On Nonlinear Shell Theories*

By Iwao OSHIMA**, Yasuyuki SEGUCHI***, and Akio SHINDO****

Generally, the geometry of a deformed body is unknown in advance except in some special cases. Then, the elastic finite deformation theory of shells subject to large displacement gradient should be dealt with in the undeformed state or the reference state. Hence, fundamental equations for general large deformation shell theory and a membrane theory in large strain are derived for arbitrary geometry of the reference state, and the physical meaning of the tensor description is shown for convenience of practical applications. The derived strain-displacement relations and equilibrium equations in tensor form of the reference state are also applicable to the general problems of finite strain such as elastic-plastic analysis and creep analysis by being rewritten in the incremental form.

1. Introduction

Many papers concerning the thin shell theory in which some of finite displacements are taken into account have been published. These were obviously motivated by the requirement in the problems of shell stability. The literature on a general nonlinear theory including the effect of large strain is, however, scanty; the well-known example is the contribution by V. V. Novozhilov⁽¹⁾. More recently L. M. Habip^{(2) (3)} and B. Budiansky⁽⁴⁾ treated the subject.

Generally, the geometry after deformation of solids should be an unknown quantity to be solved. Then, when the motion of a particle of solids is traced, the basic equation of motion has to be considered in terms of a reference state, or in terms of metrics of undeformed shell. Structures of the most nonlinear shell theories presented before the Habip's work are ambiguous, and it is difficult to make an exact interpretation of nonlinear terms included in various approximate theories, since the difference of the metrics between undeformed and deformed state can not be distinguished by the audacious assumption of small strain.

In what follows, the general nonlinear shell equations are derived in a procedure somewhat differing from the Habip's work. To diminish incomprehensibilities inherent to tensor calculus, a direct physical derivation is tried and the physical meanings of terms are manifested. Some approximations and

simplifications of the theory are carried out. The approximate nonlinear theory in a sense of Donnell-Mushtari-Vlasov simplification is derived by a calculus of variation and a large strain membrane theory is obtained by simplification of the general theory.

With the recent remarkable development of computational techniques, the behaviors of elastic-plastic shells and shells under the effect of creep are going to be analyzed in various engineering fields. The basic equations derived could be used for such problems by some rearrangements to incremental form.

2. Strain tensors

In what follows, general curvilinear coordinates in local non-holonomic space $(\theta^\alpha, \theta^3)$, $(\alpha=1, 2)$ are employed, where $\theta^3=0$ is a middle surface of shells. Then, position vectors of an arbitrary point of a shell are expressed by

$$\mathbf{r}(\theta^\alpha, \theta^3) = \bar{\mathbf{r}}(\theta^\alpha) + \theta^3 \mathbf{a}_3(\theta^\alpha) \dots\dots\dots (1)$$

where $\bar{\mathbf{r}}(\theta^\alpha)$ is a position vector of a point on the middle surface and $\mathbf{a}_3(\theta^\alpha)$ is a unit normal to the undeformed middle surface. We define covariant base vectors of an arbitrary point before and after deformation by $\mathbf{g}_i(\theta^i)$ and $\mathbf{G}_i(\theta^i)$, $(i=1, 2, 3)$ respectively and covariant base vectors of the middle surface by $\mathbf{a}_i(\theta^\alpha, 0)$, $\mathbf{A}_i(\theta^\alpha, 0)$ respectively, where $\mathbf{g}_3(=\mathbf{a}_3)$ is also a unit normal; generally, in the discussion based on the reference state, $\mathbf{G}_3$ is not the unit normal. Vectorial expressions of a line element are given by

$$\left. \begin{array}{l} \text{Before deformation: } d\mathbf{s} = d\theta^i \mathbf{g}_i \\ \text{After deformation: } d\bar{\mathbf{s}} = d\theta^i \mathbf{G}_i \end{array} \right\} \dots\dots\dots (2)$$

Then, the distance between two adjacent points is written by

* Received 18 th March, 1969.

** Engineer, Sumitomo Electric Industries Limited.

*** Associate Professor, Faculty of Engineering, Kobe University, Nada-ku, Kobe

**** Professor, Faculty of Engineering, Kobe University.

$$\left. \begin{aligned} ds^2 &= g_{ij} d\theta^i d\theta^j = g_{ij} d\theta^i d\theta^j \\ d\bar{s}^2 &= G_{ij} d\theta^i d\theta^j = G_{ij} d\theta^i d\theta^j \end{aligned} \right\} \dots\dots\dots (3)$$

where g_{ij} and G_{ij} are the covariant metric tensors before and after deformation respectively. By using the above relations, we may define the components of the Cauchy-Green strain measure in general curvilinear coordinates as follows:

$$\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}) \dots\dots\dots (4)$$

After deformation, the position vector varies by the amount of v and the new relation is obtained as

$$R = r + v \dots\dots\dots (5)$$

where v is termed a displacement vector. Substituting Eq. (5) into Eq. (4), we obtain the expressions of the strain in terms of the displacement components as follows:

$$\gamma_{ij} = \frac{1}{2}(g_{ij,j} + g_{ji,i} + v_{i,j} + v_{j,i}) \dots\dots\dots (6)$$

where $(\quad)_{,i} = \partial(\quad)/\partial\theta^i$ or using the relation $v = v^i g_i = v_i g^i$, we have

$$\gamma_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i} + v^r_{,i} v_{r,j}) \dots\dots\dots (7)$$

The notation " $|$ " means the three-dimensional covariant derivative based on the metrics in the undeformed state.

To apply the above general relations to the shell theory, it is assumed that since shell thickness is sufficiently small in comparison with the other dimensions, the displacement can be expressed as^{(2) (3) (5)}

$$v(\theta^a, \theta^3) = u(\theta^a) + \theta^3 \beta(\theta^a) \dots\dots\dots (8)$$

Then, we may have from Eq. (5),

$$R = \bar{r}(\theta^a) + \theta^3 a_3 + u(\theta^a) + \theta^3 \beta(\theta^a) \dots\dots\dots (9)$$

in which the components of u and β are defined as follows:

$$\left. \begin{aligned} u &= u^a a_a + u^3 a_3 \\ \beta &= \psi^a a_a + \psi^3 a_3 \end{aligned} \right\} \dots\dots\dots (10)$$

Using Eqs. (1), (9) and (10), Eq. (4) is rewritten in terms of ψ^a , ψ^3 , u^a and u^3 :

$$\left. \begin{aligned} \gamma_{a\beta} &= {}_0\gamma_{a\beta} + \theta^3 {}_1\gamma_{a\beta} + (\theta^3)^2 {}_2\gamma_{a\beta} \\ \gamma_{a3} &= {}_0\gamma_{a3} + \theta^3 {}_1\gamma_{a3} \\ \gamma_{33} &= {}_0\gamma_{33} \end{aligned} \right\} \dots\dots\dots (11)$$

where ${}_k\gamma_{a\beta}$ ($k=0, 1, 2$), ${}_l\gamma_{a3}$ ($l=0, 1$) and ${}_m\gamma_{33}$ ($m=0$) are the nonlinear functions of u_i and ψ_i ⁽³⁾.

