On the Number of Conditions in the Syntheses of Spatial Four-Bar Mechanisms*

By Hiroaki Funabashi**, Kiyoshi Ogawa***, and Hajime Endo****

This paper aims to make the analyses and syntheses of many spatial four-bar mechanisms possible. On the spatial four-bar mechanisms with a single degree of freedom between the driver and follower links which have pairs of one or two degrees of freedom on the fixed links, the constants which determine the form and position of each mechanism are first indicated, and the relations of the constants are analysed. Then the relations are made clear between number of characteristic constants in mechanisms and the synthetic conditions, namely, number of precision points, order of differential coefficients on the displacement curves and the constants of mechanisms given prior to syntheses. The conditions to connect the driver and follower links with the coupler links are analysed in relation to many combinations of pairs on the couplers. From the equations of these conditions, the displacement equations of all mechanisms may be easily obtained.

1. Introduction

Link mechanisms are divided into two main classes; planar and spatial ones. In spatial mechanisms, the driving and driven shafts or links can be freely arranged in three-dimensional space, and in comparison with planar mechanisms there exist many more parameters which determine the forms and positions of mechanisms, therefore the desired motions are more precisely performed, and mechanisms may be often arranged in small structure. On account of these advantages the analyses*(1)(2) and the syntheses*(3)~*(8) of spatial mechanisms have been actively researched in recent years with aid of electronic computers.

Spatial linkages are constructed with pairs having 1~5 degrees of freedom, and twenty kinds of practical pairs are known*(9). Therefore there are many more kinds of spatial linkages than those of planar ones. For example there are 95 kinds of spatial four-bar mechanisms which are constructed of revolute, prismatic, cylindric, spheric or sphere-groove pairs and have a single degree of freedom between the driver and follower links. In the traditional syntheses of spatial mechanisms, however, only several mechanisms were treated, and displacement equations between the driver and follower of each mechanism were individually derived.

In the present paper, with the purpose of making the analyses and syntheses of many spatial mechanisms possible, the quantities which determine a form and position are shown for each spatial four-bar mechanism with a single degree of freedom having revolute, prismatic or cylindric pairs on the frame, and the quantities are examined if they are characteristic constants of the mechanism or variables dependent on motion of the mechanism. Moreover the equations for these quantities are derived, and at the same time the relation between number of conditions in synthesizing a function generator, namely number of precision points, the order of differential coefficients given for the displacement equation and the characteristic constants of mechanism given prior to synthesis, is made clear.

In the paper the constants which determine the dimensions of a mechanism such as lengths of links, angles between shafts are called
"characteristic constants of mechanism".

2. Quantities determining forms and positions of mechanisms

In Fig. 1, \( J_1 \) and \( J_2 \) are revolute, prismatic or cylindrical pairs and \( J_2 \) and \( J_3 \) represent revolute, prismatic, cylindric, spheric and spherengroove pairs. Each pair is assumed to have rotational and/or translational axes. Then the forms and positions of spatial four-bar mechanisms are generally determined by the following quantities.

\( a_1 \sim a_4 \): perpendicular distances between rotational and/or translational axes of neighbouring pairs

\( b_1 \sim b_4 \): distances between common perpendiculars to the axes of pairs

\( \beta_1 \sim \beta_4 \): angles between the axes of pairs

\( \theta_1, \theta_4 \): angles which the driver and follower links (link \( \vec{a}_1 \) and \( \vec{a}_4 \)) make with \( Z-X \) plane

\( \theta_2, \theta_3 \): angles which coupler link \( \vec{a}_2 \) makes with links \( \vec{a}_2 \) and \( \vec{a}_4 \).

