Laminar Swirling Flow in the Entrance Region of a Circular Pipe

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Treated in this paper is the laminar swirling flow in the hydrodynamic entrance region of a circular pipe, in which the entering flow has uniform axial- and linearly varying tangential-velocity components. The analysis is carried out within the framework of the boundary layer theory, the solutions being obtained by means of an extended form of the finite difference method invented by Leigh and Terrill. The following results are obtained from the analysis: (1) The entrance length is significantly increased by the presence of swirl at the entrance section. (2) The additional pressure drop based on the center-line pressure at the entrance section is considerably decreased as the swirl is increased. (3) If the wall temperature is constant, the local Nusselt number is increased by a few percent at a certain distance downstream from the entrance section when Prandtl number is taken as 0.72. (4) The damping coefficients of the swirl show an exponential decay shortly after the entrance section. However, in the immediate neighbourhood of the entrance section, the damping of the swirl is a little faster than the exponential decay.

1. Introduction

A hydrodynamically developing flow in the entrance region of a circular pipe is of a practical importance concerning the viscous resistance and the heat transfer characteristics of a comparatively short pipe which has a length scores of times the pipe radius. One example of the engineering applications may be found in the analysis of the flow and the heat transfer characteristics of heat exchangers. In view of the general aspect of the technical importance of duct flows, a number of theoretical and experimental investigations have been performed about this subject. A uniform axial velocity profile at the inlet section has been assumed in these analyses to consider the development of the flow which attains a fully-developed profile at a location far downstream. To the extent of the authors' knowledge, any consideration has not yet been made concerning the effects of the swirl at the inlet on the flow development, especially on the entrance length, pressure drop and the heat transfer between the fluid and the pipe wall. The swirling pipe flows so far studied are almost concerned with the damping of swirl in the otherwise fully-developed flow regime. One can mention among others those of Talbot(1) for laminar pipe flow and Kreith & Margolis(2), Smithberg & Landis(3) and Kreith & Sonju(4) for turbulent pipe flow. One example of the entrance region flow with swirl will be found in the draining pipe installed at the bottom of a vessel.

The purpose of this paper is to theoretically describe the laminar developing flow with swirling component of velocity at the entrance of a circular pipe in order to clarify the effects of swirl on the flow development, pressure drop and heat transfer, together with the damping characteristics of the swirl. A steady and rotationally symmetrical flow of a viscous incompressible fluid in a circular pipe of constant temperature is considered. It is assumed that the fluid properties such as viscosity, thermal conductivity etc. do not change with temperature or pressure.

2. Nomenclature

\begin{itemize}
  \item $a$: radius of circular pipe
  \item $c_p$: specific heat at constant pressure
  \item $h_s$: local heat transfer coefficient $=q/(T_m' - T_w)$
  \item $k$: thermal conductivity
  \item $m_1, m_2$: damping coefficients of swirl
\end{itemize}
\[ m_1 = \int_0^1 u_0 v_0 r^2 dr / \int_0^1 w_0 r^2 dr \]
\[ m_2 = \left[ \int_0^1 v_0^2 r dr / \int_0^1 w_0^2 r dr \right]^{1/2} \]

\[ \rho' : \text{pressure} \]
\[ p : \text{dimensionless pressure} = p' / (\rho W^2) \]
\[ r' , \theta , z' : \text{cylindrical polar coordinates} \]
\[ r, z : \text{dimensionless variables} \]
\[ r = r' / a, z = (z'/a) / R_a \]
\[ u', v', w' : \text{velocity components in the direction of} \]
\[ r', \theta, z' \]
\[ u, v, w : \text{velocity components} \]
\[ u = R_a (u' / W), v = v' / W, w = w' / W \]
\[ C_a : \text{dimensionless additional pressure drop} \]
\[ C_a = \lim_{r \to 0} \left[ p(0, 0) - \rho(0, z) - 8z \right] \]

\[ K : \text{dimensionless swirling velocity} = \Omega a / W \]
\[ L_m : \text{dimensionless entrance length} \]
\[ N u_z : \text{local Nusselt number} = 2 a h_i / k \]
\[ P_r : \text{Prandtl number} = \mu c_p / k \]
\[ R_a : \text{Reynolds number} = W a / v \]
\[ T : \text{temperature} \]
\[ T_0 : \text{dimensionless temperature} = (T - T_0) / (T_0 - T_0) \]

