16)$$

The shear, moment, slope, and deflection of the intermediate beam at the contact point with $(i+1)$ th beam are given as follows

$$x=l_i, \quad X_i=Y_i \quad (17)$$

$$\frac{dX_i}{dx} = \frac{dY_i}{dx} \quad (18)$$

$$\frac{d^2 X_i}{dx^2} = \frac{d^2 Y_i}{dx^2} \quad (19)$$

$$\frac{d^3 X_i}{dx^3} + \frac{F_i}{EI_i} = \frac{d^3 Y_i}{dx^3} \quad (20)$$

The boundary conditions at the junction with the clamp are written as

$$x=l_N, \quad Y_i=Y_1 \quad (21)$$

$$\frac{dY_i}{dx} = \frac{dY_1}{dx} \quad (22)$$

Equations (15), (17) through (20) give the relation between the coefficients A_i, B_i, C_i, D_i, F_i , and the coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i$: i.e.,

$$\begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \delta_i \end{bmatrix} = \begin{bmatrix} \sinh u_i \cos z_i & \cosh u_i \cos z_i \\ -\sinh u_i \sin z_i & -\cosh u_i \sin z_i \\ -\cosh u_i \sinh z_i + \sin(u_i - z_i) & -\cosh u_i \cosh z_i \\ \sinh u_i \sinh z_i & \sinh u_i \cosh z_i + \sin(u_i - z_i) \\ \mu_i \sinh u_i & \mu_i \cosh u_i \end{bmatrix} \begin{bmatrix} A_i \\ C_i \\ F_i \end{bmatrix}$$

.....(23)

Then by making use of Eqs. (21), (22) and substituting Eqs. (23), (24), (25) into Eqs. (14) and (16), the coefficients for the i -th beam can be represented as linear functions of A_i , C_i , and F_i .

$$\begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \delta_i \end{bmatrix} = G_i \begin{bmatrix} A_i \\ C_i \\ F_i \end{bmatrix} \quad \dots\dots\dots (26)$$

where G_i is a matrix as follows:

$$G_i = \begin{bmatrix} \epsilon_{i1} & \eta_{i1} & \zeta_{i1} \\ \epsilon_{i2} & \eta_{i2} & \zeta_{i2} \\ \epsilon_{i3} & \eta_{i3} & \zeta_{i3} \\ \epsilon_{i4} & \eta_{i4} & \zeta_{i4} \end{bmatrix} \quad \dots\dots\dots (27)$$

$$= \begin{bmatrix} \sin u_i \sinh(u_i - z_i) & \cos u_i \sinh(u_i - z_i) & \sinh u_i \sin(u_i - z_i) & \cosh u_i \sin(u_i - z_i) \\ a_1 & a_2 & a_3 & a_4 \\ \sin v_i & \cos v_i & \sinh v_i & \cosh v_i \\ \cos v_i & -\sin v_i & \cosh v_i & \sinh v_i \end{bmatrix}^{-1} \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \\ p_4 & q_4 & r_4 \end{bmatrix}$$

in which

$$\begin{aligned} v_i &= k_i l_N \\ a_1 &= \sin z_i + \sin u_i \cosh(u_i - z_i) - \cos u_i \sinh(u_i - z_i) \\ a_2 &= \cos z_i + \sin u_i \sinh(u_i - z_i) + \cos u_i \cosh(u_i - z_i) \\ a_3 &= -\sinh z_i + \cosh u_i \sin(u_i - z_i) - \sinh u_i \cos(u_i - z_i) \\ a_4 &= -\cosh z_i + \sinh u_i \sin(u_i - z_i) - \cosh u_i \cos(u_i - z_i) \\ p_3 &= \sin v_i, \quad q_3 = \sinh v_i, \quad r_3 = \sigma_1 \{ \sin(z_i - v_i) + \sinh(v_i - z_i) \} \\ p_4 &= (k_i/k_i) \cos v_i, \quad q_4 = (k_i/k_i) \cosh v_i, \quad r_4 = (k_i/k_i) \sigma_1 \{ \cosh(v_i - z_i) - \cos(v_i - z_i) \} \end{aligned}$$

The boundary condition for the shortest beam are given as

$$x = l_{N-1}, \quad Y_N = Y_{N-1} \quad \dots\dots\dots (28) \quad \frac{d^2 Y_N}{dx^2} = 0 \quad \dots\dots\dots (29)$$

$$\frac{d_3 Y_N}{dx^3} = -\frac{F_{N-1}}{EI_N} \quad \dots\dots\dots (30)$$

$$x = l_N, \quad Y_N = Y_1 \quad \dots\dots\dots (31) \quad \frac{dY_N}{dx} = \frac{dY_1}{dx} \quad \dots\dots\dots (32)$$

