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A Numerical Method for the Limit Analysis of General Shells of Revolution*

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This paper is concerned with the limit analysis of shells of revolution made of a rigid-perfectly-plastic material that obeys Tresca's yield condition. The four-dimensional yield hypersurface for shells of revolution was once presented by E.T. Onat and W. Prager. Using this yield surface, a numerical solution for simply supported conical shells was obtained by R.H. Lance and E.T. Onat. Fundamentally based on their method and considering more precisely the yield hypersurface, a numerical method is presented that can be applied to general shells of revolution.

As numerical examples, the solutions for truncated conical shells, and pressure vessels with rigid circular plates are presented. These solutions verify that this method is correct.

1. Introduction

For shells of revolution of many kinds as cylindrical, conical, and spherical shells, various solutions of limit analysis have been presented as shown in ref. (1), but most of those solutions are approximate ones. A numerical method is proposed in this paper, which is applicable to limit analyses of general shells of revolution and gives numerically accurate results.

The yield surfaces of various kinds, which were used for limit analyses in most of previous papers, were approximate ones. But in the research by E.T. Onat and W. Prager⁽²⁾, an exact yield hypersurface in a four-dimensional space was presented for general shells of revolution, which was based on Tresca's yield condition. T. Nakamura⁽³⁾ proposed an approximate yield surface, which coincides with that exact surface in most parts and circumscribes it partly, and showed an analytical solution for it. J.C. Gerdeen⁽⁴⁾ solved the problem of conical shells using the Nakamura's yield surface. R.H. Lance and E.T. Onat⁽⁵⁾ also treated numerically another problem of conical shells by applying the exact yield surface mentioned above.

In this paper, adding some considerations to the solutions by R.H. Lance and E.T. Onat⁽⁵⁾, a

numerical method, which can be applied to shells of revolution of any kind, is proposed. As numerical examples, two problems of truncated conical shells and pressure vessels with rigid circular plates are treated. The results on the latter are compared with an approximate solution presented before by one of the authors.

2. Basic equations

Some main notations used in this paper and their positive directions are shown in Fig. 1.

The equilibrium of the shell element requires that⁽¹⁾

$$\left. \begin{aligned} \frac{d}{d\varphi}(N_\varphi r_0) - N_\theta r_1 \cos \varphi - Q_\varphi r_0 + Y r_1 r_0 &= 0 \\ N_\varphi r_0 + N_\theta r_1 \sin \varphi + \frac{d}{d\varphi}(Q_\varphi r_0) + Z r_1 r_0 &= 0 \\ \frac{d}{d\varphi}(M_\varphi r_0) - M_\theta r_1 \cos \varphi - Q_\varphi r_1 r_0 &= 0 \end{aligned} \right\} \dots (1)$$

The rates of strain components are related

Fig. 1 Notations

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to velocity components by the equations⁽¹⁾

$$\left. \begin{aligned} \varepsilon_\varphi &= \frac{1}{r_1} \left(\frac{dv}{d\varphi} - w \right) \\ \varepsilon_\theta &= \frac{1}{r_2} (v \cot \varphi - w) \\ \kappa_\varphi &= -\frac{1}{r_1} \frac{d}{d\varphi} \left\{ \frac{1}{r_1} \left(v + \frac{dw}{d\varphi} \right) \right\} \\ \kappa_\theta &= -\frac{\cot \varphi}{r_1 r_2} \left(v + \frac{dw}{d\varphi} \right) \end{aligned} \right\} \dots\dots\dots (2)$$

where v and w are the meridional and normal components of the middle surface velocity, ε_φ and ε_θ the meridional and circumferential rates of extension, and κ_φ and κ_θ the principal rates of curvature of the middle surface.

Elimination of v and w from Eqs. (2) leads to compatibility equations

$$\left. \begin{aligned} \frac{d\varepsilon_\theta}{d\varphi} + (\varepsilon_\theta - \varepsilon_\varphi) \frac{r_1}{r_0} \cos \varphi - \kappa_\theta r_1 \tan \varphi &= 0 \\ \frac{d\kappa_\theta}{d\varphi} + (\kappa_\theta - \kappa_\varphi) \frac{r_1}{r_0} \cos \varphi + \kappa_\theta \tan \varphi &= 0 \end{aligned} \right\} \dots\dots\dots (3)$$

3. Yield surface and flow rule

The yield surface for shells of revolution composed of a material that obeys Tresca's yield condition has been obtained by E.T. Onat and W. Prager⁽²⁾. This yield surface is obtained

by considering energy dissipation in plastic flow. The results of ref. (4) are represented in Tables 1.