\[ T_a : \text{temperature of fluid at the inlet} \]
\[ T_w : \text{temperature of pipe wall} \]
\[ T_m : \text{bulk temperature} \]
\[ = \int_0^1 T w r dr / \int_0^1 w r dr \]
\[ T_m : \text{dimensionless bulk temperature} \]
\[ = (T_m - T_0) / (T_0 - T_0) \]

\[ W : \text{mean velocity} \]
\[ \nu : \text{viscosity} \]
\[ \nu : \text{kinematic viscosity} \]
\[ \rho : \text{density} \]
\[ \Omega : \text{angular velocity of swirl} \]

3. Governing equations

The fundamental equations to describe the flow of a viscous incompressible fluid are the equations of continuity and three Navier-Stokes equations in three coordinate directions. When the heat transfer is treated, the thermal energy equation is to be added as the fifth equation. In the case of a steady and rotationally symmetrical laminar flow, these equations take the following forms:

\[ \frac{\partial (r u')}{\partial r'} + \frac{\partial (r w')}{\partial z'} = 0 \quad (1a) \]

\[ u' \frac{\partial u'}{\partial r'} - \frac{v'^2}{r'} + w' \frac{\partial u'}{\partial z'} = -\frac{1}{\rho} \frac{\partial p'}{\partial r'} + \nu \left( \frac{\partial^2 u'}{\partial r^2} \right) \quad (1b) \]

\[ u' \frac{\partial v'}{\partial r'} + u' \frac{\partial u'}{\partial z'} = \nu \left( \frac{\partial^2 u'}{\partial r^2} \right) - \frac{1}{\rho} \frac{\partial p'}{\partial z'} + \nu \frac{\partial^2 v'}{\partial z^2} \cdots (1c) \]

\[ u' \frac{\partial w'}{\partial r'} + w' \frac{\partial u'}{\partial z'} = \nu \left( \frac{\partial^2 u'}{\partial r^2} \right) \]

\[ \rho c_p \left( u' \frac{\partial T'}{\partial r'} + w' \frac{\partial T'}{\partial z'} \right) = k \frac{\partial^2 T'}{\partial z^2} + \mu \frac{\partial^2 T'}{\partial z^2} \cdots (1e) \]

where

\[ \frac{\partial T'}{\partial z^2} + \frac{1}{r'} \frac{\partial}{\partial r'} \left( r' \frac{\partial T'}{\partial r'} \right) + \frac{\partial^2 T'}{\partial z^2} \]

\[ \Phi' = 2 \left[ \left( \frac{\partial u'}{\partial r'} \right)^2 + \left( \frac{\partial w'}{\partial r'} \right)^2 \right] \]

\[ + \left( \frac{\partial u'}{\partial z'} \right)^2 + \left( \frac{\partial u'}{\partial z'} \right)^2 + \left( \frac{\partial w'}{\partial z'} \right)^2 \]

The flow configuration is shown in Fig. 1 in which \( z' \) corresponds to the pipe axis and \( u', v', w' \) denote the velocity components in the \( r', \theta, z' \) directions, respectively. The origin of \( z' \) is taken at the inlet section. The radius of the pipe is denoted by \( a \), and \( w', v' \) at the inlet section by \( w' = W g(r') \), \( v' = W f(r') \). \( f(r') \) and \( g(r') \) are the functions of \( r' \) which define the distributions of the velocity profile at the inlet. \( T_0 \) and \( T_w \) which are assumed to be constant denote the fluid temperature at the inlet and the temperature of the pipe wall, respectively. Making use of these variables, one introduces the dimensionless variables and parameters as follows:

\[ r = r' / a, z = (z'/a) / R_a, u = R_a (u' / W) \]

\[ v = v' / W, w = w' / W, p = p' / (\rho W^2) \]

\[ T = (T - T_0) / (T_0 - T_0), E_c = W^2 / (\rho c_p (T_0 - T_w)) \]

\[ R_s = W a / v, P_r = \mu c_p / k \]