3. Stress tensors

We consider the equilibrium of deformed infinitesimal tetrahedron (Fig. 1). Denoting a unit normal by n , a force acting on a surface, the area of which is S , by T , and a force acting on a surface $\theta^i = \text{constant}$ (area S_i) by T_i , we may have the following equilibrium condition:

$$T = \sum_i T_i \dots\dots\dots (12)$$

Using stress vectors per unit area before deformation ${}_0t_i$, ${}_0t_i$ and areas before deformation ${}_0S$, ${}_0S_i$, Eq. (12) reduces to

$${}_0t_0 S = \sum_i {}_0t_{i0} S_i \dots\dots\dots (13)$$

Now, considering the conditions that the unit vector on the surface before deformation is expressed by ${}_0n (= {}_0n_i g^i)$ and that the following relation exists between areas ${}_0S$ and ${}_0S_i$:

$${}_0S_0 n = \sum_i {}_0S_i g^i / \sqrt{g^{ii}}$$

Eq. (13) is converted to

$${}_0t = \sum_i {}_0t_{i0} n_i \sqrt{g^{ii}} \dots\dots\dots (14)$$

Since ${}_0t$ should have the property of the contravariant component, it can be resolved in the directions of g_i and G_i as

$${}_0t_i \sqrt{g^{ii}} = t^{ij} g_j = s^{ij} G_j \dots\dots\dots (15)$$

where

$$t^{ij} = s^{ir} (\delta_r^j + v^j_{,r}) \dots\dots\dots (16)$$

and δ_r^j is termed Kronecker delta. Hence, substitution of Eq. (15) into Eq. (14) yields

$${}_0t = s^{ij} {}_0n_i G_j = t^{ij} {}_0n_i g_j \dots\dots\dots (17)$$

s^{ij} and t^{ij} are the stress tensors per unit area before deformation. It should be noted that s^{ij} is the component acting on the surface $\theta^i = \text{constant}$ parallel to the direction of G_j , and t^{ij} is the component on the same surface parallel to the direction of g_j . By the similar process of derivation, a stress vector t per unit area after deformation is given by the components of the stress tensor after deformation, τ^{ij} and π^{ij} , as follows^{(5) (6)}:

$$t = \tau^{ij} n_i G_j = \pi^{ij} n_i g_j \dots\dots\dots (18)$$

where n_i is defined by $n = n_i G^i$. Furthermore, the relation between s^{ij} and τ^{ij} is given by

$$s^{ij} = \tau^{ij} \sqrt{G/g} \dots\dots\dots (19)$$

where $G = \det[G_{ij}]$, $g = \det[g_{ij}]$

Next, we shall consider the physical meaning of the components. The physical expression of the stress vectors on the surface $\theta^i = \text{constant}$ is

$${}_0t_i = \sum_j \bar{s}^{ij} G_j / \sqrt{G_{jj}} = \sum_j \bar{t}^{ij} g_j / \sqrt{g_{jj}} \dots\dots\dots (20)$$

and consequently, we have the physical components $\bar{s}^{ij}$ and $\bar{t}^{ij}$ as

Fig. 1 Equilibrium of an infinitesimal tetrahedron

$$\bar{s}^{ij} = s^{ij} \sqrt{G_{jj}/g^{ii}}, \quad \bar{t}^{ij} = t^{ij} \sqrt{g_{jj}/g^{ii}} \dots (21)$$

where $\bar{s}^{ij}$ and $\bar{t}^{ij}$ are not tensorial components.

The equilibrium conditions about the stress tensors s^{ij} and t^{ij} are considered (Fig. 2). In the infinitesimal parallelepiped in equilibrium condition, denoting the force acting on the surface $\theta^1 = \text{constant}$ ($G_2 d\theta^2 \times G_3 d\theta^3$) by $\bar{T} d\theta^2 d\theta^3$, $\bar{T}$ is generally expressed as

$$\bar{T} = {}_0 t_i \sqrt{g} g^{ii} \dots (22)$$

or using Eq. (15),

$$\bar{T} = \sqrt{g} s^{ij} G_j = \sqrt{g} t^{ij} g_j \dots (23)$$

Defining the volume element before deformation by $d_0 \tau (= \sqrt{g} d\theta^1 d\theta^2 d\theta^3)$ and the body force per unit volume before deformation by ${}_0 F (= {}_0 F^i g_i)$, we obtain

$${}_0 F d_0 \tau = \sqrt{g} {}_0 F^i g_i d\theta^1 d\theta^2 d\theta^3 \dots (24)$$

and hence, the equations of equilibrium for the infinitesimal parallelepiped after deformation are given as follows:

$$\sum_i T_{,i} + \sqrt{g} {}_0 F = 0 \dots (25)$$

or in scalar form⁽²⁾

$$t^{ij}_{,i} + {}_0 F^j = 0 \dots (26)$$

or

$$[s^{ir}(\delta_r^j + v^j_{,r})]_{,i} + {}_0 F^j = 0 \dots (27)$$

The equilibrium condition of the moment with respect to the center of gravity of the parallelepiped

$$\sum_i G_i \times \bar{T} = 0 \dots (28)$$

assures the symmetry property of the stress tensors s^{ij} , while t^{ij} is not always a symmetric tensor.

4. Resultant stresses and stress couples

If $\bar{N} d\theta^2$ is the stress resultant vector per length $A_2 d\theta^2$ acting on the surface $\theta^1 = \text{constant}$ of shells after deformation, we have the following relations:

$$\bar{N} d\theta^2 = \int_{-h/2}^{h/2} \bar{T} d\theta^2 d\theta^3 \dots (29)$$

Fig. 2 Equilibrium of an infinitesimal parallelepiped

where h means shell thickness and in what follows, the integration with respect to θ^3 will be performed from $-h/2$ to $h/2$. Since the stress resultant is generally given by

$$\bar{N} = \int \bar{T} d\theta^3 \dots (30)$$

substitution of Eq. (23) into Eq. (30) yields

$$\begin{aligned} \bar{N} &= \int \sqrt{g} (t^{a\beta} g_\beta + t^{a3} g_3) d\theta^3 \\ &= \int \sqrt{a} (\mu \mu_\beta{}^\beta t^{a\beta} a_\beta + \mu t^{a3} a_3) d\theta^3 \dots (31) \end{aligned}$$

where

$$a = \det |a_{\alpha\beta}|$$

$$\mu = \sqrt{g/a}$$

$$\mu_{\alpha}{}^\beta = \delta_{\alpha}{}^\beta - \theta^3 b_{\alpha}{}^\beta$$

and $b_{\alpha}{}^\beta$ is the mixed component of the second fundamental metric tensor.

Now, using the following definitions of the stress resultant tensor:

$$n^{a\beta} = \int \mu \mu_\beta{}^\beta t^{a\beta} d\theta^3, \quad q^a = \int \mu t^{a3} d\theta^3 \dots (32)$$

and substituting them into Eq. (31), we obtain

$$\bar{N} = \sqrt{a} (n^{a\beta} a_\beta + q^a a_3) \dots (33)$$

The physical meaning of the stress resultant is interpreted as follows: from the definition of the stress resultant per unit length on the middle surface before deformation which is denoted by $\bar{N}^{(\alpha)}$, we have

$$\bar{N}^{(\alpha)} = \bar{N} / \sqrt{a a^{aa}} = (n^{a\beta} a_\beta + q^a a_3) / \sqrt{a^{aa}} \dots (34)$$

or in terms of physical components,

$$\bar{N}^{(\alpha)} = n^{a\beta} a_\beta / \sqrt{a_{\beta\beta}} + q^a a_3 \dots (35)$$

Consequently, the physical components of stress resultant tensors are given by

$$\left. \begin{aligned} n^{a\beta} &= \sqrt{a_{\beta\beta}} / \sqrt{a^{aa}} n^{a\beta} \\ q^a &= (1/\sqrt{a^{aa}}) q^a \end{aligned} \right\} \dots (36)$$

From the fact that the stress couple acting on the surface $\theta^1 = \text{constant}$ is written by

$$\bar{M} d\theta^2 = \int A_3 \theta^3 \times \bar{T} d\theta^2 d\theta^3 \dots (37)$$

the general expressions for stress couples are obtained as

$$\bar{M} = \int A_3 \theta^3 \times \bar{T} d\theta^3 \dots (38)$$

From Eq. (23), we have the following relation:

$$\begin{aligned} \bar{M} &= \int A_3 \times (\sqrt{g} t^{a\beta} \theta^3 g_\beta + \sqrt{g} t^{a3} \theta^3 g_3) d\theta^3 \\ &= \sqrt{a} \int \mu \mu_\beta{}^\beta t^{a\beta} \theta^3 A_3 \times a_\beta \\ &\quad + \sqrt{a} \int \mu t^{a3} \theta^3 A_3 \times a_3 \dots (39) \end{aligned}$$