In Tables 1, n_θ , n_φ , m_θ and m_φ are defined by equations

$$\left. \begin{aligned} n_\theta &= \frac{N_\theta}{N_0}, & n_\varphi &= \frac{N_\varphi}{N_0} \\ m_\theta &= \frac{M_\theta}{M_0}, & m_\varphi &= \frac{M_\varphi}{M_0} \end{aligned} \right\} \dots\dots\dots (4)$$

Table 1 Expression of yield surface
(a) Cases when p , q and r are different

Intermediate parameter	Stress resultants			
	$\pm n_\theta$	$\pm n_\varphi$	$\pm m_\theta$	$\pm m_\varphi$
p	$-(p+q)$	$-(q-r)$	$1-2(p^2+q^2)$	$2(r^2-q^2)$
q	$-(p+q)$	$-(q+r)$	$1-2(p^2+q^2)$	$1-2(q^2+r^2)$
r	$-(q-p)$	$-(q+r)$	$2(p^2-q^2)$	$1-2(q^2+r^2)$

Flow rule

$$N_0 \varepsilon_\theta : N_0 \varepsilon_\varphi : M_0 \kappa_\theta : M_0 \kappa_\varphi = -4p(q-r) : -4r(p-q) : (q-r) : (p-q)$$

(b) Cases when two of p , q , r coincide

Coinciding parameters	Yield surface	$N_0 \varepsilon_\theta : N_0 \varepsilon_\varphi : M_0 \kappa_\theta : M_0 \kappa_\varphi$
$p = q$	$m_\theta = \pm(1-n_\theta^2)$	$\pm 2n_\theta : 0 : 1 : 0$
$q = r$	$m_\varphi = \pm(1-n_\varphi^2)$	$0 : \pm 2n_\varphi : 0 : 1$
$p = r$	$m_\theta - m_\varphi = \pm[1 - (n_\theta - n_\varphi)^2]$	$\pm 2(n_\theta - n_\varphi) : \mp 2(n_\theta - n_\varphi) : 1 : -1$

Table 2 Yield surface

Designation of yield surface element	Yield condition	Bounding inequality	Adjoining surface element
$G_\theta^\pm$	$\pm(m_\theta - m_\varphi) + (n_\theta - n_\varphi)^2 + \left(\frac{m_\varphi}{2n_\varphi} \pm n_\theta \right)^2 - 1 = 0$	$\mp m_\varphi - 2n_\varphi(n_\theta - n_\varphi) \geq 0$ $\pm m_\varphi + 2n_\theta n_\varphi \geq 0$ $1 - \left \frac{m_\varphi}{2n_\varphi} \pm n_\theta \right \geq 0$ $1 - \left \frac{m_\varphi}{2n_\varphi} \mp n_\varphi \right \geq 0$	$H_\theta^\pm$ $H_\varphi^\pm$ $G_\theta^\pm$ $G_\varphi^\pm$
$G_\theta^\pm$	$\pm m_\theta + n_\theta^2 + \left[\frac{m_\theta - m_\varphi}{2(n_\theta - n_\varphi)} \pm n_\varphi \right]^2 - 1 = 0$	$\pm(m_\theta - m_\varphi) + 2n_\theta(n_\theta - n_\varphi) \geq 0$ $\mp(m_\theta - m_\varphi) - 2n_\varphi(n_\theta - n_\varphi) \geq 0$ $1 - \left \frac{m_\theta - m_\varphi}{2(n_\theta - n_\varphi)} \mp(n_\theta - n_\varphi) \right \geq 0$ $1 - \left \frac{m_\theta - m_\varphi}{2(n_\theta - n_\varphi)} \pm(n_\theta - n_\varphi) \right \geq 0$	$H_\varphi^\pm$ $H_\theta^\pm$ $G_\varphi^\pm$ $G_\theta^\pm$
$G_\varphi^\pm$	$\mp(m_\theta - m_\varphi) + (n_\theta - n_\varphi)^2 + \left(\frac{m_\theta}{2n_\theta} \pm n_\varphi \right)^2 - 1 = 0$	$\pm m_\theta + 2n_\theta n_\varphi \geq 0$ $\mp m_\theta + 2n_\theta(n_\theta - n_\varphi) \geq 0$ $1 - \left \frac{m_\theta}{2n_\theta} \pm n_\varphi \right \geq 0$ $1 - \left \frac{m_\theta}{2n_\theta} \mp n_\theta \right \geq 0$	$H_\theta^\pm$ $H_\varphi^\pm$ $G_\theta^\pm$ $G_\varphi^\pm$
$H_\theta^\pm$	$\pm m_\theta + n_\theta^2 - 1 = 0$	$\pm m_\theta \geq 0$ $\pm m_\varphi + 2n_\varphi(n_\theta - n_\varphi) \geq 0$ $\pm(m_\theta - m_\varphi) + 2n_\varphi(n_\theta - n_\varphi) \geq 0$	$H_\theta^\pm$ $G_\theta^\pm$ $G_\varphi^\pm$
$H_\varphi^\pm$	$\pm m_\varphi + n_\varphi^2 - 1 = 0$	$\pm m_\varphi \geq 0$ $\mp(m_\theta - m_\varphi) - 2n_\theta(n_\theta - n_\varphi) \geq 0$ $\mp m_\theta - 2n_\theta(n_\theta - n_\varphi) \geq 0$	$H_\varphi^\pm$ $G_\theta^\pm$ $G_\varphi^\pm$
$H_\theta^\pm$	$\pm(m_\theta - m_\varphi) + (n_\theta - n_\varphi)^2 - 1 = 0$	$\pm(m_\theta - m_\varphi) \geq 0$ $\mp m_\varphi - 2n_\theta n_\varphi \geq 0$ $\pm m_\theta - 2n_\theta n_\varphi \geq 0$	$H_\theta^\pm$ $G_\theta^\pm$ $G_\varphi^\pm$

where $N_0 = \sigma_0 t$, $M_0 = 1/4 \sigma_0 t^2$, t is the thickness of the shell, and σ_0 is the yield stress of the shell material in pure tension. p , q and r are parameters, the absolute values of which can not exceed $1/2$. Table 1(a) is used when the values of p , q and r are different. The first, the second or the third line in Table 1(a) is used when p , q or r is the intermediate parameter, respectively. Table 1(b) is used when two parameters coincide and the remaining parameter is indefinite.

