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On the Non-Uniform Deformation of a Block in Plane-Strain Compression Caused by Friction*

(Part 3, Analysis by Finite Element Method when Surface Sticks on Tool)

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Non-uniform deformation of an elastic, strain-hardening block caused by plane-strain compression is analyzed, using the finite element method. The surface of the block is supposed to stick on a rigid tool. Deformation at rather early stage is analyzed for 6 cases concerning height to width ratio of the block, h_0 . As the result, it becomes clear that the bulge on the free surface is single when $h_0 < 2$ but is double when $h_0 > 2$, that the load of compression increases as h_0 becomes smaller, and that the pressure is larger at the center than near the edge only when h_0 is small. Moreover, it is made clear that the plastic zone grows from the surface to the center when h_0 is small, from the center to the surface when h_0 becomes a little larger and from two points inside the block to the surface when h_0 is so large that double bulge yields, and so on.

1. Introduction

When a block is compressed under plane-strain condition with two rigid plane tools, it deforms non-uniformly, if friction exists on the boundary surface (interface) between the block and the tool. This non-uniform deformation and its influence on load or pressure at the interface were calculated theoretically in Part 1⁽¹⁾ of this series of research, using Hill's method of analysis by calculus of variations. In the analysis, the block was assumed to be a perfectly rigid, plastic material, as it was difficult to apply the method of analysis to elastic, strain-hardening materials.

In the present report, more improved theoretical analysis is performed concerning the plane-strain compression of an elastic, strain-hardening block under interface condition of sticking, using the finite element method. The finite element method has become one of the most promising methods to solve non-linear problems now in Japan, since Yamada⁽³⁾ introduced a consistent means about it together with the general matrix concerning the stress-strain relation of elastic,

strain-hardening materials. We apply this method of Yamada to the problem of elastoplastic compression with some partial improvement in the present report.

Height to width ratio of the block h_0 is taken as 0.25, 0.5, 1, 1.5, 2, and 3, and analysis is restricted to the early stages of partially plastic deformation. As the result, initial appearance and growth of plastic zones after elastic deformation and shape of bulge on the free side surface are made clear as well as distribution of normal pressure and of tangential stress on the interface, etc.

2. Method of analysis

One fourth of a block under plane-strain compression is divided into 288 triangular elements of equal shape as in Fig. 1. Number of nodal points becomes 169. Number of elements and nodal points all over the block becomes four times as large as these respectively. Generally when a continuum is divided into elements in order to solve this kind of problem by finite element method, it is recommended to divide parts near boundary surface or where the gradient of stress becomes large into finer elements than other parts. From this point of view, we intended at first to divide the part near the interface into finer elements and the central part into more coarse ones than the present division. However,

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plastic regions in early stages were expected to appear at various parts inside the block as h_0 changed. Especially it was ascertained from the Moire fringes experimentally obtained in Part 1⁽¹⁾ of this series that strain was the smallest near interface in most cases and that it became sometimes considerably large near the centroid inside the block of large h_0 . Thus, equal division of the whole block as was shown in Fig. 1 is concluded to be the best for the solution of the problems in various cases of h_0 with the same program.

Next, we explain the boundary condition in compression of the block shown in Fig. 1. Normal displacement of the interface of the block is equal to the compression value, and it is known a priori. On the other hand, tangential displacement on the interface becomes zero under assumption of sticking. This means that the relative slip-ratio introduced in Part 2⁽²⁾ becomes zero all over the interface. There is no restraint on the free side surface, so both normal and tangential forces become zero there. The other two surfaces are the sections of symmetry inside the block, so both normal displacement and tangential force become zero.

The procedures of our calculation are almost the same as those by Yamada⁽³⁾ as a whole, but some improvements and approximations are made in the present report. First, we adopt the displacement instead of the load increment used by Yamada in his heuristic work. That is, the increment of normal displacement of the interface is determined in every step by calculation

rather than the normal load under the condition that the element of largest $\bar{\sigma}^e$ (equivalent stress in an elastic element) is yielded.

Next, in the present report, two elements nearest to yield are made to yield simultaneously in each step under the condition that the mean value of $\bar{\sigma}^e$ of them becomes equal to Y . The error caused by this approximation is very small, as will be explained in detail in the next report (Part 4).

