539.388.25 : 512.8 : 531.44

On the Non-Uniform Deformation of Material in Axially Symmetric Compression Caused by Friction*

(Part 2, Theoretical Analysis of Compression of Circular Cylinder by the Finite Element Method)

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Non-uniform deformation of an elastic, strain-hardening circular cylinder in axial compression is analyzed by the finite element method. Deformation is pursued from elastic through partially plastic to totally plastic stages, allowing for the finite deformation or geometrical change of the material coordinates. As the boundary conditions, the experimentally determined values of the slip-ratio which was proposed by the authors are used instead of the conventional friction coefficient. Calculations of 23 cases are performed for various combinations of lubrication and height to diameter ratio. As the result, the followings have become clear: shape of bulge of the free side surface, relation between the load and the compressive strain, distribution of normal pressure and tangential stress together with the friction coefficient on the boundary surface, mean friction coefficient, patterns of appearance, growth of plastic zones, etc. These results are qualitatively in agreement with the theoretical results which were presented in Part 1 of this series of investigation; i.e. the results obtained by applying variational principle. Agreement with the experimental observations is also satisfactory.

1. Introduction

Axial compression of a circular cylinder is often used as the means of compression test to detect strength, yield stress or compressibility of materials. In the previous report⁽¹⁾, theoretical analysis about this was performed, making use of Hill's method based on variational principle, and the results were verified experimentally. In the present report, theoretical analysis is also performed, using the finite element method. Material of a cylinder is assumed to be elastic and linearly strain-hardening. Calculation is done to the extent of compression of several percent through elastic, partially plastic and totally plastic ranges, adopting the finite deformation theory. Yield conditions of von Mises and $[D^p]$ matrix derived from the stress-strain relation of Prandtl-Reuss by Yamada⁽²⁾ are used in the

present analysis.

It is one of the most difficult problems in analysis of this kind how to introduce the boundary condition into analysis. Authors have already proposed the characteristic value named relative slip ratio in order to solve this problem⁽³⁾. In the present report, distribution of this relative slip ratio on the interface is measured experimentally, and used, as the boundary condition. The following five conditions of lubrication are selected; namely no lubricant, spindle oil No. 60, castor oil, graphite (Prohite No. 526) and Jhonson wax No. 111. In calculation, boundary condition of sticking is added to these five.

As the result of calculation, appearance and growth of plastic regions inside the cylinder, bulge on the side surface, inside stress and strain, load of compression, distribution of normal pressure and tangential stress, etc. are obtained.

2. Method of calculation

2.1 Basic theory

Relation between stress and strain of elastic body is

* Received 19th July, 1969.

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$$\{\sigma\} = [D^e]\{\varepsilon\} \quad \dots\dots\dots (1)$$

where $\{\sigma\}$ and $\{\varepsilon\}$ are row vectors which express components of stress and strain respectively. $[D^e]$ in case of the axially symmetric problem is

$$[D^e] = \frac{E}{1+\nu_s} \begin{pmatrix} \frac{1-\nu_s}{1-2\nu_s}, & \frac{\nu_s}{1-2\nu_s}, & \frac{1-\nu_s}{1-2\nu_s}, & \text{Sym.} \\ \frac{\nu_s}{1-2\nu_s}, & \frac{1-\nu_s}{1-2\nu_s}, & \frac{\nu_s}{1-2\nu_s}, & \\ \frac{\nu_s}{1-2\nu_s}, & \frac{\nu_s}{1-2\nu_s}, & \frac{1-\nu_s}{1-2\nu_s}, & \\ 0, & 0, & 0, & \frac{1}{2} \end{pmatrix} \quad \dots\dots\dots (2)$$

Yamada showed the same relation between increment of stress and increment of strain in elasto-plastic problems as Eq. (1), changing $[D^e]$ to $[D^p]$ expressed as follows.

