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Hydrodynamic Lubrication of Journal Bearings by Pseudo-Plastic Lubricants*

(Part 1, Theoretical Studies)

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It is well known that lubricating oils, with viscosity index improver added, exhibit the same behavior as pseudo-plastic fluids. Though the theory of hydrodynamic lubrication flow for non-Newtonian fluids of this type is important, it has not been developed thoroughly due to difficulties involved in analytical and experimental treatments.

The modified Reynolds equation introducing a "nonlinear factor" quantity is presented in this paper. This equation is solved by the perturbation method in the case of a small nonlinear factor and by a digital computer in the case of a large one. Applying this theory to journal bearings, the relation between the nonlinearity of lubricants and the performance of bearings, such as film pressure, load capacity, frictional force and attitude angle, can be clarified.

1. Introduction

In recent years, hydrodynamically lubricated bearings have been used under ever more severe conditions of high speed and heavy load. Under light load and high speed the air is used as a lubricant because of the small viscous resistance. On the other hand, under heavy load and high speed, oils containing additives are used.

The viscosity of lubricating oils decreases with an increase of temperature. Present-day lubricating oils contain a large quantity of high molecular-weight polymers as the viscosity index improver in order to prevent viscosity variation with temperature change. It is known that such polymer-thickened oils have the same behavior as pseudo-plastic fluids. Though favorable results of investigation on hydrodynamic lubrication for non-Newtonian fluids of this type are important, virtually no progress has been made due to difficulties involved in analytical and experimental treatments.

Ng and his co-worker⁽¹⁾ analyzed these non-Newtonian fluids by approximating the shear stress with a cubic equation of the shear rate. On the other hand, by approximating the shear

rate with a cubic equation of the shear stress, Hsu and his co-worker⁽²⁾⁽³⁾ reported analytical treatments of infinitely long slider bearings and journal bearings. However, the derivation of the modified Reynolds equation proposed by them is somewhat doubtful.

The authors' research determined whether the flow characteristics of oils containing commercial viscosity index improver could be derived from the flow characteristics equation that Hsu and his co-worker suggested. The modified Reynolds equation derived in this paper is nonlinear, while the equation introduced by Hsu and his co-worker is linear, in spite of the same flow characteristics. The modified Reynolds equation was solved for journal bearings and then the performance of infinitely long, infinitely short and finite bearings was studied. Since the influences of the non-Newtonian characteristics were clarified by these studies, these analytical results are presented in this paper. The experimental results will be reported in a second paper.

2. Nomenclature

B : length of slider surface in sliding direction
 C : radial clearance
 D : diameter of journal
 F : frictional force

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- L : bearing width
 N : revolutions per time unit
 P : bearing pressure
 Q : rate of oil flow
 R : radius of journal
 S : Sommerfeld number
 U : velocity of slider surface in x -direction, or circumferential velocity of journal
 V : velocity of slider surface in y -direction
 W : load capacity
 f : coefficient of friction
 h : film thickness
 k : "coefficient of pseudo-plasticity"
 p : film pressure
 t : time
 u, v, w : velocity
 x, y, z : coordinate
 γ : shear strain
 δ : "nonlinear factor"
 ε : eccentricity ratio
 θ : bearing angle
 μ : "initial viscosity"
 τ : shear stress
 ϕ : attitude angle
 ψ : bearing angle in Sommerfeld transformation

Subscripts

- $1, 2$: refer to surface of $y=0$ and $y=h$, or refer to inlet and outlet
 j, b : refer to journal and bearing
 $\wedge$: dimensional quantity

The non-dimensional quantities used in this paper are given in Table 1 and Table 2. The $\hat{U}$ shown in Table 1 is the larger of velocities

Table 1 Dimensionless quantities in a slider bearing

$x = \hat{x}/\hat{B}$
$y = \hat{y}/\hat{h}_2$
$z = \hat{z}/\hat{B}$
$h = \hat{h}/\hat{h}_2$
$u = \hat{u}/\hat{U}$
$v = \frac{\hat{B}}{\hat{h}_2} \frac{\hat{v}}{\hat{U}}$
$w = \hat{w}/\hat{U}$
$\delta = \hat{k} \left(\frac{\hat{\mu} \hat{U}}{\hat{h}_2} \right)^2$
$\tau = \frac{\hat{h}_2}{\hat{\mu} \hat{U}} \hat{\tau}$
$p = \frac{\hat{h}_2^2}{\hat{\mu} \hat{U} \hat{B}} \hat{p}$

Table 2 Dimensionless quantities in a journal bearing

$\theta = \hat{x}/\hat{R}$
$y = \hat{y}/\hat{C}$
$z = \hat{z}/\hat{L}$
$h = \hat{h}/\hat{C}$
$\delta = \hat{k} \left(\frac{\hat{\mu} \hat{U}}{\hat{C}} \right)^2$
$\tau = \frac{\hat{C}}{\hat{\mu} \hat{U}} \hat{\tau}$
$p = \frac{\hat{C}^2}{\hat{\mu} \hat{U} \hat{R}} \hat{p}$
$W = \frac{\hat{C}^2}{\hat{\mu} \hat{U} \hat{L} \hat{R}^2} \hat{W}$
$F = \frac{\hat{C}}{\hat{\mu} \hat{U} \hat{L} \hat{R}} \hat{F}$
$f = \frac{\hat{R}}{\hat{C}} \frac{\hat{F}}{\hat{W}}$
$Q = \frac{1}{\hat{C} \hat{U} \hat{R}} \hat{Q}$

$\hat{U}_1$ and $\hat{U}_2$.

3. Derivation of the modified Reynolds equation

3.1 Flow characteristics of pseudo-plastic fluid

It is well known that lubricating oils, with viscosity index improver added, exhibit the same characteristics as pseudo-plastic fluids. The flow characteristics of such lubricants are shown in Fig. 1. From the flow curve, the relationship between the shear stress and the shear rate may be approximated by the following equation.

$$\hat{\tau} + \hat{k} \hat{\tau}^3 = \hat{\mu} \frac{d\hat{\gamma}}{d\hat{t}} \quad \dots\dots\dots (1)$$

Since $\hat{\mu}$ is tangent at the original point of the flow curve shown in Fig. 1, $\hat{\mu}$ is called "initial viscosity" in this paper. If values of $\hat{\mu}$ do not vary, the nonlinearity of the flow curve increases with the value of $\hat{k}$, which is called "coefficient of pseudo-plasticity". In pseudo-plastic fluids, $\hat{k} > 0$, and, in Newtonian fluids, $\hat{k} = 0$. Therefore, in Newtonian fluids, the initial viscosity becomes the viscosity by Newton's law.