In the entrance region, since the changes of velocity and temperature are much larger in the \( r' \) direction than in the \( z' \) direction and \( u' \) is much smaller than \( v' \) and \( w' \), the boundary layer approximation can be applicable to the governing Eqs. (1a) \~(1e). After writing Eqs. (1a) \~(1e) in terms of the dimensionless variables, one applies the boundary layer approximation to the resulting equations to obtain

\[ \frac{\partial (r u)}{\partial r} + \frac{\partial (r w)}{\partial z} = 0 \quad (2a) \]

\[ T_w = \text{constant} \]

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**Fig. 1 Coordinate system**
The boundary conditions are

\[ z=0: \quad v=g(r), \quad w=f(r), \quad T=1 \quad \text{and} \quad T=T=0 \quad \text{as} \quad z \to \infty \] (3a)

\[ r=1, \quad z>0: \quad u=v=w=0, \quad T=0 \quad \text{at} \quad z=0 \] (3b)

\[ \frac{\partial w}{\partial z} - \left[ \frac{\partial v}{\partial r} + \frac{v}{r} \right] \int_0^r \frac{\partial w}{\partial r} dr = \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial r} \right) + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \] (6a)

\[ \frac{\partial w}{\partial z} - \left[ \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial r} \right] \int_0^r \frac{\partial w}{\partial r} dr = - \frac{dP}{dz} - \left[ \frac{2v}{r} \frac{\partial v}{\partial r} + \frac{\partial w}{\partial r} \right] + \frac{1}{r} \frac{\partial w}{\partial r} \int_0^r \frac{\partial w}{\partial r} dr = \frac{P}{\rho} \left( \int_0^r \frac{\partial T}{\partial r} + \frac{\partial w}{\partial r} \right) + E_z \left[ \left( \frac{\partial w}{\partial r} \right)^2 + \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right)^2 \right] \] (6b)

\[ \frac{\partial T}{\partial z} - \frac{\partial T}{\partial r} \int_0^r \frac{\partial w}{\partial r} dr = \frac{P}{\rho} \left( \int_0^r \frac{\partial T}{\partial r} + \frac{\partial w}{\partial r} \right) + E_z \left[ \left( \frac{\partial w}{\partial r} \right)^2 + \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right)^2 \right] \] (6c)

These three equations together with Eq. (2.8) constitute the governing equations of a laminar developable flow with swirl in the entrance region of a circular pipe. Since Eqs. (6a), (6b) and (6c) are nonlinear partial integro-differential equations, it is quite difficult to obtain an analytic solution which is valid throughout the entire region of the developing flow.

4. Finite difference equations

In order to obtain the analytical solution of Eq. (6), one would have to employ a perturbation method or an appropriate linearization technique of the inertia terms. These procedures would necessarily introduce some errors in the resulting solutions which cannot be estimated unless the exact solutions are found. Accordingly, the finite difference method developed by Leigh and Terrill to analyze two-dimensional laminar boundary layers is applied in its extended form to the laminar entrance region flow in a circular pipe with the object of minimizing the errors which would be introduced by unavoidable assumptions in the analytical approach.

The finite difference forms of Eqs. (6a) and (6b) will now be obtained. The derivatives in the z-direction are replaced by finite differences while the other quantities such as \( v, w, \frac{\partial v}{\partial r}, \frac{\partial w}{\partial r} \) etc. are replaced by averages. Let us suppose that the solutions \( v=v_i, w=w_i, P=P_i \) at \( z=z_i \) are known and the solutions \( v=v_{i+}, w=w_{i+}, P=P_{i+} \) at \( z=z_{i+} \) are to be found. After introducing necessary finite difference approximations in Eqs. (6a) and (6b), one obtains

\[ \frac{v_i+ + v_i-}{2} \frac{v_i+ - v_{i-}}{l} - \left( \frac{v_i+ + v_j}{2r} + \frac{v_i+ + v_{i+}}{2r} \right) \int_0^r \frac{v_{i+} - v_{i-}}{l} dr = \frac{v_i+ + v_{i+}}{2r} - \frac{v_i+ + v_{i-}}{2r} \] (7a)