$$[D^p] = \frac{E}{1+\nu_s} \begin{pmatrix} \frac{1-\nu_s}{1-2\nu_s} - \frac{\sigma_r'^2}{S}, & \frac{\nu_s}{1-2\nu_s} - \frac{\sigma_r'\sigma_z'}{S}, & \frac{1-\nu_s}{1-2\nu_s} - \frac{\sigma_z'^2}{S}, & \text{Sym.} \\ \frac{\nu_s}{1-2\nu_s} - \frac{\sigma_r'\sigma_z'}{S}, & \frac{\nu_s}{1-2\nu_s} - \frac{\sigma_r'\sigma_\theta'}{S}, & \frac{\nu_s}{1-2\nu_s} - \frac{\sigma_z'\sigma_\theta'}{S}, & \\ \frac{\nu_s}{1-2\nu_s} - \frac{\sigma_r'\sigma_\theta'}{S}, & \frac{\nu_s}{1-2\nu_s} - \frac{\sigma_z'\sigma_\theta'}{S}, & \frac{1-\nu_s}{1-2\nu_s} - \frac{\sigma_\theta'^2}{S}, & \\ -\frac{\sigma_r'\tau_{rz}}{S}, & -\frac{\sigma_z'\tau_{rz}}{S}, & \frac{\sigma_\theta'\tau_{rz}}{S}, & \frac{1}{2} - \frac{\tau_{rz}^2}{S} \end{pmatrix} \quad \dots\dots\dots (3)$$

where, $S = \frac{2}{3}\bar{\sigma}^2(1+H'/3G)$, $\bar{\sigma} = \sqrt{\frac{2}{3}\sigma_{ij}'\sigma_{ij}'}$: equivalent stress, $H' = d\bar{\sigma}/d\bar{\varepsilon}^p$, E : Young's modulus, G : shear modulus, $d\bar{\varepsilon}^p$: equivalent plastic strain increment, ν_s : Poisson's ratio.

Assuming an axially symmetric body as assemblage of many ring elements of triangular section, stiffness matrix $[k]$ which indicates relation between increment of nodal displacement and increment of nodal force of elements is shown as follows:

$$[k] = 2\pi \int [B]^T [D] [B] r dr dz \quad \dots\dots\dots (4)$$

$[D]$ represents $[D^e]$ for elastic regions and $[D^p]$ for plastic regions. $[B]$ is a matrix which indicates relation between nodal displacement and strain of element, and is a function of coordinates. Zienkiewicz⁽⁴⁾ showed the next three means to calculate Eq. (4); namely, (1) approximate substitution of $[B]$ with $[\bar{B}]$ which expresses $[B]$ at the centroidal point of the section of the element, (2) numerical integration with a computer, and (3) rigorous, term by term integration. If calculation of only one step is necessary as in elastic problems, each of above methods may be used. But when repetition of more than 100 steps is necessary as in the elasto-plastic problems in the present report, accumulation of errors becomes too large by the method (1), and time of calculation becomes too long by the method (2). Thus authors used the method (3) in the present analysis.

2.2 Procedure of calculation

A half body of a cylinder is divided into 288

elements which have equal triangular sections and 169 nodal points as shown in Fig. 1. The process of the present calculation is outlined below.

First, with proper displacement of tool, elastic calculation is performed. Nodal displacement, stress and strain of elements, equivalent stress $\bar{\sigma}^e$ of each element etc. are obtained. Displacement of tool is increased until the mean value of $\bar{\sigma}^e$ of four elements where $\bar{\sigma}^e$ is the largest becomes equal to the yield stress Y , and these four elements are regarded as yielded. As for yielded elements, $[D^p]$ is calculated from Eq. (3), using the stress in the previous stage. Calculat-

288 elements of equal section
169 nodal points

Fig. 1 Elements in a circular cylinder (Half body of cylinder)

ing $[B]$ in all elements again taking into account of the deformation of all elements till the previous stage, $[k]$ is calculated from Eq. (4). Increment of nodal displacement and increment of stress and strain of elements are calculated, with a proper increment of displacement of tool. Increment of compression value is increased until four elements which have the largest values of $\bar{\sigma}^e$ out of all elastic elements yield. $[D^p]$ of all elements that have yielded at the previous stage and $[B]$ of all elements are calculated again. These procedures are repeated until all elements yield. After all elements have yielded, matrices $[D^p]$ and $[B]$ of all elements are reformed at

each stage, where increment of displacement of tool is decided as 0.002. Calculation is repeated until total displacement of tool becomes equal to several percent.