Eliminating p , q and r from the equations of each surface element in Table 1, the surface is expressed in terms of the stress resultants. It is shown in Table 2, in which the surface by Lance and Onat is modified and supplemented in some aspects*. The lines in Tables 1 and 2 correspond with each other. The yield surface is composed of twelve surface elements, the designations and the equations of which are given in the first and the second column. The third column indicates the boundary conditions for each surface element, and the fourth gives the adjoining surface elements to which the stress point moves when one of the bounding conditions becomes unsatisfied.

The relations between these twelve surface elements are represented schematically in Fig. 2, in which the signs of the rates of curvatures (κ_θ and κ_φ) and their relative amounts are clarified**. Also the ways of connections between the adjoining surface elements are shown in Fig. 2, i.e. the broken lines represent smooth

connections and the solid lines connections with a ridge.

Further, Fig. 2 shows the situations where the hinge circles or the free edges can exist: the mark of $-\times-$ or $-\cdot-$ indicates a hinge circle or a free edge, respectively.

Now when the stress point is on a surface element, the equation of which is $F_i(n_\theta, n_\varphi, m_\theta, m_\varphi) = 1$, the following relations should be satisfied from the flow law:

$$\left. \begin{aligned} N_0 \varepsilon_\varphi &= \lambda \frac{\partial F_i}{\partial n_\varphi}, & N_0 \varepsilon_\theta &= \lambda \frac{\partial F_i}{\partial n_\theta} \\ M_0 \kappa_\varphi &= \lambda \frac{\partial F_i}{\partial m_\varphi}, & M_0 \kappa_\theta &= \lambda \frac{\partial F_i}{\partial m_\theta} \end{aligned} \right\} \dots\dots\dots (5)$$

where λ is a positive undetermined multiplier. Substituting these relations into Eqs. (3) and eliminating λ and $d\lambda/d\varphi$, the following equation is obtained as the necessary condition for each yield surface element:

$$\begin{aligned} & \frac{\partial F_i}{\partial m_\theta} \frac{d}{d\varphi} \left(\frac{\partial F_i}{\partial n_\theta} \right) - \frac{\partial F_i}{\partial n_\theta} \frac{d}{d\varphi} \left(\frac{\partial F_i}{\partial m_\theta} \right) \\ & - \left(\frac{\partial F_i}{\partial m_\theta} \frac{\partial F_i}{\partial n_\varphi} - \frac{\partial F_i}{\partial n_\theta} \frac{\partial F_i}{\partial m_\varphi} \right) \frac{r_1}{r_0} \cos \varphi \\ & - \frac{\partial F_i}{\partial n_\theta} \frac{\partial F_i}{\partial m_\theta} \tan \varphi - \frac{N_0}{M_0} \left(\frac{\partial F_i}{\partial m_\theta} \right)^2 r_1 \tan \varphi = 0 \dots (6) \end{aligned}$$

This is a compatibility equation expressed in terms of the stress resultants.

4. Analysis

It is understood from the preceding chapters that if the state of stress in any section of an axisymmetric shell is upon the yield condition of the material throughout the thickness, then the five stress resultants, N_θ , N_φ , M_θ , M_φ and Q_φ are specified by the five equations, i.e. Eqs. (1), Eq. (6) and one of the yield equations in the second column of Table 2.

4.1 Hinge circle

As the mode of collapse of any axisymmetric shell, at least one parallel circle exists, at which v and/or $dw/d\varphi$ is discontinuous, except in such cases as the dilatation of a complete spherical shell due to internal pressure. These parallel circles are called hinge circles. On these circles, κ_φ or κ_θ and ε_φ become infinite as seen from Eqs. (2). These circumstances are possible on the surface elements $H_\varphi^\pm$ from the flow law. These surface elements express that $\varepsilon_\theta = \kappa_\theta = 0$, and Eqs. (3) lead to $\varepsilon_\varphi = \kappa_\varphi = 0$, therefore only rigid displacements are possible on $H_\varphi^\pm$. Hence the hinge circles can exist on the boundaries of the surface elements $H_\varphi^\pm$. On the other hand it is concluded from Eqs. (2) that if v or v and $dw/d\varphi$ are discontinuous, ε_θ and κ_θ become discontinuous, and if $dw/d\varphi$ is discontinuous, κ_θ becomes

Fig. 2 Arrangement of yield surface element

* The expressions in the column of the inequalities are modified for easy understanding, and the column of the adjoining surface elements is added.

** Lance and Onat⁽⁵⁾ showed only the connecting relations between each surface element, but in Fig. 2, considerations about the signs of κ_θ and κ_φ , the hinge circle, and free edge are also included.

discontinuous. This fact means that the hinge circles can exist where the surface elements $H_{\varphi}^{\pm}$ are connected with the other surface elements with ridges. There are four cases as shown in Table 3, for which the conditions mentioned above are possible.