\[ \frac{w_i+ + w_i-}{2} \frac{w_i+ - w_{i-}}{l} - \left( \frac{w_i+ + w_j}{2r} + \frac{w_i+ + w_{i+}}{2r} \right) \int_0^r \frac{w_{i+} - w_{i-}}{l} dr = - \frac{P_i+ - P_i-}{l} \int_0^r \frac{2(v_i+ + v_j)}{2r} \frac{v_i+ - v_{i-}}{l} dr + \frac{w_{i+} + w_{i+}}{2r} + \frac{w_i+ + w_{i+}}{2r} \] (7b)

where dashes denote differentiation with respect to \( r \) and \( l=z_{i+}-z_i \). Substituting the relations defined by

\[ \phi=v_i+ + w_i+ \] (8a)

\[ \chi=v_i+ + w_i+ \] (8b)

and rearranging the terms in Eqs. (7a) and (7b), one gets

\[ \phi(\chi-2w_i) - \frac{1}{r} \left( \phi \frac{\chi}{r} + \frac{\chi}{r} \right) \int_0^r (\phi-2w_i) dr = l \left( \chi + \frac{\chi}{r} \right) \] (9a)

\[ \phi(\phi-2w_i) - \phi \left( \phi-2w_i \right) r dr = -2(P_i+ - P_i-) \int_0^r \phi(\phi-2w_i) dr + l \left( \phi^2 + \frac{\phi^2}{r} \right) \] (9b)
Since the flow rate per unit time through the pipe must be constant, the following relation holds:
\[ \int_0^1 \psi_r \, dz = 2 \pi \int_0^1 w_r \, dz \]  
(10)

Once \( \psi, \chi \) and \( P_z \) are determined as solutions of Eqs. (9a)~(10), one can easily obtain \( w_z = \psi - w_i \) and \( w_z = \chi - v_i \).

Equations (9) and (10) are nonlinear ordinary integro-differential equations for \( \psi, \chi \) and \( P_z \) which will be solved by an iterative process. If \( \psi_m = \chi_m \) and \( P_{z,m} \) define the \( m \)th iterative approximation to the solution of Eqs. (9) and (10), then the \((m+1)\)th approximations \( \psi_{m+1}, \chi_{m+1}, P_{z,m+1} \) are given by
\[
\psi_m(\chi_{m+1} - 2w_i) - \left( \frac{\chi_{m+1}}{r} + \frac{\chi_m}{r^2} \right) \int_0^r (\psi_{m+1} - 2w_i) \, dz \, dr = l \left( \frac{\chi_{m+1}}{r} + \frac{\chi_m}{r^2} \right) \]  
(11a)
\[
\psi_m(\psi_{m+1} - 2w_i) - \left( \frac{\psi_{m+1}}{r} + \frac{\psi_m}{r^2} \right) \int_0^r (\psi_{m+1} - 2w_i) \, dz \, dr \]  
\[-2(P_{z,m+1} - P_i) - 2\int_0^r \chi_m(\chi_{m+1} - 2w_i) \, dz \, dr + l \left( \frac{\psi_{m+1}}{r} + \frac{\psi_m}{r^2} \right) \]  
(11b)
\[
\int_0^1 \psi_{m+1} \, dz = 2 \pi \int_0^1 w_i \, dz \]  
(11c)

A rectangular mesh of dimensions \((l, h)\) is now formed by introducing differences in the \( r \)-direction. If the suffix \( k \) refers to the \( k \)th mesh-point in this direction, Eqs. (11a) and (11b) can be arranged as
\[
l(\chi_{m+1}) = \frac{l}{r_k} (\chi_{m+1}) - \frac{l}{r_k} \int_0^r (\chi_{m+1}) \, dz \, dr \]  
(12a)
\[
l(\psi_{m+1}) = \frac{l}{r_k} (\psi_{m+1}) + 2 \int_0^r \psi_{m+1} \, dz \, dr \]  
(12b)

where \( r_k = r_k \).