3.2 Derivation of the modified Reynolds equation

Let us consider the slider surfaces shown in Fig. 2. After substituting $d\hat{\gamma}/d\hat{t} = \partial\hat{u}/\partial\hat{y}$ into Eq. (1), nondimensionalizing is accomplished by the methods shown in Table 1.

$$\partial u / \partial y = \tau + \delta \tau^3 \quad \dots\dots\dots (2)$$

Fig. 1 Flow curve

Fig. 2 Geometry of sliding surfaces

From the relationship between the shear stress and the film pressure,

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} \quad (3)$$

Since the change of the film pressure in the direction of the film thickness can be ignored, integrating the above equation results in

$$\tau = \frac{\partial p}{\partial x} y + \tau_1 \quad (4)$$

Substituting Eq. (4) into Eq. (2), and using boundary conditions $u(y=0)=U_1$ and $u(y=h)U_2$,

$$u = \frac{y^2}{2} \frac{\partial p}{\partial x} + \tau_1 y + \delta \left\{ \frac{y^4}{4} \left(\frac{\partial p}{\partial x} \right)^3 + \tau_1 y^3 \left(\frac{\partial p}{\partial x} \right)^2 + \frac{3}{2} \tau_1^2 y^2 \frac{\partial p}{\partial x} + \tau_1^3 y \right\} + U_1 \quad (5)$$

where integration constant τ_1 is evaluated from

Substituting Eq. (7) into Eq. (5) and simplifying results in

$$u = \frac{1}{2} \frac{\partial p}{\partial x} y(y-h) + \delta \left(\frac{\partial p}{\partial x} \right)^3 \left(\frac{y^4}{4} - \frac{h}{2} y^3 + \frac{3}{8} h^2 y^2 - \frac{h^3}{8} y \right) + \frac{U_1 - U_2}{h \left\{ 1 + \delta \left(\frac{h}{2} \frac{\partial p}{\partial x} \right)^2 \right\}} \times \left\{ -y + \delta \left(\frac{\partial p}{\partial x} \right)^2 \left(-y^3 + \frac{3}{2} h y^2 - \frac{3}{4} h^2 y \right) \right\} + U_1 \quad (8)$$

Similarly, velocity w is derived from boundary conditions $w(y=0)=0$ and $w(y=h)=0$.

$$w = \frac{1}{2} \frac{\partial p}{\partial z} y(y-h) + \delta \left(\frac{\partial p}{\partial z} \right)^3 \left(\frac{y^4}{4} - \frac{h}{2} y^3 + \frac{3}{8} h^2 y^2 - \frac{h^3}{8} y \right) \quad (9)$$

As the pseudo-plastic fluids are assumed to be incompressible, the continuous equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (10)$$

Substituting Eq. (8) and Eq. (9) into Eq. (10), and integrating by using boundary conditions $v(y=0)=0$ and $v(y=h)=V$, we arrive at

$$\frac{\partial}{\partial x} \left\{ \frac{h^3}{12} \frac{\partial p}{\partial x} + \delta \frac{h^5}{80} \left(\frac{\partial p}{\partial x} \right)^3 \right\} + \frac{\partial}{\partial z} \left\{ \frac{h^3}{12} \frac{\partial p}{\partial z} + \delta \frac{h^5}{80} \left(\frac{\partial p}{\partial z} \right)^3 \right\} = \frac{1}{2} (U_1 - U_2) \frac{\partial h}{\partial x} + \frac{h}{2} \frac{\partial (U_1 + U_2)}{\partial x} + V \quad (11)$$

This is the modified Reynolds equation for pseudo-plastic fluids. If $\hat{k}=0$, the above equation is identical to the so-called Reynolds equation. Equation (11) is a more general equation, including the case of Newtonian fluids.

Each term on the right-hand side of Eq. (11) indicates, respectively, the action of wedge, stretch or squeeze. In hydrodynamically lubricated bearings using pseudo-plastic fluids, the film pressure is developed by the same three terms as are used for Newtonian fluids.

4. Analysis of the modified Reynolds equation in the case of a small nonlinear factor

By applying the above mentioned modified Reynolds equation to journal bearings, the bearing performance can be obtained. Dimensionless film thickness h in Eq. (11) is expressed with respect to the minimum film thickness. The minimum film thickness for a journal bearing is given by $\hat{h}_2 = \hat{C}(1-\varepsilon)$, where the film thickness is dependent on the eccentricity ratio. Thus, it is inconvenient that the film thickness is non-dimensionalized by h_2 . In journal bearings, it is possible to use the dimensionless expression with respect to the radial clearance $\hat{C}$. In consideration of the dimensionless expression mentioned above, multiplying both sides of Eq. (11) by $(1-\varepsilon)$ and introducing coordinate transformations $\theta = \hat{x}/\hat{R}$ and $z = \hat{z}/\hat{L}$, the modified Reynolds equation takes the following form for a steady load:

$$\frac{\partial}{\partial \theta} \left\{ \frac{h^3}{12} \frac{\partial p}{\partial \theta} + \delta \frac{h^5}{80} \left(\frac{\partial p}{\partial \theta} \right)^3 \right\} + \frac{1}{4} \left(\frac{D}{L} \right)^2 \frac{\partial}{\partial z} \left\{ \frac{h^3}{12} \frac{\partial p}{\partial z} + \frac{1}{4} \left(\frac{D}{L} \right)^2 \delta \frac{h^5}{80} \left(\frac{\partial p}{\partial z} \right)^3 \right\} = \frac{1}{2} \frac{\partial h}{\partial \theta} \quad (12)$$

the following equation.