But for circular cylinders, κ_{θ} vanishes identically, and the compatibility equation specifies only the relation between ε_{φ} and κ_{φ} , hence the conditions mentioned above for ε_{θ} and κ_{θ} are not to be applied. On the four-dimensional yield surface, κ_{θ} can vanish on the boundary of G_{φ}^{+} and $G_{\theta\varphi}^{+}$ or G_{φ}^{-} and $G_{\theta\varphi}^{-}$. Therefore, the conditions of the hinge circle for a circular cylinder are satisfied for the following two cases:

Boundary of H_{φ}^{+} , G_{φ}^{+} and $G_{\theta\varphi}^{+}$:

$$m_{\varphi} + n_{\varphi}^2 - 1 = 0, \quad 2n_{\theta} - n_{\varphi} \pm 1 = 0$$

Boundary of H_{φ}^{-} , G_{φ}^{-} and $G_{\theta\varphi}^{-}$:

$$-m_{\varphi} + n_{\varphi}^2 - 1 = 0, \quad 2n_{\theta} - n_{\varphi} \pm 1 = 0$$

4.2 Boundary conditions

Boundary conditions of two kinds are considered here.

(a) Boundary with a rigid part

The stress point in this case is on the boundary between the surface elements $H_{\varphi}^{\pm}$ and one of their adjoining elements. When a hinge circle is yielded in this point, one of the boundary conditions in Table 3 is to be considered.

(b) Free boundary

At the free boundary, n_{φ} and m_{φ} vanish, and considering the relations in Table 2, it turns out that the conditions in Table 4 are to be applied for the free boundary.

4.3 Stress trajectory and solution

A principal subject in the limit analysis is to know on which part of the yield surface the stress point passes. This subject is easily solved

by referring to Fig. 2, Table 3 and Table 4 as follows:

If a hinge circle is yielded at the boundary of the collapsing part, the corresponding position of the stress point on the yield surface is given by Table 3 as the intersection of the four-dimensional hypersurfaces, (although it is still unknown which line in Table 3 is to be applied). Also if the edge of the shell is free, the boundary conditions are given by either of the two lines in Table 4. Next, by presuming the signs of κ_{θ} and κ_{φ} from the supposed collapsing mode of the shell, the surface element on which the stress point passes is selected, and the position on the yield surface which satisfies the boundary conditions is uniquely determined. For instance, if a hinge circle is yielded at the boundary, and if $\kappa_{\theta} < 0$ and $\kappa_{\varphi} > 0$ in the neighbourhood of the boundary, then the state of the stress in the hinge circle corresponds to the intersection of H_{φ}^{+} and $H_{\theta\varphi}^{-}$ (see Fig. 2). Considering the position of the hinge circle on the yield surface as the starting point of the stress trajectory, the surface element on which the calculation is to be carried out is determined uniquely. Therefore, using the starting values of the solutions which are assumed by considering the conditions of the hinge circle, the fundamental equations (the yield condition, the equilibrium equations and the compatibility equation) are numerically integrated by the Runge-Kutta-Gill method, and the values of the stress resultants are obtained. This process is continued until the bounding conditions in Table 2 become unsatisfied. Then the same procedures are continued for the adjoining surface element, and so on. Thus, if the assumed values of the solution happen to be correct, the boundary conditions for the end boundary would be satisfied. The trial and error method is applied and the procedures mentioned above are repeated by assuming different starting values until the boundary conditions on the end boundary become satisfied. But, if the stress trajectory becomes the intersection of two surface elements with a ridge, the calculation should be continued on the intersection until the condition of continuity for strain rates becomes satisfied.

5. Velocity field

From the first, the second and third equation of Eqs. (2), we obtain

$$\left. \begin{aligned} \frac{dv}{d\varphi} &= w + \frac{r_1}{r_0} \frac{\varepsilon_{\varphi}}{\varepsilon_{\theta}} (v \cos \varphi - w \sin \varphi) \\ \frac{dw}{d\varphi} &= -v - r_1 \frac{\kappa_{\theta}}{\varepsilon_{\theta}} (v - w \tan \varphi) \end{aligned} \right\} \dots\dots\dots (7)$$

Table 3 Conditions at hinge circle

Position of hinge circle	Equations of condition
Boundary between H_{φ}^{+} and H_{θ}^{+}	$n_{\theta} = n_{\varphi}, \quad m_{\theta} = m_{\varphi}, \quad m_{\varphi} + n_{\varphi}^2 - 1 = 0$
Boundary between H_{φ}^{-} and H_{θ}^{-}	$n_{\theta} = n_{\varphi}, \quad m_{\theta} = m_{\varphi}, \quad -m_{\varphi} + n_{\varphi}^2 - 1 = 0$
Boundary between H_{φ}^{+} and $H_{\theta\varphi}^{-}$	$n_{\theta} = 0, \quad m_{\theta} = 0, \quad m_{\varphi} + n_{\varphi}^2 - 1 = 0$
Boundary between H_{φ}^{-} and $H_{\theta\varphi}^{+}$	$n_{\theta} = 0, \quad m_{\theta} = 0, \quad -m_{\varphi} + n_{\varphi}^2 - 1 = 0$