The following finite difference approximations are now made:
\[
\chi_{m+1} = (\chi_{m+1} - \chi_{m+1} - 2\psi_{m+1}, \psi_{m+1}, \psi_{m+1})/(2h) \]  
(13a)
\[
\psi_{m+1} = (\psi_{m+1} - \psi_{m+1}, \psi_{m+1}, \psi_{m+1})/(2h) \]  
(13b)
\[
\int_0^r \psi_{m+1} \, dz = h\, k \]  
(13c)
\[
\int_0^r \psi_{m+1} \, dz = h\, k \]  
(13d)
\[
\int_0^r \chi_{m+1} \, dz = h\, k \]  
(13e)

Substitution of these approximations into Eqs. (12b), (12a) and (11c) yields
\[
-2P_{z,m+1} + \alpha_m k \psi_{m+1} + \ldots + \alpha_m k \psi_{m+1} + \beta k \psi_{m+1} + \gamma k \psi_{m+1} = \frac{2h \int_0^r \psi_{m+1}}{r_k} - \frac{2h \int_0^r \psi_{m+1}}{r_k} \]  
(14a)
\[ \xi_m, k, r, \varphi_{m+1, 1} + \xi_m, k, r, \varphi_{m+1, 1} + \xi_m, k, r, \varphi_{m+1, k-1} + 1/2 \xi_m, k, r, \varphi_{m+1, k} \]
\[ + \left( \beta - \frac{\Delta h}{2r_k} \right) \lambda_{m+1, k-1} + D_m, k, \lambda_{m+1, k} + \left( \beta + \frac{\Delta h}{2r_k} \right) \lambda_{m+1, k+1} = E_{m, k} \]  
\[ (14b) \]
\[ r_1 \varphi_{m+1, k} + r_2 \varphi_{m+1, k} + \ldots + r_{n-1} \varphi_{m+1, k} + - 2r_n \]  
\[ (14c) \]

where \( \beta = l/h, \xi_m, k = \sigma_m + \frac{\Delta h}{r_k}, A_m, k = \beta - \frac{\Delta h}{2r_k}, a_m, k, r_{k-1} - B_m, k = -2\beta + \frac{1}{n} \sigma_m, a_m, k, r_k - \varphi_m, k \)
\[ C_{m, k} = -2P_2 - 2w_1, k, \varphi_m, k + 2a_m, k, r_k - 4h_k, k, D_m, k = -2\beta - \frac{\Delta h}{r_k}, \varphi_m, k, E_{m, k} = -2v_1, k, \varphi_m, k + 2\xi_m, k, r_k \]
Since \( \varphi_{m+1, n} = 0, \lambda_{m+1, n} = 0 \)

from the boundary conditions (3b) and (3c), the equations for \( k=1 \) are
\[ -2P_2 + A_m, 1, \varphi_{m+1, 0} + B_m, 1, \varphi_{m+1, 1} + \beta + \frac{\Delta h}{2r_1} \lambda_{m+1, 1} = C_{m, 1} \]  
\[ (14d) \]
\[ 1/2 \xi_m, 1, \varphi_{m+1, 1} + D_m, 1, \varphi_{m+1, 1} + \beta + \frac{\Delta h}{2r_1} \lambda_{m+1, 1} = E_{m, 1} \]  
\[ (14e) \]

where \( n \) is taken such that \( n = l/h \). Also the equations for \( k=n-1 \) become
\[ -2P_2, n-1 + A_m, n-1, \varphi_{m+1, n-1} + \ldots + A_m, n-1, \varphi_{m+1, 1} + B_m, n-1, \varphi_{m+1, 1} + B_m, n-1, \varphi_{m+1, 1} + \lambda_{m+1, 1} = C_{m, n-1} \]  
\[ (14f) \]
\[ \xi_m, n-1, \varphi_{m+1, 1} + \ldots + \xi_m, n-1, \varphi_{m+1, 1} + \lambda_{m+1, 1} + 1/2 \xi_m, n-1, \varphi_{m+1, 1} \]  
\[ (14g) \]

When \( z_1 = 0 \), some modifications must be made of Eq. (15) which are given in the Appendix.