Defining the stress couple tensors by

$$m^{a\beta} = \int \mu \mu_\beta{}^\beta t^{a\beta} \theta^3 d\theta^3, \quad t^a = \int \mu t^{a3} \theta^3 d\theta^3 \dots (40)$$

Eq. (39) becomes

$$\vec{M} = \sqrt{a} (m^{\alpha\beta} A_3 \times a_\beta + t^\alpha A_3 \times a_3) \dots \dots \dots (41)$$

To establish the definition of the physical component of the stress couples, the following relation is substituted into Eq. (41):

$$A_3 = a_3 + \beta \equiv X_3^\alpha a_\alpha + X_3^3 a_3,$$

where $X_3^\alpha = \phi^\alpha$, $X_3^3 = 1 + \phi^3$

which is obtainable by partial differentiation of Eq. (9) with respect to θ^3 . Then, Eq. (41) may be transformed as

$$\vec{M} = \sqrt{a} \{ m^{\alpha\beta} X_3^\delta \epsilon_{\delta\beta} a^\alpha + (m^{\alpha\delta} X_3^3 - t^\alpha X_3^\delta) \epsilon_{\delta\beta} a^\beta \} \dots \dots \dots (42)$$

where $\epsilon_{\delta\beta}$ is the two-dimensional Eddington symbol defined by

$$\epsilon_{11} = \epsilon_{22} = 0, \quad \epsilon_{12} = -\epsilon_{21} = \sqrt{a}.$$

If the stress couple per unit length of the middle surface before deformation is denoted by $\vec{M}^{(\alpha)}$, then

$$\vec{M}^{(\alpha)} = \frac{1}{\sqrt{a^{\alpha\alpha}}} \{ m^{\alpha\beta} X_3^\delta \epsilon_{\delta\beta} a^\alpha + (m^{\alpha\delta} X_3^3 - t^\alpha X_3^\delta) \epsilon_{\delta\beta} a^\beta \} \dots \dots \dots (43)$$

which is also written in terms of unit base vectors by

$$\vec{M}^{(\alpha)} = m_\beta^\alpha a^\beta / \sqrt{a_{\beta\beta}} + t^\alpha a_3 \dots \dots \dots (44)$$

Therefore, we conclude that the physical component of the stress couple can be expressed by

$$\left. \begin{aligned} m_\beta^\alpha &= \sqrt{\frac{a^{\beta\beta}}{a^{\alpha\alpha}}} (m^{\alpha\delta} X_3^3 - t^\alpha X_3^\delta) \epsilon_{\delta\beta} \\ t^\alpha &= \frac{1}{\sqrt{a^{\alpha\alpha}}} m^{\alpha\beta} X_3^\delta \epsilon_{\delta\beta} \end{aligned} \right\} \dots \dots \dots (45)$$

5. Equilibrium equations of shells

Integrating the general equilibrium equation of body in the case of no body force

$$\sum_i \vec{T}_i d\theta^1 d\theta^2 d\theta^3 = 0 \dots \dots \dots (46)$$

with respect to θ^3 , we have

$$\int \sum \vec{T}_i d\theta^3 + \int \vec{T}_3 d\theta^3 = 0 \dots \dots \dots (47)$$

Using Eqs. (23), (30) and (31), the above equation is transformed into

$$\left. \begin{aligned} &\int (\sqrt{a} \mu \mu_\beta^\beta t^{\alpha\delta} a_\beta + \sqrt{a} \mu t^{\alpha 3} a_3)_{,\alpha} d\theta^3 \\ &+ \int (\sqrt{a} \mu \mu_\beta^\beta t^{\alpha\delta} a_\beta + \sqrt{a} \mu t^{\alpha 3} a_3)_{,3} d\theta^3 \\ &= \int \{ \sqrt{a} (\mu \mu_\beta^\beta t^{\alpha\delta})_{||\alpha} a_\beta + \sqrt{a} (\mu \mu_\beta^\beta t^{\alpha\delta}) b_{\alpha\beta} a_3 \\ &+ \sqrt{a} (\mu t^{\alpha 3})_{||\alpha} a_3 - \sqrt{a} (\mu t^{\alpha 3}) b_{\alpha\beta} a_\beta \} d\theta^3 \\ &+ [\sqrt{a} \mu t^{\alpha 3} \mu_{\alpha\beta} a_\beta]_{-h/2}^{h/2} \\ &+ [\sqrt{a} \mu t^{\alpha 3} a_3]_{-h/2}^{h/2} = 0 \end{aligned} \right\} \dots \dots \dots (48)$$

where the notation " $||$ " is related to the two-dimensional covariant derivative based upon the metrics of the middle surface before deformation.

Resolving the last equation in the directions of a_β and a_3 , the following equilibrium equations of shells are obtained with the help of the definition for the stress tensors, Eq. (32):

$$\left. \begin{aligned} n^{\alpha\beta}_{||\alpha} - b_{\alpha\beta} q^\alpha + p^\beta &= 0 \\ q^\alpha_{||\alpha} + b_{\alpha\beta} n^{\alpha\beta} + p^3 &= 0 \end{aligned} \right\} \dots \dots \dots (49)$$

where p^α and p^3 are related to the surface traction and are defined by

$$p^\alpha = [\mu t^{\alpha\beta} \mu_\beta^\beta]_{-h/2}^{h/2}, \quad p^3 = [\mu t^{\alpha 3}]_{-h/2}^{h/2} \dots \dots \dots (50)$$

Applying the similar procedure to the equilibrium of the stress couple, we obtain

$$\int A_3 \theta^3 \times \vec{T}_{,3} d\theta^3 + \int A_3 \theta^3 \times \vec{T}_3 d\theta^3 = 0 \dots \dots \dots (51)$$

From Eqs. (23), (38) and (39), complicated tensor calculus leads to

$$\begin{aligned} &\int A_3 \times \sqrt{a} \{ (\mu \mu_\beta^\beta t^{\alpha\delta})_{||\alpha} a_\beta + (\mu \mu_\beta^\beta t^{\alpha\delta}) b_{\alpha\beta} a_3 \\ &+ (\mu t^{\alpha 3})_{||\alpha} a_3 - (\mu t^{\alpha 3}) b_{\alpha\beta} a_\beta \} \theta^3 d\theta^3 \\ &+ A_3 \times \sqrt{a} \{ [\theta^3 \mu \mu_\alpha^\beta t^{\alpha 3}]_{-h/2}^{h/2} a_\beta \\ &+ [\theta^3 \mu t^{\alpha 3}]_{-h/2}^{h/2} a_3 \} \\ &- A_3 \times \sqrt{a} \{ \int \mu t^{\alpha 3} \mu_\alpha^\beta d\theta^3 a_\beta + \int \mu t^{\alpha 3} d\theta^3 a_3 \} = 0 \end{aligned} \dots \dots \dots (52)$$

With the help of Eq. (40), the definition of stress couple, and resolving Eq. (52) in the directions of $A_3 \times a_\beta$ and $A_3 \times a_3$, the following relations are obtained:

$$\left. \begin{aligned} m^{\alpha\beta}_{||\alpha} - t^\alpha b_{\alpha\beta} + c^\beta - \int \mu t^{\alpha 3} \mu_\alpha^\beta d\theta^3 &= 0 \\ t^\alpha_{||\alpha} + m^{\alpha\beta} b_{\alpha\beta} + c^3 - \int \mu t^{\alpha 3} d\theta^3 &= 0 \end{aligned} \right\} \dots \dots \dots (53)$$

where c^β and c^3 are the terms by surface couple and are defined by

$$c^\beta = [\theta^3 \mu \mu_\alpha^\beta t^{\alpha 3}]_{-h/2}^{h/2}, \quad c^3 = [\theta^3 \mu t^{\alpha 3}]_{-h/2}^{h/2} \dots \dots \dots (54)$$

The equilibrium equations thus obtained are coincident with the results derived by Habip from the direct integration of Eq. (27) with respect to θ^3 .