When the coefficients α_{N-1} , β_{N-1} , γ_{N-1} , δ_{N-1} , and F_{N-1} for the $(N-1)$ th beam are linear homogeneous first-order equations in A_1 , C_1 , F_1 , we can assume

$$Y_{N-1} = e_1 A_1 + f_1 C_1 + g_1 F_1 \quad \dots\dots\dots (33)$$

$$-\frac{F_{N-1}}{EI_N k_N^3} = e_2 A_1 + f_2 C_1 + g_2 F_1 \quad \dots\dots\dots (34)$$

Then the coefficients α_N , β_N , γ_N , δ_N can also be written as linear functions of A_1 , C_1 , F_1 by making use of Eqs. (28), (30), (31), and (32)

$$\begin{bmatrix} \alpha_N \\ \beta_N \\ \gamma_N \\ \delta_N \end{bmatrix} = G_N \begin{bmatrix} A_1 \\ C_1 \\ F_1 \end{bmatrix} \quad \dots\dots\dots (35)$$

where G_N is a matrix as follows

$$G_N = \begin{bmatrix} \epsilon_{N1} & \eta_{N1} & \zeta_{N1} \\ \epsilon_{N2} & \eta_{N2} & \zeta_{N2} \\ \epsilon_{N3} & \eta_{N3} & \zeta_{N3} \\ \epsilon_{N4} & \eta_{N4} & \zeta_{N4} \end{bmatrix} \quad \dots\dots\dots (36)$$

$$= \begin{bmatrix} 2 \sin u_N & 2 \cos u_N & 0 & 0 \\ -\cos u_N & \sin u_N & \cosh u_N & \sinh u_N \\ \sin z_N & \cos z_N & \sinh z_N & \cosh z_N \\ \cos z_N & -\sin z_N & \cosh z_N & \sinh z_N \end{bmatrix}^{-1} \begin{bmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \\ \sin v_1 & \sinh v_1 & g_3 \\ \frac{k_1}{k_N} \cos v_1 & \frac{k_1}{k_N} \cosh v_1 & g_4 \end{bmatrix}$$

and $g_3 = \sigma_1 \{ -\sin(v_1 - z_1) + \sinh(v_1 - z_1) \} = r_3$
 $g_4 = \sigma_1 \{ -\cos(v_1 - z_1) + \cosh(v_1 - z_1) \} \left(\frac{k_1}{k_N} \right) = r_4$

The shear, moment, slope, and deflection on the right side of the clamp are related to these quantities on

the left side by the following equations.

$$Y_1|_{x=l_N} + l_0 \frac{dY_1}{dx} \Big|_{x=l_N} - Y_1'|_{x=l_N'} = 0 \quad (37)$$

$$\frac{dY_1}{dx} \Big|_{x=l_N} + \frac{dY_1'}{dx'} \Big|_{x'=l_N'} = 0 \quad (38)$$

$$R_1 Y_1|_{x=l_N} + R_2 \frac{dY_1}{dx} \Big|_{x=l_N} + \sum_{j=1}^N EI_j \left(\frac{d^3 Y_j}{dx^3} \Big|_{x=l_N} + \frac{d^3 Y_j'}{dx'^3} \Big|_{x'=l_N'} \right) = 0 \quad (39)$$

$$-R_3 Y_1|_{x=l_N} + R_4 \frac{dY_1}{dx} \Big|_{x=l_N} - \sum_{j=1}^N EI_j \left(\frac{d^2 Y_j}{dx^2} \Big|_{x=l_N} - \frac{d^2 Y_j'}{dx'^2} \Big|_{x'=l_N'} + \frac{l_0}{2} \left(\frac{d^3 Y_j}{dx^3} \Big|_{x=l_N} - \frac{d^3 Y_j'}{dx'^3} \Big|_{x'=l_N'} \right) \right) = 0 \quad (40)$$

where $R_1 = M_0 \omega^2 - (k_s + k_t)$, $R_2 = R_1 l_0 / 2 - R_3$, $R_3 = k_s l_s$, $R_4 = I_0 \omega^2 - R_3 (l_s + l_0 / 2)$