In terms of \( \varphi \), the boundary condition \( \partial \varphi / \partial r \rangle_{r=0} = 0 \)
which can be written in the finite difference form
\[ -2.74 \varphi_{m+1, 0} + 6 \varphi_{m+1, 1} - 6 \varphi_{m+1, 2} + 4 \varphi_{m+1, 3} - 1.5 \varphi_{m+1, 4} + 0.24 \varphi_{m+1, 5} = 0 \]  
\[ (14h) \]
when Lagrange's six-point interpolation formula for differentiation is employed. Eqs. (14a)~(14h) are a set of \( 2n \) simultaneous linear algebraic equations for the \( 2n \) unknown quantities \( P_2, m, \varphi_{m+1, 0}, \varphi_{m+1, 1}, \varphi_{m+1, 2}, \ldots, \varphi_{m+1, n-1}, \lambda_{m+1, 1}, \lambda_{m+1, 2}, \ldots, \lambda_{m+1, n-1} \). These simultaneous linear equations can be written in matrix form:
\[ A_m, n, \lambda_{m+1} = C_m \]
where \( A_m \) is a \( 2n \times 2n \) matrix, while \( \lambda_{m+1} \) and \( C_m \) are column matrices given by
\[ \lambda_{m+1} = \{ \lambda_{m+1, 0}, \lambda_{m+1, 1}, \lambda_{m+1, 2}, \ldots, \lambda_{m+1, n-1}, \lambda_{m+1, 1}, \lambda_{m+1, 2}, \ldots, \lambda_{m+1, n-1} \} \]
\[ C_m = \{ C_{m, 1}, C_{m, 2}, \ldots, C_{m, n-1}, E_{m, 1}, E_{m, 2}, \ldots, E_{m, n-1}, 2r_m, 0 \} \]
The matrix \( A_m \) can be written easily from the left-hand sides of Eqs. (14a)~(14h). The solution of these equations is then obtained by usual methods of matrix inversion and is given by
\[ \lambda_{m+1} = A_m^{-1} C_m \]
This procedure is repeated until the following condition is satisfied:
\[ \text{max} \{ |P_2, m, \varphi_{m+1, 0} - P_2, m, \varphi_{m, 1}, | \varphi_{m+1, n} + \varphi_m, n, | \lambda_{m+1, 1} - \lambda_m, 1 | \} \leq e \]
where max means the maximum value and \( e \) depends on the accuracy desired. This gives the values of \( P_2, \varphi \), and \( \gamma \) at \( z = z_2 \). The same process can then be applied to find the solution at \( z = z_2 + l \) and so on. The value of \( \varphi \) thus obtained will be denoted by \( \varphi_m \) for later reference.