6. Boundary condition

6.1 Mechanical boundary condition

From the definitions

s : intersecting line of the surface $\theta^3 = \text{constant}$ and a boundary surface σ

λ : unit tangential vector along the line s on the surface $\theta^3 = \text{constant}$

the following relation is confirmed:

$$\lambda = \frac{d\mathbf{r}}{ds} = \lambda^\alpha g_\alpha = \lambda^\alpha \mu_\alpha^\beta a_\beta \dots \dots \dots (55)$$

where $\lambda^\alpha = d\theta^\alpha/ds$.

The unit tangential vector $\bar{\lambda}$ which is parallel to the intersecting line $\bar{s}$ of the surface $\theta^3 = 0$ and the boundary surface σ is expressed by means of the position vector of a point on the middle surface as follows:

$$\bar{\lambda} = \frac{d\bar{\mathbf{r}}}{d\bar{s}} = \bar{\lambda}^\alpha a_\alpha \dots \dots \dots (56)$$

Hence, we have

$$\bar{\lambda}^a = \lambda^{\beta} \mu_{\beta}^a \quad (57)$$

Since the unit normal vector on the boundary surface is given by

$${}_0n = \lambda \times a_3 = {}_0n_{\beta} g^{\beta} = {}_0\bar{n}_{\beta} a^{\beta} \quad (58)$$

the following relations are obtained:

$$\lambda^a ds = \bar{\lambda}^a d\bar{s} \quad (59)$$

$${}_0n_a ds = \mu_a \bar{n}_a d\bar{s} \quad (60)$$

Resultant force per unit length on the middle surface before deformation N and stress vector per unit area before deformation acting on the boundary surface ${}_0t$ are related to each other by the expression

$$\int_c N d\bar{s} = \int_a t da \quad (61)$$

where c is the length of the whole boundary line on which the external force is prescribed and a is the area of the boundary given by

$$da = |a_3 d\theta^3 \times \lambda ds| = d\theta^3 ds$$

Consequently, from Eq. (17), the right hand side of Eq. (61) becomes

$$\begin{aligned} \int_c N d\bar{s} &= \int_a (t^{\alpha\beta} {}_0n_a g_{\beta} + t^{\alpha 3} {}_0n_a a_3) d\theta^3 ds \\ &= \int_a \mu (t^{\alpha\beta} {}_0\bar{n}_a \mu_{\beta}^{\alpha} a_3 + t^{\alpha 3} {}_0\bar{n}_a a_3) d\theta^3 d\bar{s} \end{aligned} \quad (62)$$

and finally,

$$N = {}_0\bar{n}_a \int \mu \mu_{\beta}^{\alpha} t^{\alpha\beta} d\theta^3 a_3 + {}_0\bar{n}_a \int \mu t^{\alpha 3} d\theta^3 a_3 \quad (63)$$

Similarly, the stress couple per unit length of the undeformed middle surface is expressed by

$$\begin{aligned} K &= {}_0\bar{n}_a \int \mu \mu_{\beta}^{\alpha} t^{\alpha\beta} d\theta^3 A_3 \times a_3 \\ &+ {}_0\bar{n}_a \int \mu t^{\alpha 3} d\theta^3 A_3 \times a_3 \end{aligned} \quad (64)$$

These relations are simplified by the definitions (32) and (40), as follows:

$$\left. \begin{aligned} N &= {}_0\bar{n}_a n^{\alpha\beta} a_{\beta} + {}_0\bar{n}_a q^{\alpha} a_3 \\ K &= {}_0\bar{n}_a m^{\alpha\beta} A_3 \times a_{\beta} + {}_0\bar{n}_a t^{\alpha} A_3 \times a_3 \end{aligned} \right\} \quad (65)$$

6.2 Geometrical boundary condition

On the boundary in which the displacements are prescribed,

$$\tilde{u}_i = u_i, \quad \tilde{\psi}_i = \psi_i \quad (66)$$

where $\tilde{u}_i$ and $\tilde{\psi}_i$ are the prescribed components of the displacement.

7. Distributed force

When, in a finite deformation problem, the distributed force is taken into account, the difference between a load by hydrostatic pressure and a load by dead weight has to be clarified; it is noted that the measured hydrostatic pressure is per unit area after deformation applied to deformed surface, while dead weight load is measured in the quantity per unit area before deformation.

7.1 Hydrostatic distributed load

Since the distributed pressure $P d\theta^1 d\theta^2$ acting on

a surface element after deformation $G_1 d\theta^1 \times G_2 d\theta^2$ is related to load per unit area after deformation by

$$P d\theta^1 d\theta^2 = p \sqrt{G} d\theta^1 d\theta^2 \quad (67)$$

we obtain the following relation for ${}_0p$, the load per unit area acting on the undeformed middle surface:

$${}_0p = \frac{P}{\sqrt{a}} = p \sqrt{\frac{G}{a}} = p^3 \bar{G}_3 \sqrt{\frac{G}{a}} \quad (68)$$

where $\bar{G}_3$ is a unit normal to the deformed shell surface. By defining the components of the base vectors after deformation as

$$G_a = A_a^{\beta} a_{\beta} + A_a^3 a_3$$

A_a^{β} and A_a^3 can be expressed as

$$A_a^{\beta} = \mu_a^{\beta} + (n^{\beta} |_{|a} - b_a^{\beta} u_3) + \theta^3 (\psi^{\beta} |_{|a} - b_a^{\beta} \psi_3)$$

$$A_a^3 = u_{3,a} + b_a^{\beta} u_{\beta} + \theta^3 (\psi_{3,a} + b_a^{\beta} \psi_{\beta})$$

And consequently,

$$\bar{G}_3 = (B_1 a^1 + B_2 a^2 + B_3 a^3) \sqrt{a} / \sqrt{G}$$

where

$$B_1 = A_1^2 A_2^3 - A_1^3 A_2^2, \quad B_2 = A_1^3 A_2^1 - A_1^1 A_2^3$$

$$B_3 = A_1^1 A_2^2 - A_1^2 A_2^1$$

Finally, Eq. (68) leads to

$${}_0p = p^3 (B_1 a^1 + B_2 a^2 + B_3 a^3) \quad (69)$$

7.2 Distributed force by dead weight

As stated in the previous article, the load per unit area acting parallel to the vertical direction of the undeformed shell, ${}_0q$ can be measured. If the load acting on the surface element $G_1 d\theta^1 \times G_2 d\theta^2$ is defined by $Q d\theta^1 d\theta^2$, we have

$$Q d\theta^1 d\theta^2 = {}_0q \sqrt{g} d\theta^1 d\theta^2 \quad (70)$$

From Eq. (70), the relation for the load per unit undeformed area of the middle surface ${}_0Q$ is obtained as follows:

$${}_0Q = {}_0q \sqrt{\frac{g}{a}} \equiv q^a a_a + q^3 a_3 \quad (71)$$

Using the vector component resolved in the directions of the unit base vectors

$${}_0q = q^a a_a / \sqrt{a_{aa}} + q^3 a_3 \quad (72)$$

the physical tensor components of the load are given by

$$q^a = {}_0q^a \sqrt{\frac{g}{a a^{aa}}}, \quad q^3 = \sqrt{\frac{g}{a}} {}_0q^3 \quad (73)$$

8. Constitutive equation

Only elastic isotropic linear materials are considered⁽⁷⁾. This is, no doubt, a special case but, for the most engineering purposes, a linear constitutive equation is available even for the case of finite displacement problems.