$$\Phi^3 + A\Phi + B = 0 \quad (6)$$

where

$$\Phi = \tau_1 + \frac{h}{2} \frac{\partial p}{\partial x}$$

$$A = \frac{1}{\delta} + \left(\frac{h}{2} \frac{\partial p}{\partial x} \right)^2$$

$$B = \frac{U_1 - U_2}{\delta h}$$

$|B/A|$ is smaller than unity because $h > 1$. Neglecting small quantities of higher order of B/A , τ_1 is expressed as

$$\tau_1 = - \frac{U_1 - U_2}{h \left\{ 1 + \delta \left(\frac{h}{2} \frac{\partial p}{\partial x} \right)^2 \right\}} - \frac{h}{2} \frac{\partial p}{\partial x} \quad (7)$$

where $h=1+\varepsilon \cos \theta$, and, in the derivation of the above equation, $V=U_2 \partial h / \partial x$ is taken into consideration from the configuration of journal bearings shown in Fig. 3. The dimensionless quantities in this equation are given in Table 2.

In this paper, δ is called "nonlinear factor". Its value depends on the radial clearance and the velocity of the journal and on the initial viscosity and the coefficient of pseudo-plasticity of the lubricant. The value of nonlinear factor increases with the coefficient of plasticity. In Newtonian fluids, $\delta=0$ because $k=0$. Thus, it is considered that the nonlinear factor is a dimensionless quantity which indicates the non-Newtonian characteristics.

In analyzing Eq. (12), the following two cases are considered and the equation is solved on each case. The first solution is for a small nonlinear factor and the second is for a large one. For the second case, this equation is solved by the successive over-relaxation method using a digital computer as described later. It is possible, from the numerically calculated results, to understand the effects of the non-Newtonian characteristics on the bearing performance. But it is less analytically interesting because these effects can not be clarified in the process of calculation. Considering the nonlinear factor is small in lubricants with usual viscosity index improver added, Eq. (12) is manually solved in this paper on the case of a small nonlinear factor and the effects of the non-Newtonian characteristics on the bearing performance are analytically clarified.

4.1 Infinitely long bearing

The pressure distribution of an infinitely long bearing is considered here. In such a bearing, the second term on the left-hand side of Eq. (12) vanishes because $\partial p / \partial z = 0$. Thus, Eq. (12) is

$$\frac{d}{d\theta} \left\{ \frac{h^3}{12} \frac{dp}{d\theta} + \delta \frac{h^5}{80} \left(\frac{dp}{d\theta} \right)^3 \right\} = \frac{1}{2} \frac{dh}{d\theta} \quad (13)$$

Fig. 3 Geometry of a journal bearing

If nonlinear factor $\delta < 1$ for the above mentioned reasons, then the film pressure can be assumed to be $p = p_0 + \delta p_1 + \delta^2 p_2 + \dots$. By using the perturbation method, the following two equations are obtained.

$$\frac{d}{d\theta} \left(\frac{h^3}{12} \frac{dp_0}{d\theta} \right) = \frac{1}{2} \frac{dh}{d\theta} \quad (14)$$

$$\frac{d}{d\theta} \left\{ \frac{h^3}{12} \frac{dp_1}{d\theta} + \frac{h^5}{80} \left(\frac{dp_0}{d\theta} \right)^3 \right\} = 0 \quad (15)$$

Introducing the Sommerfeld transformation,

$$1 + \varepsilon \cos \theta = \frac{1 - \varepsilon^2}{1 - \varepsilon \cos \psi} \quad (16)$$

into Eq. (14) and using the boundary conditions $p_0(\theta=0) = p_0(\theta=\pi) = 0$,

$$p_0 = \frac{6\varepsilon}{(2+\varepsilon^2)(1-\varepsilon^2)^{3/2}} \left\{ (2-\varepsilon^2) \sin \psi - \frac{\varepsilon}{2} \sin 2\psi \right\} \quad (17)$$

Applying the Sommerfeld inverse transformation to Eq. (17),

$$p_0 = \frac{6\varepsilon(2+\varepsilon \cos \theta) \sin \theta}{(2+\varepsilon^2)(1+\varepsilon \cos \theta)^2} \quad (18)$$

Integrating Eq. (15), we have

$$\frac{dp_1}{d\theta} = -\frac{3}{20} \left\{ (1+\varepsilon \cos \theta)^2 \left(\frac{dp_0}{d\theta} \right)^3 + \frac{C_1}{(1+\varepsilon \cos \theta)^3} \right\} \quad (19)$$

where C_1 is an integration constant.

Introducing the Sommerfeld transformation into Eq. (19),

$$\frac{dp_1}{d\psi} = -\frac{3}{20} \left\{ (1-\varepsilon^2) \left(\frac{dp_0}{d\psi} \right)^3 + \frac{(1-\varepsilon \cos \psi)^2}{(1-\varepsilon^2)^{5/2}} C_1 \right\} \quad (20)$$

while, from Eq. (17), $dp_0/d\psi$ is

$$\frac{dp_0}{d\psi} = \frac{6\varepsilon}{(2+\varepsilon^2)(1-\varepsilon^2)^{3/2}} \times \{ (2-\varepsilon^2) \cos \psi - \varepsilon \cos 2\psi \} \quad (21)$$

After substituting Eq. (21) into Eq. (20), integrating and using the boundary conditions $p_1(\theta=0) = p_1(\theta=\pi) = 0$,

$$p_1 = -\frac{27}{25} \frac{\varepsilon^3 \sin \psi}{(2+\varepsilon^2)^4 (1-\varepsilon^2)^{7/2}} \{ 30(2-\varepsilon^2)(8-4\varepsilon^2 + 4\varepsilon^4 + \varepsilon^6) - 10(4-\varepsilon^2)(4+8\varepsilon^2 + \varepsilon^4) \cdot \sin^2 \psi - 15\varepsilon(12-8\varepsilon^2 + 5\varepsilon^4) \cos \psi - 5\varepsilon(2+\varepsilon^2) \times (36-44\varepsilon^2 + 9\varepsilon^4) \cos^3 \psi + 72(4-\varepsilon^4) \varepsilon^2 \sin^4 \psi - 40\varepsilon^3(2+\varepsilon^2) \cos^5 \psi \} \quad (22)$$