In the same manner, finite difference approximation can be applied to the thermal energy Eq. (6c). The energy equation which corresponds to Eq. (7) becomes
\[ \frac{w_1 + w_2}{2} T_{z_1} - T_{z_1} = \frac{\Delta h}{2r} \int_0^l \left( T'''' + T'''' + T'''' + T'''' + T'''' + T'''' \right) dr \]
where the viscous dissipative term has been neglected. The inclusion of this term will not add any difficulty to the finite difference procedure adopted here and the same process can be applied in order to evaluate the viscous dissipation effect. Let
\[ \varphi_n = w_0, \quad \theta = T_0 + T_k \]
then one obtains
\[ \varphi_n (\theta - 2T_k) - \frac{\theta'_n}{r} \int_0^r (\varphi_m - 2w_0)r dr = \frac{1}{P_r} \left( \theta'_n + \theta'_k \right) \]
which can be written with respect to the kth mesh point from the pipe axis as follows:
\[ \frac{1}{P_r} \left( \theta''_k \right) + \frac{1}{P_r} \left( \theta'_k \right)_k + \frac{1}{r_k} \int_0^{r_k} (\varphi_m - 2w_0)r dr - \varphi_m, k \theta_k = -2T_{1, k} \varphi_m, k \quad \ldots \quad (16) \]
Substitution of finite difference approximations
\[ (\theta''_k)_k = (\theta_{k+1} - 2\theta_k + \theta_{k-1}) / h^2, \quad (\theta'_k)_k = (\theta_{k+1} - \theta_{k-1}) / (2h) \]
\[ \int_0^r \varphi_m r dr = h S_{m, k}, \quad S_{m, k} = r_1 \varphi_{m, 1} + r_2 \varphi_{m, k} + \ldots + r_{k-1} \varphi_{m, k-1} + \frac{1}{2} r_k \varphi_{m, k} \quad \ldots \quad (17) \]
into Eq. (16) yields
\[ H_{m, k} \theta_{k+1} + O_{m, k} \theta_k + Q_{m, k} \theta_{k-1} = X_{m, k} \quad \ldots \quad (18a) \]
where
\[ H_{m, k} = -2 \beta \frac{r_1}{P_r} \frac{\varphi_m, k}{2r_k} + S_{m, k} - 2f_k \]
\[ O_{m, k} = -2 \beta \frac{r_1}{P_r} \frac{\varphi_m, k}{2r_k} \]
\[ Q_{m, k} = -2 \beta \frac{r_1}{P_r} \frac{\varphi_m, k}{2r_k} + S_{m, k} - 2f_k \]
\[ X_{m, k} = -2T_{1, k} \varphi_m, k \]
Since the boundary condition (3b) implies that \( \theta_n = 0 \), the equation for \( k = n - 1 \) becomes
\[ H_{m, n-1} \theta_{n-1} + O_{m, n-1} \theta_{n-2} + Q_{m, n-1} \theta_{n-3} = X_{m, n-1} \quad \ldots \quad (18b) \]
Also the boundary condition (3c) gives
\[ \frac{\partial \theta}{\partial r} \bigg|_{r=0} = 0 \]
which can be reduced to
\[ -2.74\theta_n + 6\theta_1 - 6\theta_2 + 4\theta_4 - 1.5\theta_5 + 0.24\theta_3 = 0 \quad \ldots \quad (18c) \]
by Lagrange's six-points interpolation formula for differentiation. The set of n simultaneous linear Eqs. (18a)-(18c) for the n unknowns \( \theta_0, \theta_1, \ldots, \theta_{n-1} \) can be written in matrix form:
\[ \mathbf{H}_\theta \mathbf{\theta} = \mathbf{X}_\theta \]
in which \( \mathbf{H}_\theta \) is an \( n \times n \) matrix, while \( \mathbf{\theta} \) and \( \mathbf{X}_\theta \) are column matrices defined by
\[ \mathbf{\theta} = \{ \theta_0, \theta_1, \theta_2, \ldots, \theta_{n-1} \} \]
\[ \mathbf{X}_\theta = \{ X_{\theta, 1}, X_{\theta, 2}, \ldots, X_{\theta, n-1}, 0 \} \]
If the inverse of the matrix \( \mathbf{H}_\theta \) is expressed by \( \mathbf{H}_\theta^{-1} \), the solution vector can be obtained from
\[ \mathbf{\theta} = \mathbf{H}_\theta^{-1} \mathbf{X}_\theta \]
Since the energy equation is linear with respect to \( T \), the iteration process is not required.

5. Damping of swirl and heat transfer

The angular momentum \( \Gamma \) at an arbitrary cross section of the pipe is given by
\[ \Gamma = 2\pi \rho w^3 \int_0^1 w r^2 dr \]
Since the product of momentum in the z-direction and the pipe radius is
\[ 2\pi \rho w^3 \int_0^1 w r^2 dr \]
it is convenient to define the damping coefficient of swirl by the ratio of the above two quantities:
\[ m_1(z) = \int_0^1 \Gamma w r^2 dr \int_0^1 w^3 r^2 dr \quad \ldots \quad (19) \]
On the other hand, Kreith & Sonju(4) showed that the angular velocity \( \omega_{a} \) of a swirl blade meter for the measurement of the strength of swirl was given by
\[ \omega_a = \frac{\Gamma}{a} \left[ \int_0^1 \frac{r^2}{w^3 r^3} \right]^{1/3} \]
where \( r_a \) denotes the radius of the swirl blade meter. Then it is possible to define another damping coefficient of swirl in the form
\[ m_2(z) = \left[ \int_0^1 \frac{r^2}{w^3 r^3} \right]^{1/3} \quad \ldots \quad (20) \]
The bulk temperature is
\[ T_m = \int_0^1 T w r^3 dr \int_0^1 w^3 r^2 dr \quad \ldots \quad (21) \]
and the dimensionless bulk temperature \( T_m \) takes the form
\[ T_m = \frac{(T_m - T_o)/(T_i - T_o)}{2(1/T_m wr dr)_{r=0}} = 2(1/T_m wr dr)_{r=0} \quad \ldots \quad (22) \]

6. Numerical calculations

Numerical calculations are performed for
\[ f(r) = 1, \quad g(r) = K \]
with four values of \( K = 0, 0.5, 1.0 \) and 1.5. This choice of \( f(r) \) and \( g(r) \) means that the axial velocity profile is uniform at the entrance section and the initial swirl is a forced vortex. When the forced vortex is expressed as
\[ \psi = \Omega r \]
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it is easy to show that

$$K = 2a/W \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (23)$$

The energy equation is considered by assuming that the temperatures of the pipe wall and the fluid at the entrance section are constant, respectively, and the Prandtl number is taken as 0.72 (acceptable value for air).