Postulating the existence of the strain energy function in which only the quadratic terms of strains are taken into account, the following constitutive equations of homogeneous and isotropic Hookean materials are derived:

$$s^{ij} = \{\rho (g^{ia} g^{jb} + g^{ib} g^{ja}) + \lambda g^{ij} g^{ab}\} \gamma_{ab} \quad (74)$$

where ρ and λ are called the Lamé's constants and may be related to Young's modulus E and Poisson's

ratio ν by

$$\left. \begin{aligned} \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)}, \\ \rho &= \frac{E}{2(1+\nu)} \end{aligned} \right\} \dots\dots\dots (75)$$

Assuming the plane stress state, Eq. (74) reduces to

$$\left. \begin{aligned} s^{\alpha\beta} &= \frac{E}{2(1+\nu)} \left\{ g^{\alpha\lambda} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\lambda} \right. \\ &\quad \left. + \frac{2\nu}{1-\nu} g^{\alpha\beta} g^{\lambda\delta} \right\} \gamma_{\lambda\delta} \\ s^{\alpha 3} &= \frac{E}{1+\nu} g^{\alpha\beta} \gamma_{\beta 3}, \quad s^{33} = 0 \end{aligned} \right\} \dots\dots\dots (76)$$

and for the plane strain state, we have

$$\left. \begin{aligned} s^{\alpha\beta} &= \frac{E}{2(1+\nu)} \left\{ g^{\alpha\lambda} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\lambda} \right. \\ &\quad \left. + \frac{2\nu}{1-2\nu} g^{\alpha\beta} g^{\lambda\delta} \right\} \gamma_{\lambda\delta} \\ s^{\alpha 3} &= \frac{E}{1+\nu} g^{\alpha\beta} \gamma_{\beta 3} \\ s^{33} &= \frac{E}{1+\nu} \frac{\nu}{1-2\nu} g^{\alpha\beta} \gamma_{\alpha\beta} \end{aligned} \right\} \dots\dots\dots (77)$$

Thus far, the necessary basic equations of general shells subject to finite strain have been derived. In applying these equations to a practical engineering problem, it is convenient to reduce them to six nonlinear partial differential equations in terms of the six dependent variables u_α , u_3 , ψ_α and ψ_3 . If the metrics of undeformed shell are given, then, they can be solved with prescribed boundary conditions.

9. Variational principle

General nonlinear shell equations have been derived by "the direct method" in the preceding sections and it has been shown that the technique is valid for clarifying the physical meanings of the relations and quantities. Habip also derived similar results⁽⁸⁾ by using the Hellinger-Reissner principle⁽⁸⁾.

In the variational principle, the functional is shown to be

$$\begin{aligned} \Pi_R = \int_{0V} & \left[A(\gamma_{ij}) - s^{ij} \gamma_{ij} + s^{i3} \frac{1}{2} (v_{i1j} + v_{j1i} \right. \\ & \left. + v_{1i} v_{r1j}) - F^i v_i \right] dV - \int_{0s_1} \bar{s}_*^i v_i ds \\ & - \int_{0s_2} s_*^i (v_i - \bar{v}_i) ds \dots\dots\dots (78) \end{aligned}$$

where $A(\gamma_{ij})$ is the strain energy function per unit volume of undeformed body, $0V$ is the volume of undeformed body and $0s_1$, $0s_2$ are the boundaries of a body on which the mechanical boundary condition and geometrical boundary condition are prescribed respectively. The component of $0t$, the stress vector on the boundary resolved in the directions of the base vectors before deformation, is defined by s_*^i and the notation " $\sim$ " is the symbol which means that the quantity is prescribed on the boundary. F^i is the component of body force. The first variation of the functional is

$$\begin{aligned} \delta \Pi_R = \int_{0V} & \left[\left\{ \frac{1}{2} \left(\frac{\partial A}{\partial \gamma_{ij}} + \frac{\partial A}{\partial \gamma_{ji}} \right) - s^{ij} \right\} \delta \gamma_{ij} \right. \\ & - \left\{ \gamma_{ij} - \frac{1}{2} (v_{i1j} + v_{j1i} + v_{1i} v_{r1j}) \right\} \delta s_{ij} \\ & - \left[s^{ir} (\delta_r^j + v_{1r}^j) \right]_{1i} + F^j \delta v_j \Big] dV \\ & + \int_{0s_1} [s^{ir} (\delta_r^j + v_{1r}^j) n_i - \bar{s}_*^j] \delta v_j ds \\ & - \int_{0s_2} (v_i - \bar{v}_i) \delta s_*^i ds = 0 \dots\dots\dots (79) \end{aligned}$$

where the independent quantities subject to variation are γ_{ij} , s^{ij} , v^i and s_*^i , and the stationary conditions are shown to be

$$\left. \begin{aligned} \text{(I)} \quad s^{ij} &= \frac{1}{2} \left(\frac{\partial A}{\partial \gamma_{ij}} + \frac{\partial A}{\partial \gamma_{ji}} \right) \\ \text{(II)} \quad \gamma_{ij} &= \frac{1}{2} (v_{i1j} + v_{j1i} + v_{1i} v_{r1j}) \\ \text{(III)} \quad [s^{ir} (\delta_r^j + v_{1r}^j)]_{1i} + F^j &= 0 \\ \text{(IV)} \quad \bar{s}_*^j &= 0 n_i s^{ir} (\delta_r^j + v_{1r}^j) \\ \text{(V)} \quad \bar{v}_i &= v_i \end{aligned} \right\} \dots\dots\dots (80)$$

The result obtained is easily proved to coincide with the relations derived by the direct method.

10. Approximate theory

In the present chapter, an approximate shell theory by the Donnell-Mushtari-Vlasov simplification will be derived on the basis of the relations obtained in reference state and compared with the partial nonlinear theory presented by Habip. In the partial nonlinear theory, the following strain-displacement relations are used instead of Eq. (7):⁽³⁾

$$\gamma_{\alpha\beta} = \frac{1}{2} (v_{\alpha 1\beta} + v_{\beta 1\alpha} + v_{1\alpha} v_{31\beta}), \quad \gamma_{i3} = \frac{1}{2} (v_{i13} + v_{31i}) \dots\dots\dots (81)$$

Putting $\psi_3 = 0$ and expanding in terms of u_α , u_3 and ψ_α , we have

$$\left. \begin{aligned} \gamma_{\alpha\beta} &= \frac{1}{2} [\mu_\alpha^\delta (u_{\delta 1\beta} - b_{\delta\beta} u_3 + \theta^\beta \psi_{\delta 1\beta}) \\ &\quad + \mu_\beta^\delta (u_{\delta 1\alpha} - b_{\delta\alpha} u_3 + \theta^\alpha \psi_{\delta 1\alpha}) + u_{3,\alpha} u_{3,\beta}] \\ \gamma_{\alpha 3} &= \frac{1}{2} (\psi_\alpha + u_{3,\alpha} + b_{\alpha\beta} u_\beta), \quad \gamma_{33} = 0 \end{aligned} \right\} \dots\dots (82)$$

substitution of which into Eq. (78) leads to the required nonlinear theory. On the other hand, if it is assumed in the relation for $\gamma_{\alpha 3}$ of Eq. (82) that

$$\psi_\alpha + u_{3,\alpha} \gg b_{\alpha\beta} u_\beta$$

and the following approximation is performed:

$$\gamma_{\alpha 3} = \frac{1}{2} (\psi_\alpha + u_{3,\alpha}) \dots\dots\dots (83)$$

then, the variational principle is transformed into

$$\begin{aligned} \delta \Pi_R = \delta \int_{0V} & \left[A(\gamma_{ij}) - s^{ij} \gamma_{ij} + s^{\alpha\beta} \frac{1}{2} \{ \mu_\alpha^\delta (u_{\delta 1\beta} \right. \\ & - b_{\delta\beta} u_3 + \theta^\beta \psi_{\delta 1\beta}) + \mu_\beta^\delta (u_{\delta 1\alpha} - b_{\delta\alpha} u_3 + \theta^\alpha \psi_{\delta 1\alpha}) \\ & \left. + u_{3,\alpha} u_{3,\beta} \} + s^{\alpha 3} \{ \mu_\alpha^\beta (u_{\beta 3} + \theta^\beta \psi_{\beta 3}) \right. \\ & \left. + \psi_\alpha + u_{3,\alpha} + u_{3,\alpha} u_{3,\beta} \} + s^{33} (u_{3,3} - F^i v_i) \right] dV \\ & - \int_{0s_1} \bar{s}_*^i v_i ds - \int_{0s_2} s_*^i (v_i - \bar{v}_i) ds = 0 \dots\dots (84) \end{aligned}$$