Introducing the above inverse transformation into Eq. (22), and neglecting higher than third order terms of the eccentricity ratio,

$$p_1 = -\frac{864}{5} \frac{\varepsilon^3(2+\cos^2 \theta) \sin \theta}{(2+\varepsilon^2)^4 (1-\varepsilon^2)^3 (1+\varepsilon \cos \theta)^6} \quad (23)$$

Therefore, the film pressure of an infinitely long bearing with pseudo-plastic fluids is

$$p = \frac{6\varepsilon(2+\varepsilon\cos\theta)\sin\theta}{(2+\varepsilon^2)(1+\varepsilon\cos\theta)^2} - \frac{864}{5}\delta \frac{\varepsilon^3(2+\cos^2\theta)\sin\theta}{(2+\varepsilon^2)^4(1-\varepsilon^2)^3(1+\varepsilon\cos\theta)^6} \quad (24)$$

With $\delta=0$ in the above equation, this equation is identical to the expression of Newtonian fluids. It is considered that the film pressure of pseudo-plastic fluids is the same as the pressure of Newtonian fluids modified by the second term of the right-hand side of Eq. (24).

Since $\hat{k}>0$ in pseudo-plastic fluids, then $\delta>0$. And, as the half Sommerfeld boundary conditions are used in the calculation, the region of the bearing angle where the pressure develops is from 0 to π . Thus, the second term on the right-hand side of Eq. (24) is usually positive, and the film pressure in pseudo-plastic fluids becomes smaller than that of Newtonian fluids. This tendency will be more pronounced as the nonlinear factor increases.

4.2 Infinitely short bearing

The pressure distribution of an infinitely short bearing is considered here. Since $\partial p/\partial\theta \ll \partial p/\partial z$ in such a bearing, the following equation is obtained from Eq. (12).

$$\frac{1}{4}\left(\frac{D}{L}\right)^2 \frac{\partial}{\partial z} \left\{ \frac{h^3}{12} \frac{\partial p}{\partial z} + \frac{1}{4}\left(\frac{D}{L}\right)^2 \frac{h^5}{80} \left(\frac{\partial p}{\partial z}\right)^3 \right\} = \frac{1}{2} \frac{\partial h}{\partial \theta} \quad (25)$$

By assuming $\delta < 1$ and $p = p_0 + \delta p_1 + \delta^2 p_2 + \dots$ in Eq. (25), Eqs. (26) and (27) are derived as follows.

$$\frac{1}{4}\left(\frac{D}{L}\right)^2 \frac{\partial}{\partial z} \left\{ \frac{h^3}{12} \frac{\partial p_0}{\partial z} \right\} = \frac{1}{2} \frac{\partial h}{\partial \theta} \quad (26)$$

$$\frac{1}{4}\left(\frac{D}{L}\right)^2 \frac{\partial}{\partial z} \left\{ \frac{h^3}{12} \frac{\partial p_1}{\partial z} + \frac{1}{4}\left(\frac{D}{L}\right)^2 \frac{h^5}{80} \left(\frac{\partial p_0}{\partial z}\right)^3 \right\} = 0 \quad (27)$$

Solving Eqs. (26) and (27) by using the boundary conditions $p=0$ at both sides of the bearing,

$$p_0 = 12\left(\frac{L}{D}\right)^2 \frac{\varepsilon \sin \theta}{(1+\varepsilon \cos \theta)^3} z(1-z) \quad (28)$$

$$p_1 = -\frac{324}{5}\left(\frac{L}{D}\right)^4 \frac{\varepsilon^3 \sin^3 \theta}{(1+\varepsilon \cos \theta)^7} z(1-3z+4z^2-2z^3) \quad (29)$$

The film pressure of an infinitely short bearing with pseudo-plastic fluids is

$$p = 12\left(\frac{L}{D}\right)^2 \frac{\varepsilon \sin \theta}{(1+\varepsilon \cos \theta)^3} z(1-z) - \frac{324}{5}\left(\frac{L}{D}\right)^4 \frac{\varepsilon^3 \sin^3 \theta}{(1+\varepsilon \cos \theta)^7} z(1-3z+4z^2-2z^3) \quad (30)$$

Setting $\delta=0$ in Eq. (30), this equation is identical to the expression of Newtonian fluids. The second term of the right-hand side increases with the width-to-diameter ratio and the eccentricity ratio. Therefore, the film pressure in pseudo-plastic fluids becomes smaller than that of Newtonian fluids, as the width-to-diameter ratio and the eccentricity ratio increase.

4.3 Finite bearing

In this section, film pressure, load capacity, attitude angle, Sommerfeld number, frictional force, coefficient of friction and rate of oil flow for finite journal bearing are derived. When width-to-diameter ratio is from 0.5 to 1.5 in a commonly used bearing, Eq. (12) becomes pertinent.

4.3.1 Film pressure

By assuming $\delta < 1$ and $p = p_0 + \delta p_1 + \delta^2 p_2 + \dots$ in Eq. (12), the following two equations are obtained.

$$\frac{\partial}{\partial \theta} \left\{ \frac{h^3}{12} \frac{\partial p_0}{\partial \theta} \right\} + \frac{1}{4}\left(\frac{D}{L}\right)^2 \frac{\partial}{\partial z} \left\{ \frac{h^3}{12} \frac{\partial p_0}{\partial z} \right\} = \frac{1}{2} \frac{\partial h}{\partial \theta} \quad (31)$$

$$\frac{\partial}{\partial \theta} \left\{ \frac{h^3}{12} \frac{\partial p_1}{\partial \theta} + \frac{h^5}{80} \left(\frac{\partial p_0}{\partial \theta}\right)^3 \right\} + \frac{1}{4}\left(\frac{D}{L}\right)^2 \frac{\partial}{\partial z} \left\{ \frac{h^3}{12} \frac{\partial p_1}{\partial z} + \frac{1}{4}\left(\frac{D}{L}\right)^2 \frac{h^5}{80} \left(\frac{\partial p_0}{\partial z}\right)^3 \right\} = 0 \quad (32)$$

Equation (31) is the so-called Reynolds equation for Newtonian fluids and was already solved approximately by Aoki⁽⁴⁾. In this paper, p_0 is evaluated from Eq. (31) and p_1 is evaluated after substituting p_0 into Eq. (32), more simple and approximate solutions can be derived.