(') : quantities with respect to the right part of leaf spring

M_0 : additional mass attached to the clamp of leaf spring

I_0 : additional moment of inertia attached to the clamp of leaf spring

k_s, k_t : spring constant of massless spring

Up to this, it is shown that the coefficients $A_i, B_i, C_i, D_i, F_i, \alpha_i, \beta_i, \gamma_i$, and δ_i for each beam can be represented as linear functions of A_1, C_1, F_1 under the assumptions of Eqs. (24) and (25) or Eqs. (33) and (34). In the following sections, we verify the assumptions and obtain a set of linear homogenous first-order equations in terms of $A_1, C_1, F_1, A_1', C_1', F_1'$ from Eqs. (29), (37) through (40). Furthermore frequency equations are obtained covering the case of $N=1$.

Case of $N=1$. This case corresponds to a single-leaf spring. Equation (13) with $F_1=0$ may be substituted into Eqs. (37) through (40), and we have

$$L_1(\omega) \begin{bmatrix} A_1 \\ C_1 \\ A_1' \end{bmatrix} = 0 \quad (41)$$

where $L_1(\omega)$ is a matrix as follows

$$L_1(\omega) = \begin{bmatrix} \sin v_1 + k_1 l_0 \cos v_1, & \sinh v_1 + k_1 l_0 \cosh v_1, & -\sin v_1', & -\sinh v_1' \\ \cos v_1, & \cosh v_1, & \cos v_1', & \cosh v_1' \\ \tau_1 \sin v_1, & \tau_1 \sinh v_1, & -\cos v_1', & \cosh v_1' \\ +(\tau_2 - 1) \cos v_1, & +(\tau_2 + 1) \cosh v_1, & & \\ (\tau_3 + \tau_5) \sin v_1, & (\tau_3 - \tau_5) \sinh v_1, & -\tau_5 \sin v_1' - \cosh v_1', & \tau_5 \sinh v_1' + \cosh v_1' \\ +(\tau_4 + 1) \cos v_1, & +(\tau_4 - 1) \cosh v_1, & & \end{bmatrix} \quad (42)$$

and

$$\tau_1 = \{M_0 \omega^2 - (k_s + k_t)\} / (EI_1 k_1^3), \quad \tau_2 = \{M_0 \omega^2 - (k_s + k_t)\} l_0 / 2 - k_t l_s / (EI_1 k_1^2)$$

$$\tau_3 = -2k_s l_s / (EI_1 k_1^3 l_0), \quad \tau_4 = 2\{I_0 \omega^2 - k_s l_s (l_s + l_0 / 2)\} / (EI_1 k_1^3 l_0), \quad \tau_5 = 2 / (k_1 l_0)$$

A nontrivial solution of these equations is possible if and only if

$$|L_1(\omega)| = 0 \quad (43)$$

Equation (43) is the frequency equation for $N=1$. For the particular case of a symmetrical single-leaf spring, Eq. (43) with $l_1=l_1', l_s=l_0=0$, and $k_s=k_t=0$, reduces to

$$\left\{ \frac{M_0 \omega^2}{EI_1 k_1^3} (\sin v_1 \cosh v_1 - \cos v_1 \sinh v_1) - 4 \cos v_1 \cosh v_1 \right\} \\ \times \left\{ \frac{I_0 \omega^2}{EI_1 k_1} (\sin v_1 \cosh v_1 - \cos v_1 \sinh v_1) - 4 \sin v_1 \sinh v_1 \right\} = 0 \quad (44)$$

Therefore the vertical motion of the additional inertia is independent of the rotatory motion of it.

Case of $N=2$. In this case there is no intermediate beam. Substituting Eq. (13) into Eqs. (28) and (32) yields

$$x=l_1, \quad Y_{N-1}=Y_1=A_1 \sin z_1 + C_1 \sinh z_1 \quad (45)$$

$$-\frac{F_{N-1}}{EI_N k_N^3} = -\frac{F_1}{EI_2 k_2^3} \quad (46)$$

Therefore we can determine the coefficients in Eqs. (33) and (34) as follows:

$$e_1 = \sin z_1, \quad f_1 = \sinh z_1, \quad g_1 = e_2 = f_2 = 0$$

$$g_2 = -1 / (EI_2 k_2^3)$$

Substituting Eqs. (13) and (35) into Eqs. (29) and (37) through (40) leads to the same frequency equation as Eqs. (47) and (48).