In the figure, \( Y \)-axis is the common perpendicular to the axes of pairs \( J_1 \) and \( J_4 \), and \( Z \)-axis is the line which is parallel to the axis of pair \( J_4 \) and passes through the intersection point of \( Y \)-axis and the axis of pair \( J_1 \) and \( X \)-axis crosses in perpendicular with \( Y \)- and \( Z \)-axis. Putting the fundamental vectors of \( X \), \( Y \) and \( Z \) axes as \( e_1 \), \( e_2 \) and \( e_3 \) respectively, then the vectors in the figure are expressed as follows.

\[
\begin{align*}
\vec{a}_1 &= a_1 e_3 \\
\vec{a}_2 &= a_2 e_1 + a_2 e_2 + a_2 e_3 \\
\vec{a}_3 &= a_3 e_1 \\
\vec{b}_2 &= b_2 e_2 + b_2 e_3 \\
\vec{b}_3 &= b_3 e_3
\end{align*}
\]

in which

\[
\begin{align*}
a_{21} &= a_2 \cos \beta_1 \cos \theta_1 \\
&= \sin \theta_1 \\
a_{23} &= -a_2 \sin \beta_1 \cos \theta_1 \\
a_{41} &= a_4 \cos \theta_4 \\
a_{42} &= a_4 \sin \theta_4 \\
a_{43} &= 0 \\
b_{11} &= b_1 \sin \beta_1 \\
b_{12} &= 0 \\
b_{13} &= b_1 \cos \beta_1 \\
b_{21} &= -b_2 \cos \beta_2 \sin \theta_1 + b_2 \sin \beta_1 \cos \beta_2 \\
b_{22} &= b_2 \sin \beta_2 \cos \theta_1 \\
b_{23} &= b_2 \sin \beta_1 \sin \beta_2 \sin \theta_1 + b_2 \cos \beta_1 \cos \beta_2 \\
b_{31} &= -b_3 \sin \beta_1 \sin \theta_4 \\
b_{32} &= b_3 \sin \beta_3 \cos \theta_4 \\
b_{33} &= b_3 \cos \beta_4
\end{align*}
\]

Putting the position vectors of pairs \( J_2 \) and \( J_3 \) as

\[
\begin{align*}
\vec{O}_2 &= \vec{R}_2 - r_{21} e_1 + r_{22} e_2 + r_{23} e_3 \\
\vec{O}_3 &= \vec{R}_3 - r_{31} e_1 + r_{32} e_2 + r_{33} e_3
\end{align*}
\]

the following equation is derived from the figure.

\[
\begin{align*}
\vec{R}_2 &= b_1 + a_2 + a_2 \\
\vec{R}_3 &= a_1 + b_4 + a_4 + b_4
\end{align*}
\]

Therefore the coordinates of points \( J_2 \) and \( J_3 \) are expressed as follows.

\[
\begin{align*}
r_{21} &= b_1 \sin \beta_1 + a_2 \cos \beta_1 \cos \theta_1 \\
&= b_2 \cos \beta_1 \sin \beta_2 \sin \theta_1 + b_2 \sin \beta_1 \cos \beta_2 \\
r_{32} &= a_1 \sin \beta_1 + b_2 \sin \beta_2 \cos \theta_1 \\
r_{33} &= a_1 \cos \theta_1 + b_2 \sin \beta_1 \sin \beta_2 \cos \theta_1 + b_2 \cos \beta_1 \cos \beta_2 \\
r_{31} &= a_3 \cos \theta_1 - b_3 \sin \beta_1 \sin \theta_4 \\
r_{32} &= a_3 \sin \theta_4 + b_3 \sin \beta_1 \sin \beta_2 \cos \theta_4 \\
r_{33} &= b_3 + b_4 \cos \beta_4
\end{align*}
\]

Now, connecting the points \( J_2 \) and \( J_3 \) by coupler link \( \vec{a}_2 \) under the constraints of pairs \( J_2 \) and \( J_3 \), the relation between the coordinates of points \( J_2 \) and \( J_3 \) is expressed as

\[
\vec{a}_2 = \vec{R}_2 - \vec{R}_3
\]

or

\[
a_{2j} = r_{2j} - r_{3j} \quad (j = 1, 2, 3)
\]