When boundary conditions are $p(\theta, 0) = p(\theta, 1) = 0$, p_0 is assumed in the following form.

$$p_0 = \sum_{i=0}^{\infty} p_{0i} \sin(2i+1)\pi z \quad (33)$$

After substituting Eq. (33) into Eq. (31), using the transformation $p_{0i}' = (1+\varepsilon \cos \theta)^2 p_{0i}$, we have

$$(1+\varepsilon \cos \theta) \frac{d^2 p_{0i}'}{d\theta^2} + \varepsilon \sin \theta \frac{dp_{0i}'}{d\theta} + \{(2-K_i^2)\varepsilon \cos \theta - K_i^2\} p_{0i}' = -\frac{24}{\pi(2i+1)} \varepsilon \sin \theta \quad (34)$$

where

$$K_i = \frac{\pi}{2} \left(\frac{D}{L}\right) (2i+1)$$

Since the eccentricity ratio is less than unity, it can be assumed that $p_{0i}' = p_{0i0}' + \varepsilon p_{0i1}' + \varepsilon^2 p_{0i2}' + \varepsilon^3 p_{0i3}' + \dots$. Using the perturbation method and the half Sommerfeld boundary conditions,

$$p_0 = \frac{1}{(1 + \varepsilon \cos \theta)^2} \sum_{i=0}^{\infty} B_i \sin (2i+1)\pi z \left\{ \varepsilon \sin \theta + \varepsilon^2 \frac{2-K_i^2}{2(4+K_i^2)} \sin 2\theta - \varepsilon^3 \frac{K_i^2(2-K_i^2)}{4(4+K_i^2)(9+K_i^2)} \sin 3\theta - \varepsilon^3 \frac{2-K_i^2}{4(1+K_i^2)} \sin \theta \right\} \dots (35)$$

$$\text{where } B_i = \frac{24}{\pi(2i+1)(1+K_i^2)} \dots (36)$$

In the above equation, the values of B_i decrease with an increase of i . It can be considered that $B_0, B_1 > B_2, B_3, \dots$, because L/D is from 0.5 to 1.5 in normally used bearings. Introducing Eq. (35) into Eq. (32) and neglecting the higher order terms of B_i and ε ,

$$h^3 \frac{\partial^2 p_1}{\partial \theta^2} + 3h^2 \frac{\partial h}{\partial \theta} \frac{\partial p_1}{\partial \theta} + \frac{1}{4} \left(\frac{D}{L} \right)^2 h^3 \frac{\partial^2 p_1}{\partial z^2} = \frac{9}{80} \varepsilon^3 \left[\frac{\sin \theta \cos^2 \theta}{(1 + \varepsilon \cos \theta)^5} \{ 3(B_0^3 - B_0^2 B_1 + 2B_0 B_1^2) \sin \pi z + (-B_0^3 + 6B_0^2 B_1 + 3B_1^3) \sin 3\pi z + 3(-B_0^2 B_1 + B_0 B_1^2) \sin 5\pi z \} + \frac{\pi^4}{16} \left(\frac{D}{L} \right)^4 \frac{\sin^3 \theta}{1 + \varepsilon \cos \theta} \{ (B_0^3 + 3B_0^2 B_1 + 18B_0 B_1^2) \sin \pi z + (B_0^3 + 18B_0^2 B_1 + 81B_1^3) \sin 3\pi z + 15(B_0^2 B_1 + 3B_0 B_1^2) \sin 5\pi z \} \right] \dots (37)$$

In Eq. (37) p_1 is assumed in the following form,

$$p_1 = \sum_{i=0}^{\infty} p_{1i} \sin (2i+1)\pi z \dots (38)$$

It becomes clear by introducing Eq. (38) into Eq. (37) that p_{12} has higher order terms of B_i as compared with p_{10} and p_{11} . Therefore, considering the cases of $i=0, 1$, the approximating solutions are obtained as follows.

In the case of $i=0$,

$$h^3 \frac{d^2 p_{10}}{d\theta^2} + 3h^2 \frac{dh}{d\theta} \frac{dp_{10}}{d\theta} - K_0^2 h^3 p_{10} = \frac{9}{80} \varepsilon^3 \left\{ \frac{3 \sin \theta \cos^2 \theta}{(1 + \varepsilon \cos \theta)^5} (B_0^3 - B_0^2 B_1 + 2B_0 B_1^2) + K_0^4 \frac{\sin^3 \theta}{1 + \varepsilon \cos \theta} (B_0^3 + 3B_0^2 B_1 + 18B_0 B_1^2) \right\} \dots (39)$$

After assuming $p_{10}' = (1 + \varepsilon \cos \theta)^6 p_{10}$ and $p_{10}' = p_{100}' + \varepsilon p_{101}' + \varepsilon^2 p_{102}' + \varepsilon^3 p_{103}' + \dots$, and applying the perturbation method, we have

$$p_{100}' = p_{101}' = p_{102}' = 0$$

$$p_{103}' = -\frac{3}{20} \left(\frac{24}{\pi} \right)^3 \frac{\sin \theta}{(1 + K_0^2)^4 (9 + K_0^2) (1 + 9K_0^2)^2} \{ 6 + 104K_0^2 + 580K_0^4 + 320K_0^6 + 414K_0^8 - (1 + K_0^2)(2 + 34K_0^2 + 173K_0^4 - 24K_0^6 - 69K_0^8) \sin^2 \theta \} \dots (40)$$