Case of $N \geq 3$. Since Eq. (45) is valid at $x=l_1$ and

$$-\frac{\nu_2 F_1}{EI_2 k_2^3} = \left(-\frac{\nu_2}{EI_2 k_2^3} \right) F_1$$

the coefficients in Eqs. (24) and (25) for the second beam can be determined as

$$p_1 = (\nu_2/2) \sin z_1, \quad q_1 = (\nu_2/2) \sinh z_1, \quad r_1 = p_2 = q_2 = 0, \quad r_2 = -\nu_2/(EI_2 k_2^3)$$

Thus the coefficients $A_2, B_2, C_2, D_2, F_2, \alpha_2, \beta_2, \gamma_2, \delta_2$ can be written as linear functions of A_1, C_1, F_1 from Eqs. (23) and (26). Repeating this way, the coefficients in Eqs. (24) and (25) or Eqs. (33) and (34) for each beam can be expressed as linear homogeneous first-order equations in A_1, C_1, F_1 . Therefore all the terms in Eqs. (39) and (40) can be written as linear functions of $A_1, C_1, F_1, A_1', C_1', F_1'$. Substituting these relations into Eqs. (29) and (37) through (40) yields

$$L_N(\omega) \begin{bmatrix} A_1 \\ C_1 \\ F_1 \\ A_1' \\ C_1' \\ F_1' \end{bmatrix} = 0 \dots \dots \dots (47)$$

where

$$L_N(\omega) = \begin{bmatrix} \kappa_1 & \lambda_1 & \theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2 & \chi_2 & \psi_2 \\ \cos v_1 & \cosh v_1 & \theta_3 & \cos v_1' & \cosh v_1' & \psi_3 \\ \sin v_1 + k_1 l_0 \cos v_1 & \sinh v_1 + k_1 l_0 \cosh v_1 & \theta_4 & -\sin v_1' & -\sinh v_1' & \psi_4 \\ \kappa_5 & \lambda_5 & \theta_5 & \phi_5 & \chi_5 & \psi_5 \\ \kappa_6 & \lambda_6 & \theta_6 & \phi_6 & \chi_6 & \psi_6 \end{bmatrix} \dots \dots \dots (48)$$

$$\kappa_1 = -\varepsilon_{N1} \sin v_N - \varepsilon_{N2} \cos v_N + \varepsilon_{N3} \sinh v_N + \varepsilon_{N4} \cosh v_N$$

$$\lambda_1 = -\eta_{N1} \sin v_N - \eta_{N2} \cos v_N + \eta_{N3} \sinh v_N + \eta_{N4} \cosh v_N$$

$$\theta_1 = -\zeta_{N1} \sin v_N - \zeta_{N2} \cos v_N + \zeta_{N3} \sinh v_N + \zeta_{N4} \cosh v_N$$

$$\phi_2 = -\varepsilon_{N1}' \sin v_N' - \varepsilon_{N2}' \cos v_N' + \varepsilon_{N3}' \sinh v_N' + \varepsilon_{N4}' \cosh v_N'$$

$$\chi_2 = -\eta_{N1}' \sin v_N' - \eta_{N2}' \cos v_N' + \eta_{N3}' \sinh v_N' + \eta_{N4}' \cosh v_N'$$

$$\psi_2 = -\zeta_{N1}' \sin v_N' - \zeta_{N2}' \cos v_N' + \zeta_{N3}' \sinh v_N' + \zeta_{N4}' \cosh v_N'$$

$$\theta_3 = \sigma_1 \{-\cos(z_1 - v_1) + \cosh(z_1 - v_1)\},$$

$$\phi_3 = \sigma_1 \{-\cos(z_1' - v_1') + \cosh(z_1' - v_1')\}$$

$$\theta_4 = \sigma_1 [\sin(z_1 - v_1) - \sinh(z_1 - v_1) - k_1 l_0 \{\cos(z_1 - v_1) - \cosh(z_1 - v_1)\}]$$

$$\phi_4 = \sigma_1 [-\sin(z_1' - v_1') + \sinh(z_1' - v_1')]$$

$$\kappa_5 = \sum_{j=1}^N EI_j k_j^3 (-\varepsilon_{j1} \cos v_j + \varepsilon_{j2} \sin v_j + \varepsilon_{j3} \cosh v_j + \varepsilon_{j4} \sinh v_j) + R_1 \sin v_1 + R_2 k_1 \cos v_1$$