Taking into account $\psi_3=0$, the stationary conditions are shown to be

$$\left. \begin{aligned} \delta u_\beta: N^{\alpha\beta}_{||\alpha} + p^\beta + \int \mu \mu_{\alpha}^{\beta} F^{\alpha} d\theta^3 &= 0 \\ \delta \psi_\beta: M^{\alpha\beta}_{||\alpha} - Q^\beta + c^\beta + \int \mu \mu_{\alpha}^{\beta} F^{\alpha} \theta^3 d\theta^3 &= 0 \\ \delta u_3: Q^{\alpha}_{||\alpha} + b_{\alpha\beta} N^{\alpha\beta} + ({}'N^{\alpha\beta} u_{3,\beta})_{||\alpha} + p^3 \\ &+ \int \mu F^3 d\theta^3 = 0 \end{aligned} \right\} \dots (85)$$

where

$$\left. \begin{aligned} N^{\alpha\beta} &= \int \mu \mu_{\delta}^{\beta} S^{\alpha\delta} d\theta^3, \quad {}'N^{\alpha\beta} = \int \mu S^{\alpha\beta} d\theta^3 \\ Q^{\alpha} &= \int \mu S^{\alpha 3} d\theta^3, \quad M^{\alpha\beta} = \int \mu \mu_{\delta}^{\beta} S^{\alpha\delta} \theta^3 d\theta^3 \\ p^{\alpha} &= |\mu S^{\beta 3} \mu_{\beta}^{\alpha}|_{-h/2}^{h/2} \\ p^3 &= |\mu S^{\alpha 3} u_{3,\alpha} + \mu S^{33}|_{-h/2}^{h/2} \\ c^{\alpha} &= |\theta^3 \mu S^{\beta 3} \mu_{\beta}^{\alpha}|_{-h/2}^{h/2} \end{aligned} \right\} \dots (86)$$

and furthermore, the mechanical and the geometrical boundary conditions are decided as

$$\left. \begin{aligned} s^{\alpha} &= \int \mu \mu_{\beta}^{\alpha} s_{*}^{\beta} d\theta^3 = {}_0\bar{n}_{\beta} N^{\beta\alpha} \\ s^3 &= \int \mu s_{*}^3 d\theta^3 = {}_0\bar{n}_{\beta} (Q^{\beta} + {}'N^{\beta\alpha} u_{3,\alpha}) \\ t^{\alpha} &= \int \mu \mu_{\beta}^{\alpha} s_{*}^{\beta} \theta^3 d\theta^3 = {}_0\bar{n}_{\beta} M^{\beta\alpha} \end{aligned} \right\} \dots (87)$$

and

$$\bar{u}_i = u_i, \quad \bar{\psi}_\alpha = \psi_\alpha \dots (88)$$

respectively.

If the term $-b_{\alpha}^{\beta} Q$ is added to the first of the equilibrium equations (85) thus obtained, the theory may be transformed into the nonlinear partial theory. It is noted that the theory by Donnell-Mushtari-Vlasov assumption is identical with the contribution by J. L. Sanders⁽⁹⁾, in which the conventional deriving method (using the metrics of deformed state) is employed.

11. Nonlinear membrane theory

In formulating a nonlinear membrane theory which includes the effect of large strain, we shall employ the following assumption:

- (I) Shell thickness is sufficiently small.
- (II) Transverse shear stress may be neglected.

Using the procedure of the direct method, and from Eq. (49), the equilibrium equations are written as follows:

$$\left. \begin{aligned} n^{\alpha\beta}_{||\alpha} - b_{\alpha}^{\beta} q^{\alpha} + p^{\beta} &= 0 \\ q^{\alpha}_{||\alpha} + b_{\alpha\beta} n^{\alpha\beta} + p^3 &= 0 \end{aligned} \right\} \dots (89)$$

Stress resultant tensor $n^{\alpha\beta}$ and q^{α} are given in terms of $s^{\alpha\beta}$ by

$$\left. \begin{aligned} n^{\alpha\beta} &= \int \{ \mu S^{\alpha\delta} (\mu_{\delta}^{\beta} + u_{||\delta}^{\beta} - b_{\delta}^{\beta} u_3) \\ &+ \mu S^{\alpha\delta} \theta^3 (\psi_{||\delta}^{\beta} - b_{\delta}^{\beta} \psi_3) + \mu S^{\alpha 3} \psi^{\beta} \} d\theta^3 \\ q^{\alpha} &= \int \{ \mu S^{\alpha\beta} (u_{3,\beta} + b_{\beta}^{\alpha} u_3) \\ &+ \mu S^{\alpha\beta} \theta^3 (\psi_{3,\beta} + b_{\beta}^{\alpha} \psi_3) + \mu S^{\alpha 3} (1 + \psi_3) \} d\theta^3 \end{aligned} \right\} \dots (90)$$

Taking account of the preceding assumptions, substitution of Eq. (90) into Eq. (89) leads to

$$\left. \begin{aligned} \int [\mu S^{\alpha\delta} (\mu_{\delta}^{\beta} + u_{||\delta}^{\beta} - b_{\delta}^{\beta} u_3)]_{||\alpha} d\theta^3 \\ - b_{\alpha}^{\beta} \int \{ \mu S^{\alpha\delta} (u_{3,\delta} + b_{\delta}^{\alpha} u_3) \} d\theta^3 + p^{\beta} = 0 \\ \int [\mu S^{\alpha\beta} (u_{3,\beta} + b_{\beta}^{\alpha} u_3)]_{||\alpha} d\theta^3 \\ + b_{\alpha\beta} \int [\mu S^{\alpha\delta} (\mu_{\delta}^{\beta} + u_{||\delta}^{\beta} - b_{\delta}^{\beta} u_3)] d\theta^3 + p^3 = 0 \end{aligned} \right\} \dots (91)$$

or in terms of stress resultants [Eq. (86)];

$$\left. \begin{aligned} N^{\alpha\beta}_{||\alpha} + [{}'N^{\alpha\delta} (u_{||\delta}^{\beta} - b_{\delta}^{\beta} u_3)]_{||\alpha} \\ - b_{\alpha}^{\beta} [{}'N^{\alpha\delta} (u_{3,\delta} + b_{\delta}^{\alpha} u_3)] + p^{\beta} = 0 \\ [{}'N^{\alpha\beta} (u_{3,\beta} + b_{\beta}^{\alpha} u_3)]_{||\alpha} \\ + b_{\alpha\beta} [N^{\alpha\beta} + {}'N^{\alpha\delta} (u_{||\delta}^{\beta} - b_{\delta}^{\beta} u_3)] + p^3 = 0 \end{aligned} \right\} \dots (92)$$

These relations are the same as the results recently derived by B. Budiansky⁽⁴⁾.