In the case of $i=1$, similarly, by assuming $p_{11}' = (1 + \varepsilon \cos \theta)^6 p_{11}$ and $p_{11}' = p_{110}' + \varepsilon p_{111}' + \varepsilon^2 p_{112}' + \varepsilon^3 p_{113}' + \dots$,

$$p_{110}' = p_{111}' = p_{112}' = 0$$

$$p_{113}' = -\frac{1}{360} \left(\frac{24}{\pi} \right)^3 \frac{\sin \theta}{(1 + K_0^2)^4 (1 + 9K_0^2)^4} \{ 3(5 + 66K_0^2 + 42K_0^4 - 1804K_0^6 - 39K_0^8 + 10962K_0^{10}) - (1 + 9K_0^2)(5 + 51K_0^2 - 246K_0^4 - 3226K_0^6 - 3807K_0^8 - 5481K_0^{10}) \sin^2 \theta \} \dots (41)$$

From Eqs. (35), (40) and (41), the film pressure of finite bearing takes the following form.

$$p = \frac{24\varepsilon \sin \theta}{\pi(1 + \varepsilon \cos \theta)^3} \left[\frac{\sin \pi z}{1 + K^2} \left\{ 1 - \varepsilon^2 \frac{3(3 + K^2)(2 - K^2)}{(1 + K^2)(4 + K^2)(9 + K^2)} + \frac{6\varepsilon}{4 + K^2} \cos \theta + \frac{9(2 - K^2)}{(4 + K^2)(9 + K^2)} \varepsilon^2 \cos^2 \theta \right\} + \frac{\sin 3\pi z}{3(1 + 9K^2)} \left\{ 1 - \varepsilon^2 \frac{(1 + 3K^2)(2 - 9K^2)}{(1 + K^2)(1 + 9K^2)(4 + 9K^2)} + \frac{6\varepsilon}{4 + 9K^2} \cos \theta + \frac{2 - 9K^2}{(1 + K^2)(4 + 9K^2)} \varepsilon^2 \cos^2 \theta \right\} \right] - \frac{\delta}{40} \left(\frac{24}{\pi} \right)^3 \frac{1}{(1 + K^2)^4 (1 + 9K^2)^2} \times \frac{\varepsilon^3 \sin \theta}{(1 + \varepsilon \cos \theta)^7} \left[\frac{6 \sin \pi z}{9 + K^2} \{ 6 + 104K^2 + 580K^4 + 320K^6 + 414K^8 - (1 + K^2)(2 + 34K^2 + 173K^4 - 24K^6 - 69K^8) \sin^2 \theta \} + \frac{\sin 3\pi z}{9(1 + 9K^2)^2} \{ 3(5 + 66K^2 + 42K^4 - 1804K^6 - 39K^8 + 10962K^{10}) - (1 + 9K^2)(5 + 51K^2 - 246K^4 - 3226K^6 - 3807K^8 - 5481K^{10}) \sin^2 \theta \} \right] \dots (42)$$

where

$$K = \frac{\pi}{2} \left(\frac{D}{L} \right)$$

Setting $L/D \rightarrow \infty$ in Eq. (42),

$$\lim_{L/D \rightarrow \infty} p \approx \frac{6\varepsilon(2+\varepsilon \cos \theta) \sin \theta}{(2+\varepsilon^2)(1+\varepsilon \cos \theta)^2} - \frac{864}{5} \delta \frac{\varepsilon^3(2+\cos^2 \theta) \sin \theta}{(2+\varepsilon^2)^4(1-\varepsilon^2)^3(1+\varepsilon \cos \theta)^6} \dots (43)$$

This identical to the expression of the pressure of an infinitely long bearing that is given by Eq. (24).

And, setting $L/D \rightarrow 0$ in Eq. (42),

$$\lim_{L/D \rightarrow 0} p \approx 12 \left(\frac{L}{D} \right)^2 \frac{\varepsilon \sin \theta}{(1+\varepsilon \cos \theta)^3 z(1-z)} - \frac{324}{5} \left(\frac{L}{D} \right)^4 \delta \frac{\varepsilon^3 \sin^3 \theta}{(1+\varepsilon \cos \theta)^7} z(1-3z+4z^2-2z^3) \dots (44)$$

This is identical to the expression of the pressure of an infinitely short bearing that is given by Eq. (30).

Thus, expression of the film pressure of a finite bearing using pseudo-plastic fluids is concluded over a wide range of width-to-diameter ratio.

4.3.2 Load capacity, attitude angle and Sommerfeld number

From the equilibrium of the film pressure and the applied load,

$$W \sin \phi = \int_0^1 \int_0^\pi p \sin \theta d\theta dz, \\ W \cos \phi = - \int_0^1 \int_0^\pi p \cos \theta d\theta dz \dots (45)$$

Integrating after substituting Eq. (42) into Eq. (45), we have

$$W \sin \phi = \frac{24}{\pi} \frac{\varepsilon}{(1-\varepsilon^2)^{3/2}} \frac{1}{1+K^2} \left[\frac{36-86K^2+K^4}{(4+K^2)(9+K^2)} + \frac{4-86K^2+9K^4}{9(1+9K^2)(4+9K^2)} \right. \\ \left. - \varepsilon^2 \frac{3(2-K^2)(9+7K^2)}{(1+K^2)(4+K^2)(9+K^2)} - \varepsilon^2 \frac{(2-9K^2)(1+7K^2)}{3(1+9K^2)^2(4+9K^2)} + \frac{66K^2}{\varepsilon^2} \{1-(1-\varepsilon^2)^{3/2}\} \right. \\ \left. \times \left\{ \frac{1}{(4+K^2)(9+K^2)} + \frac{1}{9(1+9K^2)(4+9K^2)} \right\} \right] - \frac{\delta}{8640} \left(\frac{24}{\pi} \right)^3 \\ \times \frac{1}{(1+K^2)^4(1+9K^2)^4(9+K^2)} \frac{\varepsilon^3}{(1-\varepsilon^2)^{11/2}} [3(8+12\varepsilon^2+\varepsilon^4) \{54(1+9K^2)^2(6+104K^2 \\ +580K^4+320K^3+414K^3) + (9+K^2)(5+66K^2+42K^4-1804K^6-39K^8+10962K^{10})\} \\ - (1-\varepsilon^2)(6+\varepsilon^2) \{162(1+K^2)(1+9K^2)^2(2+34K^2+173K^4-24K^6-69K^8) \\ + (1+9K^2)(9+K^2)(5+51K^2-246K^4-3226K^6-3807K^8-5481K^{10})\}] \dots (46)$$