$$\lambda_5 = \sum_{j=1}^N EI_j k_j^3 (-\eta_{j1} \cos v_j + \eta_{j2} \sin v_j + \eta_{j3} \cosh v_j + \eta_{j4} \sinh v_j) + R_1 \sinh v_1 + R_2 k_1 \cosh v_1$$

$$\theta_5 = \sum_{j=1}^N EI_j k_j^3 (-\zeta_{j1} \cos v_j + \zeta_{j2} \sin v_j + \zeta_{j3} \cosh v_j + \zeta_{j4} \sinh v_j) \\ + R_1 \sigma_1 \{\sin(z_1 - v_1) - \sinh(z_1 - v_1)\} - R_2 k_1 \sigma_1 \{\cos(z_1 - v_1) - \cosh(z_1 - v_1)\}$$

$$\phi_5 = \sum_{j=1}^N EI_j k_j^3 (-\varepsilon_{j1}' \cos v_j' + \varepsilon_{j2}' \sin v_j' + \varepsilon_{j3}' \cosh v_j' + \varepsilon_{j4}' \sinh v_j')$$

$$\chi_5 = \sum_{j=1}^N EI_j k_j^3 (-\eta_{j1}' \cos v_j' + \eta_{j2}' \sin v_j' + \eta_{j3}' \cosh v_j' + \eta_{j4}' \sinh v_j')$$

$$\psi_5 = \sum_{j=1}^N EI_j k_j^3 (-\zeta_{j1}' \cos v_j' + \zeta_{j2}' \sin v_j' + \zeta_{j3}' \cosh v_j' + \zeta_{j4}' \sinh v_j')$$

$$\kappa_6 = \sum_{j=1}^N EI_j k_j^2 \left\{ \left(\varepsilon_{j1} - \frac{k_j l_0}{2} \varepsilon_{j2} \right) \sin v_j + \left(\varepsilon_{j2} + \frac{k_j l_0}{2} \varepsilon_{j1} \right) \cos v_j - \left(\varepsilon_{j3} + \frac{k_j l_0}{2} \varepsilon_{j4} \right) \sinh v_j \right. \\ \left. - \left(\varepsilon_{j4} + \frac{k_j l_0}{2} \varepsilon_{j3} \right) \cosh v_j \right\} + R_4 k_1 \cos v_1 - R_3 \sin v_1$$

$$\lambda_6 = \sum_{j=1}^N EI_j k_j^2 \left\{ \left(\eta_{j1} - \frac{k_j l_0}{2} \eta_{j2} \right) \sin v_j + \left(\eta_{j2} + \frac{k_j l_0}{2} \eta_{j1} \right) \cos v_j - \left(\eta_{j3} + \frac{k_j l_0}{2} \eta_{j4} \right) \sinh v_j \right. \\ \left. - \left(\eta_{j4} + \frac{k_j l_0}{2} \eta_{j3} \right) \cosh v_j \right\} + R_4 k_1 \cosh v_1 - R_3 \sinh v_1$$

$$\theta_6 = \sum_{j=1}^N EI_j k_j^2 \left\{ \left(\zeta_{j1} - \frac{k_j l_0}{2} \zeta_{j2} \right) \sin v_j + \left(\zeta_{j2} + \frac{k_j l_0}{2} \zeta_{j1} \right) \cos v_j - \left(\zeta_{j3} + \frac{k_j l_0}{2} \zeta_{j4} \right) \sinh v_j \right. \\ \left. - \left(\zeta_{j4} + \frac{k_j l_0}{2} \zeta_{j3} \right) \cosh v_j \right\} - R_4 k_1 \sigma_1 \{\cos(z_1 - v_1) - \cosh(z_1 - v_1)\} - R_3 \sigma_1 \{\sin(z_1 - v_1) - \sinh(z_1 - v_1)\}$$