Considering the homogeneous and isotropic Hookean materials, the constitutive equations of plane stress state become in this case,

$$s^{\alpha\beta} = \frac{E}{2(1+\nu)} \left\{ g^{\alpha\lambda} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\lambda} + \frac{2\nu}{1-\nu} g^{\alpha\beta} g^{\lambda\delta} \right\} \gamma_{\lambda\delta} \dots (93)$$

The assumption of membrane state suggests the possibility that the following approximation can hold:

$$g^{\alpha\beta} \approx a^{\alpha\beta}, \quad g \approx a, \quad N^{\alpha\beta} \approx {}'N^{\alpha\beta}$$

Then, we obtain the constitutive equation of membranes as

$$\begin{aligned} N^{\alpha\beta} &= {}'N^{\alpha\beta} = \int \frac{E}{2(1+\nu)} \left\{ a^{\alpha\lambda} a^{\beta\delta} + a^{\alpha\delta} a^{\beta\lambda} \right. \\ &\left. + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\lambda\delta} \right\} \gamma_{\lambda\delta} d\theta^3 \dots (94) \end{aligned}$$

The geometrical boundary condition is reduced to

$$\bar{u}_i = u_i \dots (95)$$

and the mechanical boundary condition, Eq. (63), can be specified as

$$\begin{aligned} N &= {}_0\bar{n}_{\alpha} \int \{ \mu S^{\alpha\lambda} (\mu_{\lambda}^{\beta} + u_{||\lambda}^{\beta} - b_{\lambda}^{\beta} u_3) \\ &+ \mu \theta^3 S^{\alpha\delta} (\psi_{||\delta}^{\beta} - b_{\delta}^{\beta} \psi_3) + s^{\alpha\beta} \psi^{\beta} \} d\theta^3 \alpha_{\beta} \\ &+ {}_0\bar{n}_{\alpha} \int \{ \mu S^{\alpha 3} (1 + \psi_3) + \mu S^{\alpha\beta} (u_{3,\beta} + b_{\beta}^{\alpha} u_3) \\ &+ \theta^3 \mu S^{\alpha\beta} (\psi_{3,\beta} + b_{\beta}^{\alpha} \psi_3) \} d\theta^3 \alpha_3 \dots (96) \end{aligned}$$

With the help of the assumptions (I), (II) and Eq. (86), Eq. (96) is shown to become

$$\begin{aligned} N &= {}_0\bar{n}_{\alpha} [N^{\alpha\beta} + {}'N^{\alpha\lambda} (u_{||\lambda}^{\beta} - b_{\lambda}^{\beta} u_3)] \alpha_{\beta} \\ &+ {}_0\bar{n}_{\alpha} [{}'N^{\alpha\beta} (u_{3,\beta} + b_{\beta}^{\alpha} u_3)] \alpha_3 \dots (97) \end{aligned}$$

12. Application to shells of revolution

Nonlinear asymmetric theory of shells of revolution will be formulated within the limits of Donnell-Mushtari-Vlasov approximation. Referring to Fig. 3, the coordinates of the shell are selected as

$$d\theta^1 = d\xi: \text{meridional direction}$$

$$d\theta^2 = d\theta: \text{circumferential direction}$$

Fig. 3 A coordinate system of shells of revolution

$d\theta^3 = d\zeta$: direction normal to shell surface

Then, the base vectors of the middle surface of shells in terms of unit vectors i, j and k of the rectangular Cartesian coordinate system are written as

$$\left. \begin{aligned} a_1 &= \cos \phi (\cos \theta i + \sin \theta j) + \sin \phi k \\ a_2 &= r(-\sin \theta i + \cos \theta j) \\ a_3 &= -\sin \phi (\cos \theta i + \sin \theta j) + \cos \phi k \end{aligned} \right\} \dots\dots\dots (98)$$

Using the above equations, the covariant and contravariant components of the first fundamental metric tensor are written as follows:

$$\left. \begin{aligned} a_{\alpha\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \\ a_{\alpha 3} &= 0, \quad a_{33} = 1, \quad a = r^2 \\ a^{\alpha\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \end{aligned} \right\} \dots\dots\dots (99)$$

The second fundamental metric tensor and Christoffel symbol are calculated respectively as

$$b_{\alpha\beta} = \begin{pmatrix} \partial\phi/\partial\xi & 0 \\ 0 & r \sin \phi \end{pmatrix} \dots\dots\dots (100)$$

and

$$\left. \begin{aligned} \Gamma_{12}^2 &= \Gamma_{21}^2 = \cos \phi / r \\ \Gamma_{22}^1 &= -r \cos \phi \\ \Gamma_{11}^1 &= \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{22}^2 = 0 \end{aligned} \right\} \dots\dots\dots (101)$$

Next, the equilibrium equations including the effect of distributed force, Eq. (85), if the body force is neglected, lead to

$$\left. \begin{aligned} N^{\alpha\beta}_{,\alpha} + \Gamma_{\alpha\delta}^{\alpha} N^{\delta\beta} + \Gamma_{\alpha\delta}^{\beta} N^{\alpha\delta} + p^{\beta} &= 0 \\ M^{\alpha\beta}_{,\alpha} + \Gamma_{\alpha\delta}^{\alpha} M^{\delta\beta} + \Gamma_{\alpha\delta}^{\beta} M^{\alpha\delta} - Q^{\beta} &= 0 \\ Q^{\alpha}_{,\alpha} + \Gamma_{\alpha\delta}^{\alpha} Q^{\delta} + b_{\alpha\beta} N^{\alpha\beta} \\ + ({}'N^{\alpha\beta} u_{\delta,\beta})_{,\alpha} + \Gamma_{\alpha\delta}^{\alpha} ({}'N^{\delta\beta} u_{3,\beta}) &+ p^3 = 0 \end{aligned} \right\} \dots\dots (102)$$

Constitutive equations (76) still hold in the present problem:

$$\left. \begin{aligned} s^{\alpha\beta} &= \frac{E}{2(1+\nu)} \left\{ g^{\alpha\lambda} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\lambda} + \frac{2\nu}{1-\nu} g^{\alpha\beta} g^{\lambda\delta} \right\} \gamma_{\lambda\delta} \\ s^{\alpha 3} &= \frac{E}{1+\nu} g^{\alpha\beta} \gamma_{\beta 3}, \quad s^{33} = 0 \end{aligned} \right\} \dots\dots\dots (76)$$

where $g^{\alpha\beta}$ is defined by

$$g^{\alpha\beta} = a^{\alpha\beta} + 2\theta^3 b^{\alpha\beta} + 3(\theta^3)^2 b^{\alpha\delta} b_{\delta}^{\beta} \dots\dots\dots (103)$$

It is self-evident from Eq. (82) and Eq. (83) that the strain tensors may be written as follows:

$$\left. \begin{aligned} \gamma_{\alpha\beta} &= \frac{1}{2} \left[\mu_{\alpha}^{\delta} \{ u_{\delta,\beta} - \Gamma_{\delta\beta}^{\epsilon} u_{\epsilon} - b_{\delta\beta} u_3 + \theta^3 (\psi_{\delta,\beta} \right. \\ &\quad \left. - \Gamma_{\delta\beta}^{\epsilon} \psi_{\epsilon}) \} + \mu_{\beta}^{\delta} \{ u_{\delta,\alpha} - \Gamma_{\delta\alpha}^{\epsilon} u_{\epsilon} - b_{\delta\alpha} u_3 \right. \\ &\quad \left. + \theta^3 (\psi_{\delta,\alpha} - \Gamma_{\delta\alpha}^{\epsilon} \psi_{\epsilon}) \} + u_{3,\alpha} u_{3,\beta} \right] \\ \gamma_{\alpha 3} &= \frac{1}{2} (\phi_{\alpha} + u_{3,\alpha}) \end{aligned} \right\} \dots\dots (104)$$

13. Concluding summary

So far in Chapter 2 to 8, a general nonlinear shell theory subject to finite deformation has been derived in terms of quantities defined in the reference state by using the direct method, and the physical meanings of the tensorial components have been clarified. It is briefly commented in Chapter 9 that the same results can be derived from the Hellinger-Reissner principle. In Chapter 10, the Donnell-Mushtari-Vlasov approximate shell theory in the reference state is obtained from the variational principle and compared with the partial nonlinear theory. We have, in Chapter 11, referred to the nonlinear membrane theory derived by the direct method and shown that it coincides with the theory obtained recently by Budiansky. The final Chapter is devoted to the practical application, an example of which is an asymmetric deformation problem of shells of revolution.