Besides, we neglected the change of matrix $[D^p]$ which was caused by subsequent plastic deformation after elements had been yielded. This means that we used the relation between stress and strain in a yielded element at the onset of yielding. In order to check the error due to this neglect, we performed also some calculations without this neglect, changing matrix $[D^p]$ in all elements already yielded in every stage of successive calculation. This error is found to be very small till the plastic region penetrates through the block in oblique direction, and the computing time becomes shortened by this approximation. For example, till the plastic region penetrates inside the block whose $h_0 = 0.25$ (namely till about 180 elements are yielded), both cases with and without this approximation follow the same load-displacement curve, and the difference between the predicted loads at any stage of displacement curve constituted at most 1.2% of total load. Then, this approximation is considered to be the useful method for saving the computing time, when deformation of elements in plastic regions is presumed not to be much larger than in elastic regions. However, when this approximation is adopted, the computer stops after penetration of plastic region inside the block because of rapid increase of error. In calculation without this approximation, the error does not increase even after the penetration of plastic region, though the computing time becomes a little longer than in calculation by the present approximate method.

In solving the global matrix which expresses the relation between the increment of nodal displacements and that of nodal forces, the usual decomposition method called Cholesky's square root method is used in this analysis instead of unit partitioning method introduced in FRAN⁽⁴⁾. In formation of the stiffness matrix, assumption of infinitesimal deformation is adopted, and influence of deformation of elements or the shape changes on the stiffness matrix is neglected, as this analysis is restricted to the early stages of partially plastic deformation.

Fig. 1 Triangular mesh in a quarter of a block, 288 equal elements and 169 nodal points

3. Result of calculation

Calculation was carried out for 6 cases of height to width ratio of the block h_0 (0.25, 0.5, 1, 1.5, 2 and 3). Material of block was assumed to be aluminum (Young's modulus $E=7\,200\text{ kg/mm}^2$, Poisson's ratio $\nu_s=0.33$, yield stress $Y=7.24\text{ kg/mm}^2$, modulus of strain-hardening $H'=13.5\text{ kg/mm}^2$).

Figure 2 shows growth of bulge on the free surface when $h_0=0.25$. When compression ratio is smaller than 0.656×10^{-3} , the whole block is in elastic region. So, shape of bulge is completely similar to (a) in Fig. 2, and its dimension is proportional to compression ratio. As compression advances, the elements near the edge which have been yielded first suffer large deformation, and growth of bulge in early stage depends mainly upon the deformation of these elements. Compression ratio when 180 elements are yielded is 0.001472 and it is 2.24 times as large as compression ratio when the first element is yielded.

Figure 3 gives shape of bulge on the free surface when compression ratio is 0.001. Size of bulge becomes large nearly in proportion to h_0 , for the case where the height of the block h_0 is rather small. When h_0 becomes large, however,

Fig. 2 Shape of bulge on the side surface (when $h_0=0.25$)

size of bulge approaches nearly a constant value, and shape of bulge changes from single to double when $h_0=2$. This fact was already verified by the Moire fringes experimentally obtained in Part 1⁽¹⁾ of this series of research. (f) in this figure ($h_0=3$) shows clearly shape of double bulge.

Figure 4 gives the relation between load and compression ratio. As was described above, our

when compression ratio $\delta_{yh}/h_0=0.001$
Blocks are yielded partially.

h_0 and h are height to width ratios before and during compression respectively.

Fig. 3 Shape of bulge on the side surface

Dots show the times when first 2 elements are yielded.

Load of compression denotes one per unit area of the interface before compression.

Fig. 4 Relation between load of compression and compression ratio

calculation is restricted to a relatively early stage of yielding, and an approximate value of stress is used to compose stiffness matrix after yielding. It is described above that the error due to this approximation is small till plastic region penetrate obliquely inside the block. But once it penetrates, the error becomes large immediately. Therefore in the present report, calculation is stopped as soon as penetration occurs. Relation between load and compression ratio is linear in early stage of yielding as well as in elastic compression, but it deviates from linear relation after plastic region spreads considerably. Number of yielded elements before penetration of plastic region is the smallest when $h_0=1$. This indicates that penetration occurs most easily when $h_0=1$, because the diagonal of the block is in a half angle from interface which is equal to the direction of maximum shear stress. Besides, load at the same compression ratio becomes small as h_0 becomes large. To illustrate this fact more clearly, relation between load and h_0 is given in Fig. 5. This figure indicates that load is large when h_0 is small, and that it decreases rapidly at first and gradually gently as h_0 becomes large. The same relation between load and h_0 as this was obtained both in the result of theoretical analysis by calculus of variation in Part 1⁽¹⁾ and in the experimental result of Part 2⁽²⁾.