$$\begin{aligned}\phi_6 &= \sum_{j=1}^N EI_j k_j^2 \left\{ -\left(\varepsilon_{j1}' - \frac{k_j l_0}{2} \varepsilon_{j2}' \right) \sin v_j' - \left(\varepsilon_{j2}' + \frac{k_j l_0}{2} \varepsilon_{j1}' \right) \cos v_j' + \left(\varepsilon_{j3}' + \frac{k_j l_0}{2} \varepsilon_{j4}' \right) \sinh v_j' \right. \\ &\quad \left. + \left(\varepsilon_{j4}' + \frac{k_j l_0}{2} \varepsilon_{j3}' \right) \cosh v_j' \right\} \\ \chi_6 &= \sum_{j=1}^N EI_j k_j^2 \left\{ -\left(\eta_{j1}' - \frac{k_j l_0}{2} \eta_{j2}' \right) \sin v_j' - \left(\eta_{j2}' + \frac{k_j l_0}{2} \eta_{j1}' \right) \cos v_j' + \left(\eta_{j3}' + \frac{k_j l_0}{2} \eta_{j4}' \right) \sinh v_j' \right. \\ &\quad \left. + \left(\eta_{j4}' + \frac{k_j l_0}{2} \eta_{j3}' \right) \cosh v_j' \right\} \\ \psi_6 &= \sum_{j=1}^N EI_j k_j^2 \left\{ -\left(\zeta_{j1}' - \frac{k_j l_0}{2} \zeta_{j2}' \right) \sin v_j' - \left(\zeta_{j2}' + \frac{k_j l_0}{2} \zeta_{j1}' \right) \cos v_j' + \left(\zeta_{j3}' + \frac{k_j l_0}{2} \zeta_{j4}' \right) \sinh v_j' \right. \\ &\quad \left. + \left(\zeta_{j4}' + \frac{k_j l_0}{2} \zeta_{j3}' \right) \cosh v_j' \right\}\end{aligned}$$

The condition for the existence of a nontrivial solution requires that the determinant of the coefficients of the unknowns of the system be zero.

$$|L_N(\omega)| = 0 \quad (49)$$

This determinant is the frequency equation of the system. When the angular frequency ω is evaluated to satisfy Eq. (49), the ratio $A_1 : C_1 : F_1 : A_1' : C_1' : F_1'$ can be calculated from Eq. (47) and the mode of its frequency is estimated by utilizing Eqs. (4), (5), (13), (23), (26), and (35).

Now we consider the particular case of a symmetrical multi-leaf spring illustrated in Fig. 3. Then we have

$$l_j = l_j', \quad k_j = 0 \quad (50)$$

$$\therefore u_j = u_j', \quad v_j = v_j', \quad z_j = z_j' \quad (51)$$

$$\therefore \varepsilon_{ji} = \varepsilon_{ji}', \quad \eta_{ji} = \eta_{ji}', \quad \zeta_{ji} = \zeta_{ji}' \quad (i=1, 2, 3, 4) \quad (52)$$

and

$$\left. \begin{aligned}\kappa_1 &= \phi_2, & \lambda_1 &= \chi_2, & \theta_1 &= \psi_2, & \theta_3 &= \psi_3, & \theta_4 &= -\psi_4 + k_1 l_0 g_4 \\ \kappa_5 &= \phi_5 + R_1 \sin v_1 + R_2 k_1 \cos v_1 \\ \lambda_5 &= \chi_5 + R_1 \sinh v_1 + R_2 k_1 \cosh v_1 \\ \theta_5 &= \psi_5 + (R_1 g_3 + R_2 k_1 g_4) \\ \kappa_6 &= -\phi_6 + R_4 k_1 \cos v_1 \\ \lambda_6 &= -\chi_6 + R_4 k_1 \cosh v_1 \\ \theta_6 &= -\psi_6 + R_4 k_1 g_4\end{aligned} \right\} \quad (53)$$

Substituting Eq. (53) into Eq. (49), we obtain

$$\begin{vmatrix} \phi_2, & \chi_2, & \psi_2 \\ \sin v_1 + \frac{k_1 l_0}{2} \cos v_1, & \sinh v_1 + \frac{k_1 l_0}{2} \cosh v_1, & g_3 + \frac{k_1 l_0}{2} g_4 \\ -2\phi_6 + R_4 k_1 \cos v_1, & -2\chi_6 + R_4 k_1 \cosh v_1, & -2\psi_6 + R_4 k_1 g_4 \end{vmatrix} = 0 \quad (54)$$

or

$$\begin{vmatrix} \phi_2, & \chi_2, & \psi_2 \\ \cos v_1, & \cosh v_1, & g_4 \\ \phi_5 + \frac{R_1}{2} \sin v_1, & \chi_5 + \frac{R_1}{2} \sinh v_1, & \psi_5 + \frac{R_1}{2} g_3 \end{vmatrix} = 0 \quad (55)$$

where

$$R_1 = M_0 \omega^2 - k_1, \quad R_2 = R_1 l_0 / 2, \quad R_4 = I_0 \omega^2$$

$$g_3 = \sigma_1 \{-\sin(v_1 - z_1) + \sinh(v_1 - z_1)\}$$

$$g_4 = \sigma_1 \{-\cos(v_1 - z_1) + \cosh(v_1 - z_1)\}$$

Therefore the vertical motion of the additional inertia becomes independent of the rotatory motion of it.