$$W \cos \phi = \frac{96}{\pi^2} \frac{\varepsilon^2}{(1-\varepsilon^2)^2} \frac{1}{1+K^2} \left[\frac{18-44K^2+K^4}{(4+K^2)(9+K^2)} + \frac{2-44K^2+9K^4}{9(1+9K^2)(4+9K^2)} \right. \\ \left. - 6\varepsilon^2 \frac{(2-K^2)(3+2K^2)}{(1+K^2)(4+K^2)(9+K^2)} - \frac{2}{9} \varepsilon^2 \frac{(2-9K^2)(1+6K^2)}{(1+9K^2)^2(4+9K^2)} + \frac{33K^2}{\varepsilon^2} \left\{ 1 + \frac{(1-\varepsilon^2)^2}{2\varepsilon} \log \left(\frac{1-\varepsilon}{1+\varepsilon} \right) \right\} \right. \\ \left. \times \left\{ \frac{1}{(4+K^2)(9+K^2)} + \frac{1}{9(1+9K^2)(4+9K^2)} \right\} \right] - \frac{\delta}{4050\pi} \left(\frac{24}{\pi} \right)^3 \\ \times \frac{1}{(1+K^2)^4(1+9K^2)^4(9+K^2)} \frac{\varepsilon^4}{(1-\varepsilon^2)^6} [3(35+42\varepsilon^2+3\varepsilon^4) \{54(1+9K^2)^2(6+104K^2 \\ +580K^4+320K^3+414K^3) + (9+K^2)(5+66K^2+42K^4-1804K^6-39K^8+10962K^{10})\} \\ - 2(1-\varepsilon^2)(7+\varepsilon^2) \{162(1+K^2)(1+9K^2)^2(2+34K^2+173K^4-24K^6-69K^8) \\ + (1+9K^2)(9+K^2)(5+51K^2-246K^4-3226K^6-3807K^8-5481K^{10})\}] \dots (47)$$

Therefore, the load capacity and the attitude angle are calculated from the following equations.

$$W = \{(W \sin \phi)^2 + (W \cos \phi)^2\}^{1/2} \dots (48)$$

$$\phi = \tan^{-1}(W \sin \phi / W \cos \phi) \dots (49)$$

In pseudo-plastic fluids, viscosity cannot exist in the meaning of Newtonian fluids. In this paper, the Sommerfeld number of pseudo-plastic fluids is defined as follows by using the initial viscosity.

$$S = \frac{\hat{\mu} \hat{N}}{\hat{P}} \left(\frac{\hat{R}}{\hat{C}} \right)^2 \dots (50)$$

where $\hat{P} = \hat{W} / \hat{L} \hat{D}$

Rewriting Eq. (50), we obtain

$$S = 1/(\pi W) \dots (51)$$

4.3.3 Frictional force and coefficient of friction.

Since the shear stress in an oil film is given by Eq. (4), it is

$$\tau_{j,b} = \pm \frac{h}{2} \frac{\partial p}{\partial \theta} + \frac{1}{h \left\{ 1 + \delta \left(\frac{h}{2} \frac{\partial p}{\partial \theta} \right)^2 \right\}} \quad \dots\dots\dots (52)$$

where the positive sign refers to the journal surface and the negative refers to the bearing surface.

Setting $\delta=0$ in the above equation, this equation becomes the shear stress of Newtonian fluids. Since the second term of the denominator of the right-hand side is usually positive, the shear stress of pseudo-plastic fluids is smaller than that of Newtonian fluids. Thus, the frictional force is also smaller.

Considering that $p=0$ at $\pi \leq \theta \leq 2\pi$ in Eq. (52), the frictional force is

$$F_{j,b} = \pm \frac{\varepsilon}{2} W \sin \phi + \frac{\pi}{(1-\varepsilon^2)^{1/2}} + \int_0^1 \int_0^\pi \frac{d\theta dz}{h \left\{ 1 + \delta \left(\frac{h}{2} \frac{\partial p}{\partial \theta} \right)^2 \right\}} \quad \dots\dots\dots (53)$$

The coefficient of friction is

$$f_{j,b} = F_{j,b}/W \quad \dots\dots\dots (54)$$

4.3.4 Rate of oil flow

The rate of oil flow out of both sides of the bearing is

$$Q = 2 \int_0^\pi \int_0^h (-w_{z=0}) dy d\theta = \int_0^\pi \left\{ \left(\frac{D}{L} \right) \frac{h^3}{12} \frac{\partial p}{\partial z} + \frac{\delta}{320} \left(\frac{D}{L} \right)^3 h^5 \left(\frac{\partial p}{\partial z} \right)^3 \right\}_{z=0} d\theta \quad \dots\dots\dots (55)$$

The integrations in Eq. (53) and Eq. (55) are performed using numerical integrals.

4.4 Analytical results and discussions

The analytical treatments of the modified Reynolds equation in the case of a small non-linear factor have been discussed about infinitely long, infinitely short and finite bearings. In the process of these analyses, the non-Newtonian characteristics of the following influences on the bearing performance were clarified. That is, the oil

pressure of pseudo-plastic fluids is smaller than that of Newtonian fluids whose viscosity is equal to the initial viscosity of pseudo-plastic fluids, and the load capacity is also smaller. These become more pronounced as the eccentricity ratio and the width-to-diameter ratio increase. As the shear stress is smaller than that of Newtonian fluids, the frictional force becomes smaller.