Table 4 Conditions at free edge

Position of free edge	Equation of condition
Boundary between G_{θ}^{+} , H_{θ}^{+} and $H_{\theta\varphi}^{+}$	$n_{\varphi} = 0, \quad m_{\varphi} = 0, \quad m_{\theta} + n_{\theta}^2 - 1 = 0$
Boundary between G_{θ}^{-} , H_{θ}^{-} and $H_{\theta\varphi}^{-}$	$n_{\varphi} = 0, \quad m_{\varphi} = 0, \quad -m_{\theta} + n_{\theta}^2 - 1 = 0$

Substitution of Eqs. (5) into Eqs. (7) gives

$$\left. \begin{aligned} \frac{dv}{d\varphi} &= w + \frac{r_1}{r_0} \frac{\partial F / \partial n_\varphi}{\partial F / \partial n_\theta} (v \cos \varphi - w \sin \varphi) \\ \frac{dw}{d\varphi} &= -v - r_1 \frac{\partial F / \partial m_\theta}{\partial F / \partial n_\theta} (v - w \tan \varphi) \end{aligned} \right\} \dots (8)$$

Also the multiplier in Eqs. (5) is obtained from the second equation of Eqs. (5) and the second equation of Eqs. (2) as

$$\frac{1}{N_0} \lambda = \frac{1}{r_0} \frac{v \cos \varphi - w \sin \varphi}{\partial F / \partial n_\theta} \dots (9)$$

Thus, if the stress trajectory for a problem is elucidated, v , w and λ are obtained uniquely according to the boundary conditions of them, and their values confirm the appropriateness of the solution.

6. Two examples

6.1 Conical shell

A truncated conical shell is considered as in Fig. 3, the outer edge of which is fixed at a rigid wall, and along the inner edge of which a line load Q acts in the perpendicular direction to the shell element. This problem is already treated by T. Nakamura⁽³⁾ and J.C. Gerdeen⁽⁴⁾ with the Nakamura's approximate yield surface. In order to ascertain the appropriateness of our method, this problem is treated here. It is easily supposed that $\kappa_\varphi \geq 0$ and $\kappa_\theta \leq 0$ in the whole range of the shell in case of collapse. Hence the hinge circle at the outer boundary B corresponds to the intersection of two surface elements H_φ^+ and $H_{\theta\varphi}^-$, and the free edge A to the intersection of three surface elements G_θ^- , H_θ^- and $H_{\theta\varphi}^-$. The possible stress trajectories for this problem are shown in Fig. 4. But, only the case that the stress trajectory is in the surface element $H_{\theta\varphi}^-$ is considered in this paper. This means that the conical shell is assumed as a shallow shell.

Fig. 3 Truncated conical shell

Fig. 4 Possible stress trajectories for truncated conical shell

In this case, the equilibrium equations are as follows:

$$\left. \begin{aligned} \frac{dN_\varphi}{ds} + (N_\varphi - N_\theta) \frac{1}{s} &= 0 \\ \frac{dM_\varphi}{ds} + (M_\varphi - M_\theta) \frac{1}{s} N_\varphi \cot \alpha - Q \frac{L}{s} &= 0 \\ Q_\varphi + N_\varphi \cot \alpha - Q \frac{L}{s} &= 0 \end{aligned} \right\} \dots (10)$$

The nondimensional expressions are defined as

$$\left. \begin{aligned} n_\varphi &= \frac{N_\varphi}{N_0}, \quad n_\theta = \frac{N_\theta}{N_0}, \quad m_\varphi = \frac{M_\varphi}{M_0}, \quad m_\theta = \frac{M_\theta}{M_0} \\ q_\varphi &= \frac{Q_\varphi}{N_0}, \quad q = \frac{Q}{N_0}, \quad x = \frac{s}{L}, \quad k = \frac{M_0}{N_0 L} = \frac{t}{4L} \end{aligned} \right\} \dots (11)$$

Then we obtain

$$\left. \begin{aligned} \frac{dn_\varphi}{dx} + (n_\varphi - n_\theta) \frac{1}{x} &= 0 \\ \frac{dm_\varphi}{dx} + (m_\varphi - m_\theta) \frac{1}{x} + \frac{1}{k} \left(n_\varphi \cot \alpha - \frac{q}{x} \right) &= 0 \\ q_\varphi + n_\varphi \cot \alpha - \frac{q}{x} &= 0 \end{aligned} \right\} \dots (12)$$

The compatibility equation [Eq. (6)] for conical shell is

$$\begin{aligned} \frac{\partial F}{\partial m_\theta} \frac{d}{dx} \left(\frac{\partial F}{\partial n_\theta} \right) - \frac{\partial F}{\partial n_\theta} \frac{d}{dx} \left(\frac{\partial F}{\partial m_\theta} \right) \\ = \frac{1}{x} \left(\frac{\partial F}{\partial m_\theta} \frac{\partial F}{\partial n_\varphi} - \frac{\partial F}{\partial n_\theta} \frac{\partial F}{\partial m_\varphi} \right) + \frac{\cot \alpha}{k} \left(\frac{\partial F}{\partial m_\theta} \right)^2 \end{aligned} \dots (13)$$