3. Numerical results

For the system of the leaf spring and additional inertias with properties shown in Table 1, natural frequencies and modes are calculated from the theoretical equations previously obtained. For the purpose of demonstration, the results for the samples E, F, G are illustrated in Fig. 4. We take $k_s = 1.37$

kg/mm, $l_s = 300$ mm in order to compare with experimental results. It is interesting to note that the first, second, third, fourth, and sixth natural frequencies of F and G agree fairly well with the first, second, third, fourth, and fifth natural frequencies of E respectively within a difference of 7 per

Fig. 3

cent. It is also seen that the flexural lateral vibration of the shorter leaf is predominant at 166 Hz. However the range of frequencies considered is below 250 Hz.

4. Experimental results and conclusions

With the experimental apparatus as illustrated in Fig. 6, a single-leaf spring was tested to determine

Table 1

Sample	h mm	b mm	l_0 mm	M_0 kg mm ⁻¹ sec ²	I_0 kg mm sec ²	Number of leaves	l_1 l_1'	l_2 l_2'	l_3 l_3'	l_4 l_4'
A	6	60	70	0.007 1	14.1	1	565 615			
B	6	60	70	0.013 3	14.1	2	220 240	565 615		
C	6	60	70	0.025 5	14.1	3	220 240	320 355	565 615	
D	6	60	70	0.025 5	14.1	4	220 240	320 355	415 465	565 615
E	7	60	70	0.007 1	14.1	1	565 615			
F	7	60	70	0.019 4	14.1	2	230 210	565 615		
G	7	60	70	0.025 5	14.1	3	230 210	425 420	565 615	

(a) Sample E

(b) Sample F

(c) Sample G

Fig. 4

Fig. 5

Fig. 6

the effect of the flexibility of the base supports on the natural frequency of the system. The pertinent dimensions and physical constants are:

$$l_1 = 565 \text{ mm}, l_1' = 615 \text{ mm}, l_0 = 70 \text{ mm}$$

$$b = 60 \text{ mm}, h = 7 \text{ mm}, l_s = 300 \text{ mm}$$

$$k_s = 1.37 \text{ kg/mm}, M_0 = 0.00714 \text{ kgsec}^2/\text{mm}$$

$$I_0 = 14.1 \text{ kgsec}^2\text{mm}$$

The reason for exciting the end of the lever extended through the clamp and not the center of the clamp in Fig. 6 is to distinctly detect the even mode shapes in the case of a symmetrical leaf spring.

The natural frequencies of the system obtained by varying the flexibility of the base supports A and B are compared with theoretical ones in Fig. 5. The agreement between theory and experiment is very good. With reference to the numerical results, it is found that the difference between the frequency values for vertically rigid supports and those for vertical stiffness equal to 100 kg/mm is less than 2.5 percent. Therefore it is appropriate to make the apparatus so that the vertical stiffnesses of the base supports A and B are respectively more than 100 kg/mm. All the following experimental values are measured with the zero length: i.e., $l_A = 0$, and $l_B = 0$, keeping the amplitude of the sinusoidal exciting displacement of an electro-magnetic exciter equal to 0.5 mm and recording the frequency response of displacement with barium-titanate transducers fixed on the main leaf and the exciter-head.

Next in order to determine the constraint conditions for both ends of the main leaf, resonant frequencies of the system with same properties except $h=6$ were recorded for the various cases where the rotatory displacements of both ends were free and the horizontal displacements were (a) fixed-fixed, (b) free-fixed, and (c) free-free respectively. Figure 7 shows the experimental results and the theoretical results on the basis of simple support. From Fig. 7 it is seen that comparison of theory and experiment is possible when the resonant frequencies of the system are measured with simple

base-supports which are horizontally free. It is considered that these results can be applied to the multi-leaf spring, since the highest resonant frequency in the frequency range of 250 Hz is that of the fifth mode of the main leaf and the effects of the shorter leaf on vibration of the main leaf are very small.

Here the fundamental conditions for the experiment are determined. The natural frequencies measured for various leaf springs with properties in Table 1 are shown in Table 2 for the sake of comparison of numerical results. In Table 2, Hypothesis I means the analytical results based on the "point pressure" hypothesis, while Hypothesis II represents the numerical results obtained from the stepped beam hypothesis.