Finite deformation theory is adopted in this procedure, namely all $[B]$ matrices are reformed at each stage throughout calculation. If infinitesimal deformation theory is adopted, namely if changes of $[B]$ due to deformation of elements are ignored, accumulation of errors becomes too

Fig. 2 Distribution of relative slip-ratio on the interface (Experimental result)

Fig. 3 Distribution of relative slip-ratio on the interface (Experimental result)

	h_0	Number of elements yielded
a	0.25	14
b	0.5	36
c	1	82
d	1.5	79
e	2	31
f	2.5	13

Compressive strain = 0.000 94

Sticking

Fig. 4 Shape of bulge on the side surface

when $h_0 = 2$

Without lubricant

Bulge is double at first, but changes into single gradually

Fig. 5 Shape of bulge on the side surface

large to continue calculation.

3. Boundary condition

A constant friction coefficient is generally used as the boundary condition in analysis concerning friction. But it is very difficult to measure friction coefficient in actual metal working, and moreover validity of using the constant value of friction coefficient in plastic processing is doubtful. Then, authors thought of grasping friction with deformation of worked material, and proposed relative slip ratio. Relative slip ratio ν is defined as follows in axial compression of the cylinder;

$$\nu = -2HV_R/(RV_z) \quad (5)$$

where V_R is velocity of relative slip between the tool and the surface of material at an instant of processing, V_z is normal velocity of this interface, $2H$ is height of the cylinder and R is radius at an arbitrary point on the interface. In this definition, ν is zero when the interface sticks to the surface of the tool, and 1 when there is no friction between them.

In practical measurement of ν , finite increments of relative slip and of normal displacement are measured instead of V_R and V_z , and then V_R and V_z are obtained from these increments, because direct determination of the velocity is difficult.

Seventeen kinds of experiments were done to measure distribution of ν on the interface, under a series of conditions of lubrication, that is, no lubricant, spindle oil No. 60, castor oil, graphite (Prohite No. 526) and Jhonson wax No. 111, and for various height to width ratios h_0 . Aluminum cylinders of 35 mm diameter were used in these experiments.

Some examples of the results given in Figs. 2 and 3 show that ν becomes larger in the order

when $h_0 = 0.5$

Compressive strain = 0.05

Fig. 6 Shape of bulge on the side surface

of no-lubricant, spindle oil, castor oil, graphite and Jhonson wax. This indicates that ν becomes large as friction becomes small. Besides, ν becomes large as h_0 becomes small in the same condition of lubrication. Moreover, ν seems to be small near the center, and large at the circumference in cases of no-lubricant and spindle oil. But it is nearly constant in other cases of small friction.

Figure 3 shows examples of experimental result of ν when load is 0~12 tons and 12~15 tons. ν without lubricant is a little smaller when 12~15 tons than when 0~12 tons. This fact is related to the experimental result in the previous report that friction coefficient without lubricant becomes the largest when load is 14~21 tons. With lubricants, however, ν is inversely a little larger when 12~15 tons than when 0~12 tons. Meanwhile the experimental result in the previous report has made clear that friction coefficient with lubricants becomes the largest when load is 10~11.5 tons. This indicates that ν is small when friction coefficient is large.

These experimentally measured results of ν are used as the boundary condition of the theoretical analysis as follows: 0.5ν is used in totally plastic stages. As the measurement of ν is impossible in elastic and in partially plastic stages, the boundary condition is assumed respectively as

$$\nu_s \nu, \quad (\nu_s \text{ is Poisson's ratio})$$

and

$$(\nu_s + (0.5 - \nu_s)N/N_0)\nu,$$

The two marks which cut each curve indicate the instants when first plastic region yields and when total yielding occurs respectively. Load is the smallest when $h_0 = 1$ within the scope of this figure, but when compression advances more, load becomes smaller as h_0 becomes larger.

	h_0
a	0.25
b	0.5
c	1
d	1.5
e	2
f	2.5

Fig. 7 Relation between load of compression and compressive strain

Table 1 Effect of lubrication on compression when ($h_0=1$)

Lubrication	C.S. on first yielding $\times 10^4$	Load on first yielding kg	C.S. on total yielding $\times 10^4$	Load on total yielding kg	Load when C.S.=0.005 kg	Load when C.S.=0.085 kg	M.F.C. when C.S.=0.01 kg
Sticking	8.178	20.02	118.847	25.67	23.02	58.20	0.2233
No-lubricant	8.538	20.43	105.320	25.07	23.13	55.96	0.2050
Spindle oil	8.978	21.25	60.727	23.88	23.46	55.56	0.1569
Castor oil	9.747	22.55	12.011	22.83	24.1964	55.51	0.0377
Graphite	9.794	22.62	11.734	22.86	24.1969	55.33	0.0328
Jhonson wax	9.892	22.76	11.145	22.91	24.2479	55.21	0.0213