3. Condition of connection of driver and follower with coupler and displacement equation

The conditions to connect the driver and follower links with the coupler link are now discussed in relation to the combination of pairs on the coupler. The mechanism shown in Fig. 2 is taken as an example. \( J_1 \) is a revolute pair and the others are cylindrical pairs. As the distance between cylindrical pairs \( J_2 \) and \( J_3 \) is constant, the next equation holds.

\[
J_1 J_3 = |\vec{R}_3 - \vec{R}_2| = a_3
\]

As coupler link \( \vec{a}_2 \) intersects with the axes of

![Fig. 1 Spatial four-bar mechanism](image-url)
pairs \( J_2 \) and \( J_3 \) at right angles, the next equations hold.

\[
(d_3, b_2) = (d_3, b_2) = 0
\]  

Moreover as the angle between the axes of pairs \( J_2 \) and \( J_3 \) is \( \beta_1 \), the next equation holds.

\[
(b_2, b_3) = \cos \beta_1
\]  

These equations are the conditions to connect the driving and driven links with the coupler link for the mechanism shown in Fig. 2. In the above equations the symbols capped with \( \wedge \) indicate unit vectors.

The above equations are concretely expressed as follows.

\[
\begin{align*}
(r_{21} - r_{31})^2 + (r_{22} - r_{32})^2 + (r_{23} - r_{33})^2 &= a_5^2 \\
&= 0
\end{align*}
\]  

\[
\begin{align*}
&= a_3 a_5 + a_3 b_2 + a_3 b_3 = 0 \\
&= a_3 b_1 + a_3 b_2 + a_2 b_3 = 0 \\
&= b_1 b_1 + b_2 b_2 + b_3 b_3 = b_2 b_3 \cos \beta_1
\end{align*}
\]  

The conditions to connect the driver and follower with the coupler are generally listed in Table 1 in relation to kinds of pairs \( J_2 \) and \( J_3 \). The symbols in the table express kinds of pairs, namely,

- \( R \): Revolute pair
- \( P \): Prismatic pair
- \( C \): Cylindric pair
- \( S \): Spheric pair
- \( S_g \): Sphere-groove pair

These symbols shall be also used to identify a mechanism. For example the mechanism shown in Fig. 2 is designated as \([R-C-C-C]\) and the mechanism in Fig. 3 is named as \([R-S-S-R]\) because pairs \( J_1 \) and \( J_4 \) are revolute and \( J_2 \), \( J_3 \) are spheric pairs. The symbols at both ends of the designator express the pairs on the frame; the pair of the driver shall be placed at the left end and the pair of the follower at the right when the driver and follower are known. Therefore the mechanism shown in Fig. 4 is designated as

\[
\begin{align*}
&= S-R-S \quad S-S \quad R-R \\
&= P-S \quad C-S_g
\end{align*}
\]

The double line = means that there is an idle freedom between \( S \) and \( S \); the coupler between pairs \( S \) and \( S \) can rotate about its axis freely and independently of the motion of the whole mechanism. In Table 1, two cases of \( C-S \) and \( C=S \) are listed for the combination of cylindric and spheric pairs and the latter shows a case of containing an idle freedom between \( C \) and \( S \) and having the axis of the cylindric pair passing

![Fig. 2](image)

![Fig. 3](image)

![Fig. 4](image)
through the center of the spheric pair.

Now the relations between the characteristic constants and the variables of mechanisms are derived from the conditions listed in Table 1 and Eqs. (1)~(8). Moreover variables except displacements of the driver and follower may be eliminated from the equations just as for mechanisms with a single degree of freedom. Thus the displacement equations between driver and follower may be obtained. For example the displacement equation of \( RSCC \) mechanism shown in Fig. 2, the relation between angular displacement \( \theta_1 \) and \( \theta_4 \) of links \( a_2 \) and \( a_4 \), is obtained by substituting Eq. (2) into the bottom equation of Eq. (12) as follows.