Since the surface element $H_{\theta\varphi}^-$ is related to this problem, we obtain from Table 2

$$F = -m_\theta + m_\varphi + (n_\theta - n_\varphi)^2 - 1 = 0 \dots (14)$$

Substitution of this equation into Eq. (13) gives

$$\frac{dn_\theta}{dx} = \frac{dn_\varphi}{dx} - \frac{\cot \alpha}{2k} \dots (15)$$

Substitution of the first equation of Eqs. (12) into Eq. (15) gives

$$\frac{dn_\theta}{dx} = \frac{1}{x} (n_\theta - n_\varphi) - \frac{\cot \alpha}{2k} \dots (16)$$

Also substitution of Eq. (14) into the second equation of Eqs. (12) and elimination of m_θ gives

$$\frac{dm_\varphi}{dx} = \frac{1}{x} \{ (n_\theta - n_\varphi)^2 - 1 \} - \frac{1}{k} \left(n_\varphi \cot \alpha - \frac{q}{x} \right) \dots (17)$$

Therefore the following equations should be satisfied for this problem:

$$\left. \begin{aligned} \frac{dn_\varphi}{dx} &= (n_\theta - n_\varphi) \frac{1}{x} \\ \frac{dm_\varphi}{dx} &= \frac{1}{x} \{ (n_\theta - n_\varphi)^2 - 1 \} - \frac{1}{k} \left(n_\varphi \cot \alpha - \frac{q}{x} \right) \\ \frac{dn_\theta}{dx} &= (n_\theta - n_\varphi) \frac{1}{x} - \frac{\cot \alpha}{2k} \\ m_\theta &= m_\varphi + (n_\theta - n_\varphi)^2 - 1 \\ q_\varphi &= -n_\varphi \cot \alpha + \frac{q}{x} \end{aligned} \right\} \dots (18)$$

Also the conditions that the stress point is to be on the surface element $H_{\theta\varphi}^-$ are from Table 2

$$\left. \begin{aligned} -m_{\theta} + m_{\varphi} &\geq 0 \\ m_{\varphi} - 2n_{\theta}n_{\varphi} &\geq 0 \\ -m_{\theta} - 2n_{\theta}n_{\varphi} &\geq 0 \end{aligned} \right\} \dots\dots\dots(19)$$

Further, the boundary conditions for the stress resultants are as follows:

$$\left. \begin{aligned} \text{At } x=1 \quad n_{\varphi}=m_{\varphi}=0, \quad q_{\varphi}=q \\ \text{At } x=x_b \quad n_{\theta}=m_{\theta}=0, \quad m_{\varphi}+n_{\varphi}^2-1=0 \end{aligned} \right\} \dots\dots(20)$$

where $x_b=b/L$. In calculations, the value of q and that of n_{φ} at $x=x_b$ are assumed firstly, and Eqs. (18) are numerically integrated. If either of Eqs. (19) becomes unsatisfied, different starting values of q and n_{φ} are assumed and numerical integrations are repeated. When the value of q is found for which the conditions at $x=1$ are satisfied, then that value is the collapse load we want to know.

The calculated results are shown in Fig. 5. For $\cot\alpha=0.01$ and 0.02 , the inclines of the curves at the right end vanish, and the value of the limiting load q does not decrease for greater values of x_b . When $\cot\alpha \geq 0.03$, the stress trajectory passes on the adjoining surface elements of $H_{\theta\varphi}^-$ for larger values of x_b as shown in Fig. 5.

The results obtained here coincide well with T. Nakamura's analytical solutions⁽³⁾ obtained by his approximate yield surface. This means that the effects of approximations of Nakamura's solutions do not appear in this problem, and that the accuracy of numerical calculations in our solution is enough. The increment in numerical integration (Runge-Kutta-Gill method) is taken here as $\Delta x = -0.05$.

6.2 Pressure vessel with a rigid circular plate

The problem of a pressure vessel which is composed of a rigid circular plate, a toroidal shell and a cylindrical shell as shown in Fig. 6 is treated. This problem was once solved by one of the authors⁽⁶⁾ by using the approximate three-dimensional yield surface⁽⁷⁾.

Fig. 5 Limit load of truncated conical shell

As in Fig. 6, it is assumed that the collapse develops in the range between the two points A and B, and that the hinge circles are yielded at these points. A point C is taken between the points A and B, and it is deduced that

For the range between A and C: $\kappa_{\varphi} \leq 0, \kappa_{\theta} \leq 0$

For the range between C and B: $\kappa_{\varphi} \geq 0, \kappa_{\theta} \leq 0$

Therefore it is found that the point A corresponds to the boundary of H_{φ}^- and H_{θ}^- , and the point B to the boundary of H_{φ}^+ and $H_{\theta\varphi}^-$. Hence the stress trajectory for this problem is considered to be one as shown in Fig. 7. The calculated result of the stress trajectory is shown in Fig. 8.