C.S. : Compressive strain

Load : Load of compression of circular cylinder of unit radius

M.F.C. : Mean friction coefficient

Without lubricant

Total yielding occurs when compressive strain is 0.0035~0.009

Fig. 8 Relation between load of compression and compressive strain

Jhonson wax No. 111

Total yielding occurs when compressive strain is 0.001~0.0012

Fig. 9 Relation between load of compression and compressive strain

when $h_0=2.5$

Sticking

a: first yielding

b: 144 elements yielded

c: total yielding

Fig. 10 Distribution of normal stress on the interface

Without lubricant

Compressive strain=0.001

Fig. 11 Distribution of normal pressure on the interface

where N_0 is total of elements and N is the number of yielded elements.

4. Result of calculation

Material of a cylinder is assumed to be aluminum, and the following characteristic values are given: Young's modulus $E=7200 \text{ kg/mm}^2$, Poisson's ratio $\nu_s=0.33$, yield stress $Y=7.24 \text{ kg/mm}^2$, strain hardening coefficient $H'=13.5 \text{ kg/mm}^2$.

4.1 Bulge on the free surface

Figure 4 gives bulge in early stages of compression in sticking. Bulge is small when h_0 is

	Lubrication	h_0
a	Without lubricant	1.5
b	Sticking	1.5
c	Spindle oil	1
d	Jhonson wax	0.25

Compressive strain=0.05

Fig. 12 Distribution of friction coefficient on the interface

(1) 24 elements yielded

(2) 25~48 elements yielded

(3) 73~96 elements yielded (49~72 elements yielded are omitted)

When $h_0=0.25$ Jhonson wax No. 111

Grey parts show regions which have been yielded by the previous stages.

Fig. 13 Appearance and growth of plastic region

small, because mutual interference of friction acting on both interfaces is large. Bulge becomes larger as h_0 increases till $h_0=1$, but its size does not change so much when $h_0>1$. Then, bulge changes from single to double when h_0 is a little larger than 1.5. These facts are valid also for the lubricated cylinders. It was presumed experimentally in Part 1⁽¹⁾ that bulge changes from single to double when $h_0=1.65$, which coincides well with this theoretical result.

Figure 5 shows growth of bulge when $h_0=2$ without lubricant. Bulge is double in early stage of compression, and gradually changes to single as compression advances.

Figure 6 shows relation between size of bulge and lubrication, and bulge becomes extremely small when lubrication is good.

4.2 Load of compression

Figure 7 gives load in sticking in rather early periods of compression. Figures 8 and 9 give loads without lubricant and with Jhonson wax No. 111. In Fig. 7, load becomes the smallest when $h_0=1$, because the conical shaped plastic region penetrates most easily. But even in this case, load comes to decrease as h_0 increases, when compression advances. In actual materials in which stress-strain curves are not straight but always convex to the upper direction, load is always considered to decrease as h_0 increases. Figures 8 and 9 indicate that the worse lubrication becomes, the more greatly load is influenced by h_0 .

Table 1 indicates influence of lubrication on

(4) 121~144 elements yielded (97~120 elements yielded are omitted)

(5) 169~192 elements yielded (145~168 elements yielded are omitted)

(6) 241~264 elements yielded (193~240 elements yielded are omitted)

load. The first plastic region appears earlier, but the last elastic region disappears later as lubrication becomes worse. Load becomes smaller in early stage of compression, but larger in advanced stage of compression as lubrication becomes worse. The reason for this was explained above.

4.3 Distribution of normal pressure on the interface

Figure 10 gives growth of normal pressure in sticking when $h_0=2.5$. Its distribution is con-

cave, namely pressure at the center is smaller than that at the circumference. But it turns to convex as compression advances, because hydrostatic pressure at the center increases.