Figure 6 gives distribution of normal pressure on the interface when $h_0=0.25$. Pressure increases rapidly on the whole till the first 30 elements yield, and shapes of the distribution are almost similar. Afterward, pressure near the center of the interface continues to increase because of an increasing hydrostatic pressure inside the block, but pressure near the edge of the interface scarcely increases, because resistance of material to compression decreases there due to

Fig. 5 Relation between load of compression and height to width ratio h_0

rapid growth of plastic deformation at the edge in early stages. Therefore with progress of compression, difference of pressure between at the center and near the edge increases, and tendency of convex distribution of it increases.

Figure 7 gives distribution of pressure for

	Number of yielded elements	Load per unit area kg/mm ²
a	2	6.528 8
b	30	10.384 8
c	60	11.038 7
d	90	12.050 5
e	120	12.586 5
f	150	12.871 6
g	180	12.990 7

Fig. 6 Distribution of normal pressure on the interface (when $h_0=0.25$)

	h_0	Number of yielded elements
a	0.25	12
b	0.5	18
c	1	92
d	1.5	100
e	2	104
f	3	60

when compression ratio=0.001
Blocks are yielded partially.

Fig. 7 Distribution of normal pressure on the interface

the stage when compression ratio is 0.001. Pressure is very large at the edge in every case, and pressure at the center is larger than that at other parts except the edge (this means convex distribution) when h_0 is small. As h_0 increases, pressure at the center decreases rapidly first and then gradually gently, so distribution of it changes to concave. This fact was made clear by both the theoretical analysis by calculus of variation in Part 1 and the experiment in Part 2. When h_0 is large, distributions of pressure in this figure show small ridges at the center. This is considered to be the error (within 6.6% of total pressure) at caused by the coarse division of block into elements in the direction of height as h_0 becomes large.

Figure 8 gives distribution of tangential stress on the interface when compression ratio is 0.001. Tangential stress is zero at the center, increases

Fig. 8 Distribution of tangential stress on the interface

Fig. 9 Distribution of friction coefficient on the interface

towards the edge, and becomes considerably large at the edge in any case of h_0 . The theoretical analysis by calculus of variations in Part 1 gave the same result as this.

Figure 9 gives distribution of friction coefficient on the interface. This shows nearly the same tendency as distribution of tangential stress, and also coincides qualitatively with both the theoretical result by calculus of variation in Part 1 and the experimental result of distribution of contact ratio in Part 2.

Fig. 10 Appearance and growth of plastic region

Figure 10 shows appearance and growth of plastic regions in the block in which $h_0=0.25$. Yield begins first at the edge of the block. Next, yield occurs inside the central region ($y=0$) relatively near the free surface, and both plastic

regions connect to each other immediately. This connected plastic region develops toward the center, and when about 180 elements are yielded, plastic region contains the center of the block to penetrate the block obliquely. At the stage when plastic region penetrates thus inside the block, large elastic region still remains near the interface because material is the most difficult to be deformed there due to constraint by friction force on the interface. Besides, small elastic region remains still near the center of the free surface. This is because material here is especially little restrained, and can move easily without large deformation.

Figure 11 shows appearance and growth of plastic regions when $h_0=2$. First yield begins at the edge of the block, too, but next yield occurs at an inside part on the section of $x=0$ a little apart from the center. This inside plastic region covers the central region of the block immediately, and then it grows rapidly towards the free surface. Meanwhile the plastic region which yields first at the edge scarcely grows. The reason for this phenomenon is considered as follows. As plastic region which appears inside the block is surrounded completely by elastic region, elastic restraint must be loosened by yielding neighbouring elastic region in order that plastic deformation may grow there. But on the contrary, as plastic region on the free surface near the edge is little restrained (free to move), deformation here can progress without exerting large influence on neighbouring elastic regions.

Black part : Initial plastic region
 Arrow : Dominant direction of growth
 Dotted line : Shape of bulge on the side surface
 Fig. 12 Initial plastic regions and their dominant directions of growth inside the block in compression

Now, plastic region which yields on the central section spreads rapidly to arrive at the center of the free surface. Afterwards, it penetrates inside the block obliquely, connecting with the plastic region near the edge.

Initial appearance and main direction of growth of plastic regions are shown together with rough shape of bulge on the free surface in Fig. 12. Plastic region appears first at the edge of the block in any case, but appearance and growth of plastic regions in other parts are affected largely by h_0 . When h_0 is small ($h_0=0.25, 0.5$), they grow from outside toward inside. In these cases, large hydrostatic pressure develops inside the block because of mutual interference of friction forces on the two interfaces. Thus, distribution of pressure on the interface becomes convex (refer to Figs. 6 and 7), and load becomes large (refer to Fig. 5). This strong interference of friction forces prevents growth of bulge on the side surface.