Finally, it should be remarked that Refs. (11), (12) and (13) were also useful guides for the present investigation.

References

- (1) V. V. Novozhilov: *Foundation of the Nonlinear Theory of Elasticity*, (1953), Albary.
- (2) L. M. Habip and I. K. Ebcioğlu: *Ing.-Arch.*, Vol. 34 (1965), p. 28.
- (3) L. M. Habip: *Ing.-Arch.*, Vol. 34 (1965), p. 228.
- (4) B. Budiansky: *Trans. ASME*, Ser. E, Vol. 35, No. 2 (1968), p. 393.
- (5) P. M. Naghdi: *Progress in Solid Mechanics*, Vol. 4 (1963), North-Holland.
- (6) A. E. Green and J. E. Adkins: *Large Elastic Deformations and Nonlinear Continuum Mechanics*, (1960), Oxford.
- (7) A. E. Green and W. Zerna: *Theoretical Elasticity*, (1954), Oxford.
- (8) K. Washizu: *Variational Methods in Elasticity and Plasticity*, (1968), Pergamon.
- (9) J. L. Sanders, Jr.: *Quart. Appl. Math.*, Vol. 21, No. 1 (1963), p. 21.
- (10) Y. Tsuboi et al.: *Applied Mathematics* (in Japanese), (1967), Corona.
- (11) P. M. Naghdi and R. P. Nordgren: *Quart. Appl. Math.*, Vol. 21, No. 1 (1963), p. 49.
- (12) Y. Yamamoto: *Proc. 7th Japan Nat. Congr. Appl. Mech.*, (1957), p. 129.
- (13) Y. Yoshimura: *Bulletin of the Aeronautical Research Institute, University of Tokyo*, Vol. 2, No. 2 (1960), p. 31 (in Japanese).

Discussion

M. SAITO: (1) Point out the problems confronting the shell investigation and show the

originality of this report in connection with it.

(2) Don't you think it is unjustifiable to regard

as a general shell theory the theory obtained by the authors which is constructed by the combination of the linear constitutive equation and geometrical nonlinearity. I think that which of nonlinearities can be neglected should be decided by the order estimation encountered in the practical problems.

(3) It is difficult to solve a practical problem by using the basic equations formulated in the Green's fashion. For example, in the Green's monograph, only the special case to which a separation of variables is applicable is treated. How do you think the basic equation of shells should be solved?

(4) What is your opinion about the ambiguity of the physical meanings and the difficulties of solving a problem when the convected coordinates are taken into account?

K. MIZOGUCHI: (5) Equation (54) is said to be the expression related to the surface couple. Indicate materially to what kind of the surface couples it is equivalent.

(6) On the occasion of solving a practical problem, how are Eqs. (50) and (54) dealt with; namely,

a) are the conditions of both upper and lower surfaces of shells used as boundary conditions?
or

b) as a known quantity, is it only substituted into the equilibrium equations?

If (b) is affirmative, the stress tensors are naturally determined as a function of the coordinates. Then, is it guaranteed that the quantity solved should coincide with the prescribed value? When these quantities are not identical (in my opinion, the theory suggests the possibility that both quantities do not coincide), it seems to be difficult to examine the accuracy of the theory unless the physical meaning of Eqs. (50) and (54) can be clarified.

(7) What does the boundary condition of Eq. (79) imply? Conjecturing from the term "boundary", it seems that there is no surface integral corresponding to both upper and lower surfaces of shells in Eq. (79). Give the reason why it could not be.

(8) If the approximate expression of Eq. (83) were used, could this strain tensors vanish as to a rigid motion?

(9) Don't you think it is difficult to apply the theory to practical use, if any techniques solving the unknown variables u_a , u_3 , ψ_a and ψ_3 under the prescribed boundary condition are not shown?

Authors' closure

(1) In the essential assumption that the shell thickness is small compared with the other dimensions, a shell theory is nothing but an approximation of the rigorous theory of three-dimensional elasticity.

Consequently, it is well-known that we often encounter, for example, such inconsistencies that uniform heating and rigid motion give rise to a certain strain. Such situation becomes more complicated when the nonlinear effect should be considered. Particularly, in the problems of shell stability that is very sensitive to which of nonlinear terms should be taken account of or not, it is important to choose appropriately the higher order terms in basic equation, and the previous nonlinear thin shell theories should be re-examined*. This is the reason why the investigation of shell theory, especially, the theory including the geometrically nonlinear effect is considered to be important, not to speak of practical requirements for the engineering purposes.

From this point of view, the authors, unsatisfied with the previous nonlinear shell theory in which the origin of geometrical nonlinearity was not definite, investigated the theory rigorously in the light of the reference state. Habip also discussed the problem by integrating the equation of the three-dimensional elasticity with respect to the independent variable in the direction of shell thickness. By this procedure, however, it seems to be difficult to understand the physical meaning of the quantities. The authors derived the same theory by the "direct method" referring to the shell element itself which was acceptable intuitively and, at the same time, revealed physical meanings of the quantities. They also obtain some approximate theory and nonlinear membrane theory by rearranging the present general result. Finally, the authors would like to remark that their work was promoted by Habip's papers.

(2) The authors think it might be so. But one of the purposes of this paper does not consist in the investigation of the physical nonlinearity but of the geometrical nonlinearity. Naturally, if the nonlinear constitutive equation is taken account of, generality of the theory is lost. Even if it is not so, the linear constitutive equation is very important in the practical problem even in the anelastic problem, if the equations are used in incremental form.

(3) The authors are going to use a numerical procedure. The investigation on the buckling problem of shallow spherical shells carried out by them** could be referred to.

(4) The authors do not think that the deriving process is troublesome but it is true that the physical meanings are very difficult to understand.

* There appeared some papers concerning it. For example, see L. J. Hart-Smith: *Int. Jour. Mech. Sci.*, Vol. 10 (1968), p. 665.

** To be published. Presented at the 11th Conference on Strength of Structures, July, 1969, Japan Society of Mechanical Engineers and Japan Society for Aeronautical and Space Sciences. See the preprint, p. 61.

(5) c^3 corresponds physically to the component of the surface couple in the direction of $A_3 \times a_3$. Since it is related simultaneously to both base vectors after and before deformation, it is difficult to understand as a vectorial quantity in the three-dimensional space.

(6) It should be substituted into the equilibrium equation as a prescribed quantity. Consequently, as stated in the question (6), both quantities do not always agree with each other. This inconsistency must be attributed to the essential assumption of thin shells stated in the discussion (1) and be inevitable. It should be noted that the Kirchhoff-Love assumption on which the most of thin shell theories is based, is a kind of physical inconsistencies in the sense that it is ingeniously used in place of the above-mentioned mathematical inconsistency.

(7) The boundary now discussed means the

edge boundary of shells. The terms corresponding to the conditions of both shell surfaces are included in the term of the volume integral of the first term in Eq. (79).

(8) It could not vanish even in linearized form.

(9) The debater could refer to the authors' investigation stated in Discussion (3) as an example about how to solve a practical problem.

Particular thanks are due to Prof. T. Nakamura of Kyoto Univ. who pointed out that the work done by W. T. Koiter* was not referred to in this report and the authors would express their regrets at overlooking it throughout the investigation and having no time to discuss about it.

---

* W. T. Koiter: Proc. Koninkl. Nederl. Akademie van Wetenschappen, Series B, Vol. 69, No. 1 (1966), p. 1.