\[
A(\theta_1) \sin \theta_4 + B(\theta_1) \cos \theta_4 = C(\theta_1)
\]

in which

\[
A(\theta_1) = \sin \beta_4 (\cos \beta_1 \sin \beta_2 \sin \theta_1 - \sin \beta_1 \cos \beta_2)
\]
\[
B(\theta_1) = \sin \beta_3 \sin \beta_4 \cos \beta_1
\]
\[
C(\theta_1) = -\sin \beta_1 \sin \beta_3 \sin \beta_4 \sin \theta_1 - \cos \beta_1 \cos \beta_2 \cos \beta_4 + \cos \beta_3
\]

The displacement equation of \( RSCC \) mechanism shown in Fig. 3 shall be derived as another example. Comparing Fig. 1 and Fig. 3 the next equations are obtained.

\[
a = a_2, \quad c = a_1, \quad d = a_4
\]
\[
l = b_3, \quad m = a_2, \quad n = b_3
\]
\[
\alpha = 90^\circ - \beta_1, \quad \theta = \theta_1, \quad \phi = \theta_4
\]

The conditions to connect the driver and the follower with \( S = S \) are obtained from Table 1 as follows.

\[
J_3J_5 = a_2, \quad b_2 = b_3 = 0
\]

Substituting Eqs. (15) and (16) into Eq. (5), we obtain

\[
\begin{align*}
r_{21} &= l \cos \alpha + b \sin \alpha \cos \theta \\
r_{22} &= b \sin \theta \\
r_{23} &= l \sin \alpha - b \cos \alpha \cos \theta
\end{align*}
\]

and

\[
\begin{align*}
r_{31} &= d \cos \phi \\
r_{32} &= d \sin \phi + m \\
r_{33} &= n
\end{align*}
\]

Substituting Eqs. (17) and (18) into the first equation of Eq. (16), namely

\[(r_{21} - r_{31})^2 + (r_{22} - r_{32})^2 + (r_{23} - r_{33})^2 = a_2^2 = c^2
\]

the next equation is obtained.

\[
A(\theta) \sin \phi + B(\theta) \cos \phi = C(\theta)
\]

in which

\[
A(\theta) = d \sin \theta - m
\]
\[
B(\theta) = d \sin \alpha \cos \theta + l \sin \alpha
\]
\[
C(\theta) = -bm \sin \theta - ln \sin \alpha + (b^2 + d^2 + B^2 + m^2 + n^2)/2
\]

Equation (19) is the displacement equation of \( RSCC \) mechanism of Fig. 3 and it coincides with Eq. (4) in Reference (8).

4. Number of characteristic constants of mechanism and the synthetic conditions

Though the form and position of a mechanism can be expressed by the quantities shown in Fig. 1, it depends on kinds of mechanisms which of those quantities are the characteristic constants or which are the variables.

\[
(\text{RSCC})\text{ mechanism shown in Fig. 2 is taken up again as an example. As } J_1 \text{ is a revolute pair and the others are cylindric pairs, there are nine quantities which do not vary with the motion of the mechanism, namely } a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, \text{ and } b_5. \text{ Consider the mechanism as a function generator and put}
\]

\[
\begin{align*}
\theta_3 &= \theta_1 + \theta \\
\theta_4 &= \theta_1 + \phi \\
b_4 &= b_1 + Z
\end{align*}
\]

in which constants \( \theta_1, \theta_4 \) and \( b_4 \) are called the initial conditions of the driver and follower, and satisfy the function \( f(\theta, \phi, Z) = 0 \) generated by \( \theta, \phi \) and \( Z \) coincident with the ideal or desired function so that \( \theta_1, \theta_4 \) and \( b_4 \) are the quantities which determine the displacement curve of the mechanism and they may be considered as the characteristic constants of the mechanism. Therefore the total number of characteristic constants becomes 9+3=12.

Such an investigation has been carried out with 44 kinds of mechanisms whose pairs on the frame are revolute, prismatic or cylindric and which have a single degree of freedom between the driver and follower. The result is listed in columns of “Variable” and “Characteristic constant” in Table 2.