Table 2 shows that the natural frequencies calculated from both hypotheses agree closely with the experimental values up to the fourth mode. However for the natural frequencies of the fifth and sixth modes, the percentage error between experimental and theoretical values based on Hypothesis II grows about 60 per cent, while the agreement between Hypothesis I and experiment is very good. The interpretation for this result is obtained from comparison of the vibrational modes in Figs. 4 and 8. In these figures, the first, second, third, and fourth mode shapes and their frequencies of the multi-leaf springs are similar in appearance to those

Fig. 7

Table 2 Percent-error of theoretical natural frequencies for experimental ones

Sample	2nd mode			3rd mode			4th mode			5th mode			6th mode		
	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C
A	Hz 23	% 0	%	Hz 60	% 2.5	%	Hz 70	% 5.7	%	Hz 195	% -1.3	%	Hz	%	%
B	31	-19.0	-19.0	58	0	5.5	73	1.4	5.5	165	0.6	17.5	190	2.5	9.9
C	33	-24.0	-24.0	62	-3.3	1.6	87	-8.9	-2.3	162	-0.6	21.0	175	2.9	31.4
D	38	-31.6	-31.6	72	-14.0	-5.6	104	-19.0	-10.1	145	7.5	58.5	165	11.5	73<
E	27	0		69	4.4		88	0		204	9.8				
F	36	-20.0	-20.0	65	1.5	3.1	85	8.3	3.5	165	0.6	35.0	196	15.3	52<
G	40	-33.0	-33.0	73	-5.5	2.8	99	-5.1	3.1	165	0.6	41.0	208	8.1	52<

A: measured value B: based on Hypothesis I C: based on Hypothesis II

Fig. 8 Numerical solutions from Hypothesis 11 for sample G

of the single-leaf spring. Therefore it may be considered that in case of a multi-leaf spring the main leaf vibrates predominantly in the neighbourhood of its natural frequency and forces the shorter leaves to have similar mode shape. On the other hand, both Hypotheses I and II treat the motion of the main leaf. It is the reason for small difference between theory and experiment. However, the fifth normal mode shape in Fig. 4 shows that the flexural vibration of the shorter leaves becomes very large and differs from that of the single-leaf spring. It may also be seen that the fifth normal mode and natural frequency in Fig. 8 correspond to those of the single-leaf spring or the sixth mode and frequency of the multi-leaf spring in Fig. 4. We are therefore justified in concluding that Hypothesis II fails to evaluate the resonant frequencies of the shorter leaves because of neglecting the flexural lateral vibration of the shorter leaves and that Hypothesis I can be more favourably applied to vibrational analysis of a multi-leaf spring.

5. Summary

The important results can be summarized briefly as follows:

(1) On the basis of the "point pressure" hypothesis stated, the frequency equations are obtained to predict accurately the frequency of flexural lateral vibration of a multi-leaf spring involving a single-leaf spring.

(2) For leaf springs with characteristics as shown in Table 1, the natural frequencies and normal modes are calculated from two theories; i.e., one is presented in this paper and the other is based on the stepped beam hypothesis. The results are compared with the experimental values obtained with the apparatus as illustrated in Fig. 6.

(3) For the vibrational analysis of a multi-leaf spring, it is reasonable to deal with such a theoretical model as shown in Fig. 1 (a). In special case where no resonant phenomenon of the shorter leaf appears in the frequency range considered, it is allowable in practice to consider a multi-leaf spring as a stepped beam.

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Discussion

S. NISHIMURA: This investigation treats a system connected by a spring of stiffness k_s as illustrated in Fig. 1 (a). I would like to know if the effect of stiffness k_s could be neglected.

Author's closure

The spring which has a spring constant of 1.37 kg/mm is designed in order to reduce the load for the exciter and enable a stable sinusoidal excitation. In the range of this spring constant, the effect on the third and higher-order frequencies is negligibly small. For the sake of comparison, the numerical

results for the sample G are shown as

Mode	Table of natural frequencies					
	1st	2nd	3rd	4th	5th	6th
						(Hz)
Case of $k_s=1.37$	1.9	29.9	69.6	94.0	167.6	225.7
Case of $k_s=0$	1.5	28.1	69.5	93.6	167.5	225.7

It is conceivable that, for higher frequency, the influence of inertia grows more effective and makes the vertical and rotatory displacements of the clamp smaller in comparison with the vibrational deformation of the leaves.