The equilibrium equations are

$$\left. \begin{aligned} \frac{dN_{\varphi}}{d\varphi} + N_{\varphi} \tan \varphi + \frac{r}{r_0} (N_{\varphi} - N_{\theta}) \cos \varphi \\ - \frac{1}{2} \frac{pr_0}{\cos \varphi} = 0 \\ \frac{dM_{\varphi}}{d\varphi} + N_{\varphi} r \tan \varphi + \frac{r}{r_0} (M_{\varphi} - M_{\theta}) \cos \varphi \\ - \frac{1}{2} \frac{pr r_0}{\cos \varphi} = 0 \\ Q_{\varphi} \cos \varphi + N_{\varphi} \sin \varphi - \frac{1}{2} pr_0 = 0 \end{aligned} \right\} \dots\dots(21)$$

where $r = \text{constant}$, $r_0 = D/2 - r(1 - \sin \varphi)$.

The following nondimensional expressions are defined:

Fig. 6 Pressure vessel with rigid circular plate

Fig. 7 Possible stress trajectories for pressure vessel with rigid circular plate

Fig. 8 Stress trajectory for pressure vessel with rigid circular plate

$$\left. \begin{aligned} n_\varphi &= \frac{N_\varphi}{N_0}, & n_\theta &= \frac{N_\theta}{N_0}, & m_\varphi &= \frac{M_\varphi}{M_0}, & m_\theta &= \frac{M_\theta}{M_0} \\ q_\varphi &= \frac{Q_\varphi}{N_0}, & P &= \frac{pD}{2N_0}, & R &= \frac{r}{D}, & R_0 &= \frac{r_0}{D} \\ T &= \frac{t}{D} \end{aligned} \right\} \dots\dots\dots (22)$$

Then the equilibrium equations are

$$\left. \begin{aligned} \frac{dn_\varphi}{d\varphi} + n_\varphi \tan \varphi + \frac{R}{R_0} (n_\varphi - n_\theta) \cos \varphi \\ - \frac{PR_0}{\cos \varphi} &= 0 \\ \frac{dm_\varphi}{d\varphi} + 4n_\varphi \frac{R}{T} \tan \varphi + \frac{R}{R_0} (m_\varphi - m_\theta) \cos \varphi \\ - 4 \frac{PRR_0}{T \cos \varphi} &= 0 \\ n_\varphi \sin \varphi + q_\varphi \cos \varphi - PR_0 &= 0 \end{aligned} \right\} \dots\dots\dots (23)$$

where $R_0 = 1/2 - R(1 - \sin \varphi)$.

Next, the necessary conditions for each yield surface element are given by

For H_θ^- :

The yield condition:

$$m_\theta - n_\theta^2 + 1 = 0 \dots\dots\dots (24)$$

The compatibility condition:

$$\frac{dn_\theta}{d\varphi} = \left(n_\theta - 2 \frac{R}{T} \right) \tan \varphi \dots\dots\dots (25)$$

The bounding inequalities:

$$\left. \begin{aligned} I_{1,1} &= -m_\theta \geq 0 \\ I_{1,2} &= -m_\varphi + 2n_\varphi(n_\theta - n_\varphi) \geq 0 \\ I_{1,3} &= -m_\theta + m_\varphi + 2n_\varphi(n_\theta - n_\varphi) \geq 0 \end{aligned} \right\} \dots\dots\dots (26)$$

For G_θ^- :

The yield condition:

$$-m_\theta + m_\varphi + (n_\varphi - n_\theta)^2 + \left(\frac{m_\varphi}{2n_\varphi} - n_\theta \right)^2 - 1 = 0 \dots\dots\dots (27)$$

The compatibility condition:

$$\begin{aligned} \frac{dn_\theta}{d\varphi} &= \left(1 - \frac{m_\varphi}{2n_\varphi^2} \right) \frac{dn_\varphi}{d\varphi} + \frac{1}{4n_\varphi} \frac{dm_\varphi}{d\varphi} \\ &- \frac{1}{n_\varphi} \left(\frac{m_\varphi}{2n_\varphi} - n_\theta \right) \left(\frac{m_\varphi}{2n_\varphi} - n_\theta + n_\varphi \right) \frac{R}{R_0} \cos \varphi \\ &+ \frac{1}{2} \left(2n_\theta - n_\varphi - \frac{m_\varphi}{2n_\varphi} \right) \tan \varphi - \frac{R}{T} \tan \varphi \dots\dots\dots (28) \end{aligned}$$

The bounding inequalities:

$$\left. \begin{aligned} I_{2,1} &= m_\varphi - 2n_\varphi(n_\theta - n_\varphi) \geq 0 \\ I_{2,2} &= -m_\varphi + 2n_\theta n_\varphi \geq 0 \\ I_{2,3} &= 1 - \left| \frac{m_\varphi}{2n_\varphi} - n_\varphi \right| \geq 0 \\ I_{2,4} &= 1 - \left| \frac{m_\varphi}{2n_\varphi} + n_\varphi \right| \geq 0 \end{aligned} \right\} \dots\dots\dots (29)$$

For $H_{\theta\varphi}^-$:

The yield condition:

$$-m_\theta + m_\varphi + (n_\theta - n_\varphi)^2 - 1 = 0 \dots\dots\dots (30)$$

The compatibility condition:

$$\frac{dn_\theta}{d\varphi} = (n_\theta - n_\varphi) \tan \varphi - 2 \frac{R}{T} \tan \varphi \dots\dots\dots (31)$$