When h_0 is a little larger, yield occurs at the center of the block after at the edge. This internal plastic region decreases the inside hydrostatic pressure. Thus, distribution of pressure becomes concave and load decreases. When h_0 is considerably large ($h_0=2$), second yield occurs on two parts on $x=0$ a little apart from the center, and the resultant plastic region contains the center immediately. When h_0 is very large ($h_0=3$), plastic regions on two parts on $x=0$ grow to reach the free surface before covering the central

part. This causes double bulge on the side surface. It is interesting that early plastic region appears inside the block instead of on its surface, when h_0 is large.

Figure 13 gives the slip-lines inside perfectly rigid plastic blocks that are yielded by plane-strain compression with rough plane tools, and shows the ultimate state of compression of elastic, strain-hardening block in Fig. 12. Close relation between Fig. 12 and Fig. 13 can be seen at a glance, and the result of our analysis by the finite element method verifies validity of the solution by slip-line theory well.

4. Conclusions

The authors analysed deformation inside the elastic, strain-hardening block when it is compressed under the condition of plane-strain, using the finite element method. The surface of the block was assumed to stick to the surface of two parallel rigid plane tools. Appearance and growth of plastic region, shape of bulge, load of compression, distribution of normal pressure and of tangential stress on the interface, etc. were calculated at early stages of deformation. As the result, the following matters were made clear.

(1) When height to width ratio h_0 is small, size of bulge becomes large nearly in proportion to h_0 , but when h_0 becomes considerably large, it scarcely changes.

(2) When $h_0 < 2$, bulge is single, but when $h_0 > 2$, it is doubled.

(3) Relation between load and compression ratio deviates from the linear relation, after yield advances considerably.

(4) Load is large when h_0 is small, and as h_0 becomes large, it decreases at first rapidly and gradually gently.

(5) Plastic region penetrates inside the block most easily, when $h_0=1$.

(6) Distribution of normal pressure on the interface is convex when h_0 is small, but it is concave when h_0 is large.

(7) Tangential stress and friction coefficient on the interface are zero at the center, and they increase toward the edge.

(8) Yield occurs first at the edge of the block regardless of h_0 . Plastic region which appears next differs much according to h_0 . That is, when h_0 is small, it appears near the free surface and grows toward the inside. When h_0 is a little larger, it appears near the center, and when h_0 becomes still larger, two plastic regions appear at two points of the inside.

(9) The above phenomena also validate

Grey parts are slip zones, and other parts are rigid regions.

Fig. 13 Slip lines inside perfectly rigid, plastic blocks in compression when friction force is large

the solutions obtained formerly by the slip line theory well.

References

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Discussion

S. TAKAHASHI (Hitachi Ltd.):

(1) In this report, analysis is performed only within the scope of small strain (within 0.2%). Is it possible to extend the analysis to larger strain (for instance several percentage)? If it is not possible, please tell me the reason for it, limit of application of the finite element method and future prospect.

(2) Elastic, strain-hardening material is used in this analysis. In what part of the result does the effect of strain-hardening appear?

(3) Distribution of tangential stress on the interface given in Fig. 8 (a) agrees qualitatively with the experimental result by Backofen, but it seems to be inconsistent with the solution of slip-line theory in Fig. 13 (a). Please tell me the reason for this inconsistency.

Authors' closure

(1) The finite element method can be used to analyse strain larger than certain percentage, and we will put this into practice in the next report (Part 4). In the present report, analysis is restricted to small strain, because we want to know especially early deformation in detail, and because we want to prove validity of neglect of change of the matrix $[D^p]$ after yield occurs for saving of computing time.

(2) We performed analysis also in case of elastic, perfectly plastic material. Comparing the result of the analysis with that of the present analysis, the former load is only a little smaller than the latter one, and no large difference is found in appearance and early growth of plastic regions in both cases. Thus effect of strain-hardening is not so large in early deformation. But after plastic region penetrates the block, difference of loads between both cases becomes large rapidly. After the block is yielded totally, load does not change at all in case of elastic, perfectly plastic material, but load continues to increase in case of elastic, strain-hardening material.

(3) Figure 13 (a) gives slip-line in a perfectly rigid, plastic block in which grey parts slip at once and white parts are not deformed at all. Meanwhile in case of an elastic, strain-hardening block, plastic region grows gradually towards the centroid, and even unyielded region is deformed elastically as is shown in Fig. 10. Therefore, Fig. 8 (a) differs from Fig. 13 (a) mainly because of elastic deformation and secondarily because of strain-hardening. As the actual material is an elastic one, not Fig. 13 (a) but Fig. 8 (a) coincides with the experimental result by Backofen.