In the case where the pair \( J_1 \) on the frame is prismatic, for example, \( \theta_1 \) and \( a_1 \) may be put 90 deg and 0 respectively without losing the generality. Therefore the number of characteristic constants of mechanisms having a prismatic pair on the frame decreases by two. Such relation is shown in the column of “Other condition” in Table 2.

Though the number of characteristic constants varies from 5 to 13 depending on the kinds of mechanisms, many of them are between 8 and 11. These numbers are fairly large in comparison with the number of 5~6 of the planar four-bar mechanisms.

In order to design a mechanism which generates an ideal function, the next method seems to be widely adopted: the precision points and/or
<table>
<thead>
<tr>
<th>Type</th>
<th>Mechanism</th>
<th>No. of links</th>
<th>No. of joints</th>
<th>Variable</th>
<th>Characteristic constant</th>
<th>Condition to connect variable except input and output</th>
<th>Condition for other constant</th>
<th>Other condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>R-R-S₁-R₁</td>
<td>12</td>
<td>11</td>
<td>b₁, θ₁, b₂, θ₂, b₃, θ₃</td>
<td>a₁, a₂, a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>0 = a₁, 0 = a₂, 0 = a₃</td>
</tr>
<tr>
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<td>R-P-S₁-R₁</td>
<td>12</td>
<td>11</td>
<td>b₁, θ₁, b₂, θ₂, b₃, θ₃</td>
<td>a₁, a₂, a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>0 = a₁, 0 = a₂, 0 = a₃</td>
</tr>
<tr>
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<td>P-R-S₁-R₁</td>
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<td>10</td>
<td>b₁, θ₁, b₂, θ₂, b₃, θ₃</td>
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<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>0 = a₁, 0 = a₂, 0 = a₃</td>
</tr>
<tr>
<td>A₁</td>
<td>P-P-S₁-R₁</td>
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<td>10</td>
<td>b₁, θ₁, b₂, θ₂, b₃, θ₃</td>
<td>a₁, a₂, a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>0 = a₁, 0 = a₂, 0 = a₃</td>
</tr>
<tr>
<td>A₁</td>
<td>R-R-S₁-P₁</td>
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<td>10</td>
<td>b₁, θ₁, b₂, θ₂, b₃, θ₃</td>
<td>a₁, a₂, a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>0 = a₁, 0 = a₂, 0 = a₃</td>
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<td>R-P-S₁-P₁</td>
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<td>10</td>
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<td>a₁, a₂, a₃</td>
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<td>8</td>
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<tr>
<td>A₁</td>
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<td>8</td>
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<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>0 = a₁, 0 = a₂, 0 = a₃</td>
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<td>9</td>
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<td>a₁, a₂, a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
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</tr>
<tr>
<td>A₂</td>
<td>R-S-C₂-R₂</td>
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<td>8</td>
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<td>b₁ = a₁, b₂ = a₂, b₃ = a₃</td>
<td>0 = a₁, 0 = a₂, 0 = a₃</td>
</tr>
</tbody>
</table>