Figure 11 gives an example of relation between pressure and h_0 . Distribution of pressure is convex when h_0 is small, but changes to concave as h_0 increases. This fact is valid in all lubrications, and agrees well with the theoretical result by calculus of variations and with the experimental result both in the previous report⁽⁹⁾.

when $h_0=1$, Without lubricant, Grey parts show regions which have been yielded by the previous stages.
Fig. 14. Appearance and growth of plastic region

4.4 Distribution of friction coefficient on the interface

In the present analysis, the experimental results of relative slip ratio are used as the boundary condition, as was related above. Thus distributions of friction coefficient on the interface are obtained as the result of analysis. Figure 12 shows some examples of the results. Generally speaking, the results are divided into two types: One in which friction coefficient increases from the center to the circumference, and the other in which it increases first, decreases then and increases again from the center to the circumference. The former is relevant when friction is large, and the latter is relevant when lubrica-

tion is good. Existence of these two types has already been verified in the previous report⁽¹⁾.

The readers looking at Fig. 12 may think that friction force without lubricant is larger than in sticking, but the values of mean friction in Table 1 indicate this is false. It is because friction coefficient in sticking amounts to more than 2 at the circumference. Moreover, Table 1 indicates that mean friction coefficient becomes smaller in the order of sticking, no-lubricant, spindle oil, castor oil, graphite and Jhonson wax.

4.5 Appearance and growth of plastic regions

When a circular cylinder is compressed in the axial direction, plastic zones appear in several parts, and grow until they cover the whole block

when $h_0=2$,

Sticking, Grey parts show regions which have been yielded by the previous stages.

Fig. 15 Appearance and growth of plastic region

at last. Figures 13, 14 and 15 give some examples of them. In all these cases, plastic region appears first at the edge. In Fig. 13, the next plastic region appears on the mid-section ($z=0$) near the side surface, and then at the centroid and at a part on the interface. But these appear nearly at the same time, because lubrication is good in this case and deformation is nearly uniform. In Fig. 14, the second plastic region appears at the centroid, and this grows much faster than the first one at the edge. In Fig. 15, the second ones appear at two parts on z -axis, and these cause a double bulge on the side surface.

5. Conclusions

Non-uniform deformation of an elastic, strain-hardening cylinder in axial compression was analysed, using the finite element method. In the present analysis, the theory of finite deformation was adopted, and calculation was continued through

elastic, partially plastic and totally plastic deformation. Experimentally measured values of the relative slip ratio are used as the boundary condition. As the result, many new matters are made clear concerning shape of bulge on the side surface, load of compression, distributions of normal pressure, of tangential stress and of friction coefficient on the interface, appearance and growth of plastic regions, etc.

Lastly, Authors thank Prof. Y. Yamada at the Institute of Industrial Science, University of Tokyo who guided them in their present research greatly.

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Discussion

A. SINDO (Kobe University):

(1) A circular cylinder is divided into elements of equal section in Fig. 1, but finer division than this seems to be necessary near the circumference of the interface, because this place is a singular point of deformation. If this operation is impossible, more accurate calculation by some way is desirable about deformation near this place, for normal pressure changes rapidly there as is shown in Fig. 11, and for friction coefficient becomes larger than 2 there under the condition of sticking.

(2) Is it not necessary to take a change in value of relative slip-ratio during compression into account? Are there some means to correct experimentally measured values of relative slip-ratio using theoretical result?

(3) Is it impossible to find any method to correct experimentally measured stress-strain relation using results of this theoretical analysis, for instance shape of bulge?

(4) Theoretical results may change considerably by changing the way to divide the body into elements, for instance, by only changing direction of all oblique sides of elements inversely.

Authors closure

(1) We also intended to make finer division

near the edge. But computing time is shortened and necessary volume of core is decreased by equal division. Besides, the location where strain is large changes much according to h_0 and condition of lubrication. So, we adopt the way of equal division in order to perform calculation under various conditions in one program.

As we use cores of 95% of capable volume now, more accurate calculation is impossible even partially.

(2) It is desirable to take change of relative slip-ratio into account. But we did neither take it into account nor correct relative slip-ratio because of limits in computing time, in volume of core and in accuracy of measurement of relative slip-ratio.

(3) Before looking for such a method, change of H' during compression should be taken into account in future.

(4) The way of division in Fig. 1 is suitable, because oblique side of rectangular elements coincides with the direction of the oblique plastic zone of large deformation which penetrates the block. The number of elements in the present analysis is so large that theoretical result may not be affected much by way of division.