The calculated examples described here are for a finite bearing. The pressure distributions in pseudo-plastic fluids are shown in Fig. 4 and Fig. 5, in which the thick lines indicate the results based on Eq. (42) and the dashed lines are the numerical results of the later mentioned finite-difference methods. At a small eccentricity ratio as shown in Fig. 4, the results of Eq. (42)

Fig. 5 Pressure distribution

Fig. 4 Pressure distribution for small ε

Fig. 6 W/W_n versus nonlinear factor

Fig. 7 F_j/F_{jn} versus nonlinear factor

agree well with the numerical ones based on the finite-difference method. But, when increasing the eccentricity ratio as shown in Fig. 5, some differences exist in spite of a small nonlinear factor. Figure 6 shows the ratio of the load capacity of pseudo-plastic fluids W and Newtonian fluids W_n with respect to the nonlinear factor. Similarly, Fig. 7 indicates the frictional force. The load capacity and the frictional force of pseudo-plastic fluids are smaller than that of Newtonian fluids as shown. The effects of the non-Newtonian characteristics, which have been clarified in the process of the above analysis, are reconfirmed from these figures. Analytical expressions can be applied only for the cases of a small eccentricity ratio and a small nonlinear factor. Of course, the numerical results which are described later on are available for quantitative discussions with respect to a large eccentricity ratio.

The above mentioned analytical treatments are available for qualitative discussions.

5. Analysis of the modified Reynolds equation in the case of a large nonlinear factor

The nonlinear factor is very small in practical lubricants with a viscosity index improver and a viscosity thickener added. In such a case, the bearing performance is given by the above mentioned theoretical results.

But the above analysis can not be used when the nonlinear factor is greater than unity or is nearly unity. So, for the case of a large nonlinear factor, the performance of the journal bearing is obtained by using the following numerical analysis.

Since the modified Reynolds equation for pseudo-plastic fluids is nonlinear with respect to the film pressure, the following cubic algebraic equation is obtained by rewriting Eq. (12) into the finite-difference equation.

$$a_1 p_{i,j}^3 + a_2 p_{i,j}^2 + a_3 p_{i,j} + a_4 = 0 \quad (56)$$

where $p_{i,j}$ is the pressure at the grid points that are i -th and j -th, respectively, in the θ, z -direction. Coefficients a_1, a_2, a_3 and a_4 depend on $p_{i+1,j}, p_{i-1,j}, p_{i,j+1}, p_{i,j-1}, L/D, \epsilon$ and δ .

In general, the cubic equation has a real root or three real roots. It is known that the real root is always one obtained by solving this equation for various width-to-diameter ratios, eccentricity ratios and nonlinear factors. It gives the film pressure uniquely. The Reynolds equation for Newtonian fluids is linear, and the finite-difference equation is also linear with respect to $p_{i,j}$.

Calculating Eq. (56) by applying the successive over-relaxation method under the half Sommerfeld boundary conditions, the film pressure of a finite bearing can be obtained for a large nonlinear factor.

The load capacity, the attitude angle and the Sommerfeld number are calculated numerically by Eqs. (45), (48), (49) and (51). The frictional force and the coefficient of friction are obtained by Eqs. (53) and (54) and the rate of oil flow is obtained by Eq. (55).

Fig. 8 Circumferential pressure distribution

Fig. 9 Axial pressure distribution

Fig. 10 Dimensionless load capacity versus eccentricity ratio

Based on these numerical results, the film pressure in the circumferential direction is shown in Fig. 8. It is clear from this figure that the pressure in pseudoplastic fluids is smaller as compared with Newtonian fluids whose viscosity is equal to the initial viscosity. This becomes more pronounced as the nonlinear factor increases. The shear rate is large at a large bearing angle ($\theta \cong \pi$). Since the flow curve indicates clearly the nonlinear characteristic in the region of large shear rates as in Fig. 1, the film pressure decreases abruptly at a large bearing angle. The pressure distribution in the axial direction is shown in Fig. 9, from which it is also understood that the film pressure in pseudo-plastic fluids is smaller than that of Newtonian fluids, so that the load capacity also decreases, as shown in Fig. 10. The relationship between the Sommerfeld number and the eccentricity ratio is shown in Fig. 11. In these figures, the load capacity of a journal bearing using pseudo-plastic fluids is smaller and the Sommerfeld number becomes larger at the same eccentricity ratio. These tendencies become more pronounced as the nonlinear factor and the eccentricity ratio increase. The relationship between the eccentricity ratio and the attitude angle is shown in Fig. 12.

Fig. 11 Eccentricity ratio versus Sommerfeld number

Fig. 12 Eccentricity ratio versus attitude angle

The relationship between the frictional force and the eccentricity ratio is given in Fig. 13. In this figure, the frictional force is smaller than that of Newtonian fluids, and the coefficient of

Fig. 13 Dimensionless frictional force versus eccentricity ratio

Fig. 14 Coefficient of friction versus Sommerfeld number

Fig. 15 Dimensionless flow versus Sommerfeld number

Fig. 16 W/W_n , F_j/F_{jn} versus width-to-diameter ratio

friction decreases, as shown in Fig. 14, under the same value of the Sommerfeld number. Figure 15 indicates an increase in the rate of oil flow. From the above facts, the heat generated by the frictional force in the journal bearing using pseudo-plastic fluids is known to be small. In addition to this, the rate of oil flow is large, so that any rise in oil temperature is small as compared with the bearing using Newtonian fluid. It is interesting that decrease of the viscosity due to the temperature is difficult to bring about.

Figure 16 indicates the ratio of the load capacity of pseudo-plastic fluids W to that of Newtonian fluids W_n . Also, the frictional force is given similarly. From these factors, the effects of the non-Newtonian characteristics on the

bearing performance are found to become larger as the width-to-diameter ratio increases.

6. Conclusions

The modified Reynolds equation of pseudo-plastic fluids is derived so that the performance of the journal bearing can be calculated and the effects of the non-Newtonian characteristics on the bearing performance can be clarified.

Consequently,

(1) The modified Reynolds equation for the journal bearings using pseudo-plastic fluids is given by Eq. (12).

(2) The pressure distribution of pseudo-plastic fluids decreases below that of Newtonian fluids whose viscosity is equal to the initial viscosity of pseudo-plastic fluids. And the load capacity and the frictional force also decrease. These tendencies become more pronounced as the nonlinear factor, the eccentricity ratio and the width-to-diameter ratio increase.

(3) Under the same load, the coefficient of friction decreases somewhat and the rate of oil flow increases. Therefore, it is considered that the change of the viscosity is small because the temperature rise in lubricants is small as compared with that of Newtonian fluids.

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