| A₂  | P-S-C₁-P₁ | 6            | 6             | b₁, b₂, b₃, θ₁, b₄ | a₂, a₃ | b₁ = a₂, b₂ = a₃ | b₁ = a₂, b₂ = a₃ | 0 = a₂, 0 = a₃ |
| A₂  | P-S-C₁-P₂ | 6            | 6             | b₁, b₂, b₃, θ₁, b₄ | a₂, a₃ | b₁ = a₂, b₂ = a₃ | b₁ = a₂, b₂ = a₃ | 0 = a₂, 0 = a₃ |
| A₂  | P-S-C₂-P₁ | 6            | 6             | b₁, b₂, b₃, θ₁, b₄ | a₂, a₃ | b₁ = a₂, b₂ = a₃ | b₁ = a₂, b₂ = a₃ | 0 = a₂, 0 = a₃ |
| A₂  | R-S-S₁-R₁ | 9            | 8             | b₁, θ₁, b₂, θ₂, b₃, θ₃ | a₁, a₂, a₃ | b₁ = a₁, b₂ = a₂, b₃ = a₃ | b₁ = a₁, b₂ = a₂, b₃ = a₃ | 0 = a₁, 0 = a₂, 0 = a₃ |
| A₂  | P-S-S₁-R₁ | 5            | 5             | b₁, θ₁, b₂, θ₂, b₃, θ₃ | a₁, a₂, a₃ | b₁ = a₁, b₂ = a₂, b₃ = a₃ | b₁ = a₁, b₂ = a₂, b₃ = a₃ | 0 = a₁, 0 = a₂, 0 = a₃ |
| A₃  | R-R-S₁-C₁ | 11           | 11            | b₁, b₂, θ₁, b₃, θ₃ | a₁, a₂, a₃ | b₁ = a₁, b₂ = a₂, b₃ = a₃ | b₁ = a₁, b₂ = a₂, b₃ = a₃ | 0 = a₁, 0 = a₂, 0 = a₃ |
| B₁  | R-P-S₁-C₁ | 9            | 9             | b₁, b₂, θ₁, b₃, θ₃ | a₁, a₂, a₃ | b₁ = a₁, b₂ = a₂, b₃ = a₃ | b₁ = a₁, b₂ = a₂, b₃ = a₃ | 0 = a₁, 0 = a₂, 0 = a₃ |

Note: The table represents various mechanisms with their corresponding variables and conditions. The conditions include the lengths of link and the angles between axes, etc., and are indicated by mathematical expressions and parameters.
<table>
<thead>
<tr>
<th>Code</th>
<th>B_1</th>
<th>B_2</th>
<th>B_3</th>
<th>E_1</th>
<th>E_2</th>
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<td>9</td>
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<td>b_2</td>
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<td>(9)</td>
<td>b_1</td>
<td>b_2</td>
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<td>R-S-P-C</td>
<td>11</td>
<td>10</td>
<td>(9)</td>
<td>b_1</td>
<td>b_2</td>
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<td>P-S-R-C</td>
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the differential coefficients at the points of
the ideal function are first substituted into the
displacement equations and/or their derivatives
in which the characteristic constants are con-
sidered as unknowns, and then the characteristic
constants are determined by solving these sim-
ultaneous equations\(^{(9)}\). Therefore the number
of synthetic conditions must be equal to the
number of characteristic constants of the mech-
anism. In the case where all of the synthetic
conditions are given as precision points on the
ideal displacement curve, the number of preci-
sion points must be equal to the number of
characteristic constants. In the case where
some of synthetic conditions are given as differ-
cential coefficients of the displacement curve,
the maximum order of the coefficient is equal to the
number of characteristic constants minus one,
because the coefficients of order 1, 2, ..., \(N-1\)
must be given in order to give a coefficient of
order \(N\).

The relation just mentioned may be expres-
sed by the following equation.

\[
C = K + U = K + \sum_{N=0,1,2,\ldots} M_N (N+1) \cdots (22)
\]

in which

- \(C\): number of characteristic constants of
  mechanism
- \(K\): number of characteristic constants given
  prior to synthesis (number of known
  constants)
- \(M_N\): number of precision points at which dif-
  ferential coefficients of order 1, 2, ..., \(N\)
  are given as synthetic conditions
- \(N\): the highest order of differential coeffi-
  cients given at each precision point; \(N=0\)
  in the case where differential coefficients
  are not given
- \(U\): number of characteristic constants to be
determined (number of unknowns)

The maximum number of precision points and
the maximum order of differential coeffi-
cients are listed in Table 2 for each mechanism.
In the case where both input and output are
angular quantities, length of a link may be
arbitrarily determined because the size of a
mechanism has no influence on the angular dis-
placement. Therefore the maximum number of
precision points and the maximum order of dif-
ferential coefficients become less than the number
of characteristic constants by one and two
respectively.