The bounding inequalities:

$$\left. \begin{aligned} I_{3,1} &= -m_\theta + m_\varphi \geq 0 \\ I_{3,2} &= m_\varphi - 2n_\theta n_\varphi \geq 0 \\ I_{3,3} &= -m_\theta - 2n_\theta n_\varphi \geq 0 \end{aligned} \right\} \dots\dots\dots (32)$$

Further, the boundary conditions are

$$\left. \begin{aligned} \text{At point A: } n_\theta &= n_\varphi, m_\theta = m_\varphi, -m_\varphi + n_\varphi^2 - 1 = 0 \\ \text{At point B: } n_\theta &= 0, m_\theta = 0, m_\varphi + n_\varphi^2 - 1 = 0 \end{aligned} \right\} \dots\dots\dots (33)$$

In calculations, the values of P and n_φ at point A are assumed firstly, and the values of n_φ , n_θ , m_φ , m_θ and q_φ on the H_θ^- are obtained by integrating numerically Eqs. (23), Eqs. (24) and Eq. (25). If the inequality $I_{1,2}$ in Fqs. (26) becomes unsatisfied, the stress point is considered to pass to G_θ^- and the calculations are continued for Eqs. (23), Eq. (27) and Eq. (28). If the inequality $I_{2,2}$ becomes unsatisfied, the calculations are carried out for Eqs. (23), Eq. (30) and Eq. (31). And when the inequality $I_{3,3}$ becomes unsatisfied, it is checked whether the conditions of point B are satisfied at the point where $I_{3,3} = 0$. When one of the inequalities $I_{1,1}$, $I_{1,3}$, $I_{2,1}$, $I_{2,3}$, $I_{2,4}$, $I_{3,1}$ and $I_{3,2}$ becomes unsatisfied, or the conditions of point B are not filled when the

Fig. 9 Limit load of pressure vessel with rigid circular plate

Fig. 10 Mode of collapsing of pressure vessel with rigid circular plate

inequality $I_{3,3}$ becomes unsatisfied, different starting values of P and n_θ are assumed and the calculations are repeated. Among the values of P which fill the conditions at point B, the minimum of such values of P is sought out by varying the assumed value of n_θ . This minimum value is the collapse pressure we want to know.

The calculated results are shown in Fig. 9. Fig. 10 shows an example of the mode of displacement. Fig. 11 shows the comparison between the calculated results by our method, and those by the approximate method⁽⁶⁾. These approximate results are obtained by one of the authors using the approximate yield surface of D.C. Drucker and R.T. Shield⁽⁷⁾ where the effects of M_θ on yielding are neglected.

7. Conclusions

Many approximate solutions for limit analyses of axisymmetric shells made of perfectly plastic material have been hitherto presented, but in this paper a numerical method is pursued

Fig. 11 Comparison between our results and approximate results⁽⁶⁾ for pressure vessel with rigid circular plate

which makes it possible to obtain a numerically correct solution for general axisymmetric shells. E.T. Onat and W. Prager⁽²⁾ once presented a four-dimensional yield surface, and R.H. Lance and E.T. Onat⁽⁵⁾ proposed a numerical method using their surface. In this paper, in order to make it easier to apply the method by Lance and Onat to general shells of revolution, considerations are made on four-dimensional yield surface by Onat and Prager: the relations between each surface element are precisely studied and the conditions at the hinge circle and the free edge are investigated. And two examples are solved by this method to verify its appropriateness. Although further considerations are still needed on the special conditions in each problem when we solve a problem of a shell of revolution of any shape with this method, it seems that the calculations become passably easier by our research, and many problems will be solved by this method in future.

We wish to express our thanks to Assistant Professor T. Nakamura of Kyoto University for introducing us two references⁽³⁾⁽⁴⁾, which were very helpful to our research.

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Discussion

Y. YAMAMOTO (University of Tokyo):

The limit analysis is a mathematically interesting subject of study, but its practical significance is questionable, though the debater himself once tried to relate the design problem of steel pipes subjected to internal pressure with the limit analysis. Buckling due to internal pressure may occur in the shell, and after buckling the shell may be in a stable state in some cases. Also it is possible that a shell expands like a balloon due to the effects of finite displacements. The debater wishes to know the significance in practice of the limit

analysis.

Authors' closure

If the limit load is defined as one at which considerable deformation in any structure begins, the practical significance of the limit analysis for structures composed of perfectly plastic materials can not be denied. For instance, the behaviours of a simply supported circular plate subjected to the uniformly distributed load, as shown in Append-Fig. 1⁽¹⁾ indicate the practical meaning of the collapse load. Also the experimental results of the pressure vessel with a circular

Append.-Fig. 1 Problem of simply supported circular plate subjected to uniformly distributed load

rigid plate, as shown in Append.-Fig. 2⁽⁶⁾ indicate it.

As the debater mentions, there is no practical significance of the limit analysis for any structure in which tolerable deformations are permitted. But its significance should be recognized for the structures like the U-shaped bellows, in which the quantity of the deformation is important.

The method in this research is likely to

Append.-Fig. 2 Problem of pressure vessel with rigid circular plate

develop the method of analysis which allows for the effects of strain hardening of materials and the effects of large deformations of the structures. On the other hand, it is meaningful at the present time to obtain numerically accurate results for various problems of the limit analysis which have been hitherto solved approximately by various methods.