Then for mechanisms having a cylindric
pair on the frame, the maximum number of
precision points and the maximum order of dif-
ferential coefficients vary depending on whether

the displacement distance of the cylindric pair
is adopted as input or output or only angular
displacement is adopted. For example, as
the number of characteristics of \(RCSC\)
mechanism is 11, when the displacement distance
of the cylindric pair is adopted as input or output,
the maximum number of precision points is
11 and the maximum order of differential coeffi-
cients is \(11-1=10\), and when only angular dis-
placement is adopted as input or output, the
maximum number of precision points is \(11-1=10\)
and the maximum order of differential coeffi-
cients is \(11-2=9\). When two combinations of
the maximum number of precision points and
the maximum order of differential coefficients
are allowable as just mentioned, the number of
them in case of only angular displacements be-
ing adopted as input and output is listed in
parentheses in Table 2.

Now numerical relations among precision
points, order of differential coefficients and
characteristic constants given prior to synthesis
may be exactly grasped from Table 2 and Eq.
(22). Consider for example the case where
\(RCSC\) mechanism shown in Fig. 2 is used
as a function-generator. When the input and
output are both angular, the maximum number
of precision points and the maximum order of
differential coefficients are found from Table 2
to be 11 and 10 respectively. Assume that the
mechanism will be synthesized under the condi-
tion shown in Fig. 5. As the condition is ex-
pressed as \(C=12, M_2=2, M_1=1\) and \(M_3=1\) in Eq.
(22), the next equation is obtained.

\[
12 = K + 2 \times (2+1) + 1 \times (1+1) + 1 \times (0+1)
\]
\[
\therefore K = 3
\]

Therefore it is known that three characteristic
constants of a mechanism should be given prior
to the synthesis. It is also known that at least
one of the three constants must be a constant
which can be a unit of dimension of the mech-
nism such as length of a link, distance between
axes of pairs, etc., because both of input and

![Fig. 5 An example of conditions of synthesis for an ideal function](image-url)
output are angular.

5. Conclusions

In order to make analyses and synthesises of spatial mechanisms possible, the quantities which determine the forms and positions of mechanisms have been discussed for forty-four spatial four-bar mechanisms which have pairs of one or two degrees of freedom on fixed links and have a single degree of freedom between the driver and follower links and the relations among the quantities have been derived, and numerical relations between the characteristic constants of mechanism and conditions under which mechanisms are synthesized have also been analysed.

The conclusions are summarized as follows.

1. The forms and the positions of spatial four-bar mechanisms which have revolute, prismatic and/or cylindric pair on the frame and have these pairs, spheric and/or sphere-groove pairs on the coupler link are determined by the perpendicular distances and angles between axes of neighbouring pairs, distances between common perpendiculars to axes of pairs, and angles which the driver and follower links make with the coupler link and with Z-X plane in Fig. 1. These quantities have been examined if they are the characteristic constants of mechanisms or variables dependent on positions of the driver link, the results being listed in Table 2. Though the number of characteristic constants ranges between 5 and 13, many of them are between 8 and 11, which are fairly large in comparison with numbers of 5–6 of the planar four-bar mechanisms.

2. Equation (22) can express the relation between number of characteristic constants of mechanism and number of conditions in dimensional synthesises, namely number of precision points, order of differential coefficients of displacement equations and number of characteristic constants given prior to synthesises. Therefore the relation of numerical conditions in synthesises may be exactly determined from Table 2 and Eq. (22).

3. The maximum number of precision points and the maximum order of differential coefficients of displacement equations are listed in Table 2. The maximum number of precision points is equal to number of the characteristic constants of mechanisms when one or both of input and output are given in length, and it is equal to number of the constants minus one when both of input and output are angular. The maximum order of differential coefficients is equal to the maximum number of precision points minus one.

4. The conditions to connect the driver and follower with the coupler are listed in Table 1 and 2 in relation to combinations of pairs on the coupler. The displacement equations of the driver and follower may be derived from these conditions and Eqs. (1) ~ (8). Therefore, on analysing or synthesizing spatial four-bar mechanisms, the displacement equations may be easily obtained.

Conditions for variables except input and output such as pressure angles are also listed in Table 2.

References