In the immediate neighbourhood of the entrance section, the mesh size of $n = 36$ and $l = 10^{-4}$ was employed and the values of $n$ and $l$ were varied in the downstream direction as are shown in Table 1. The value of $e$ for convergence was chosen to be $10^{-4}$ and the number of iterations of 8 in the first step was changed to 2 or 3 after a few steps, when the first approximate solution at $z = z_1$ was chosen to be the same at $z = z_2$. The computations were carried out for four values of $K$ on the HITAC 5020E electronic digital computer installed at the Computer Center of University of Tokyo, which took about forty minutes to proceed from $z = 0$ to $z = 0.4$. Library program No. 25, FL/TC/MINV of the Computer Center was utilized in order to obtain the inverse matrices $A_n^{-1}$ and $H_n^{-1}$.

7. Results and discussion

Figure 2 shows the profiles of axial and tangential velocity components and temperature at several $z$ sections for $K = 1.0$. The damping process of the swirl and the asymptotic approach of $w$ to the fully-developed parabolic velocity profile $w = 2(1 - r^2)$ are clearly seen. It should be noted that the otherwise flat part of the inviscid core near the entrance section is a little distorted by the presence of the swirl. The variations of $w$ in the axial direction for $r = 0, 0.5, 0.7, 0.8$ and 0.9 are shown in Fig. 3 for $K = 0$ and 1.0. It is obvious from this figure that the flow is accelerated in the neighbourhood of the pipe wall and is decelerated near the pipe axis due to the effect of the swirl when $z > 0.02$. This tendency becomes more significant as the value of $K$ increases from 1.0 to 1.5.

The results for $K = 0$ correspond to the flow in the entrance region without swirl and its overall development agrees well with previous calculations(7)–(10) and measurements(11). Although a number of numerical analyses have been reported since the work of Bodoia & Osterle(12) for the laminar developing flow in the entrance region of a circular pipe, the inertia terms of the equation for $z = z_1$ are in most cases linearized by the velocity at $z = z_1$ and these linearized equations are solved. This process corresponds roughly to the first iteration in the scheme presented in the present paper. So far as the number of iterations in the vicinity of the entrance section and the value of $e$ employed in the present computation are concerned, it could be judged that the present result is more accurate than previous calculations.

The flow in the entrance region approaches asymptotically the fully-developed regime and then the theoretical fully-developed flow can be attained at a distance infinitely downstream. However, for prac-
tical purposes, it may be reasonable to define the entrance length as the axial distance where the center-line velocity reaches 99% of its fully-developed value. Table 2 contains the dimensionless entrance length $L_{eq}$ thus defined, together with the value obtained by other investigators. As would physically be expected, the entrance length increases as $K$ increases. For instance, the value of $L_{eq}$ for $K=1.0$ is larger by 27% than that for $K=0$ which means there is no swirl at the inlet.

Figure 4 shows the variation of the tangential velocity component in the axial direction for $K=1.0$ and $r=0.5$, 0.7, 0.8 and 0.9. Figure 5 shows the damping coefficients of the swirl $m_1$, $m_2$ defined by Eqs. (19) and (20). It can clearly be seen in Fig. 5(a) and (b) that $m_1$ and $m_2$ decrease exponentially in the range $z>0.15$ and $z>0.08$. The damping process can be written in the following form:

$$K=0.5: m_1=0.1255 e^{-10.72z}$$
$$m_2=0.2647 e^{-11.61z}$$

$$K=1.0: m_1=0.255 e^{-10.72z}$$
$$m_2=0.522 e^{-10.66z}$$

$$K=1.5: m_1=0.376 e^{-10.72z}$$
$$m_2=0.763 e^{-10.68z}$$

It is noteworthy that the damping factor $-(1/m_1) (dm_1/dz)$ is about 10.7 irrespective of the value of $K$ at least in the range covered in this paper. Therefore, both $m_1$ and $m_2$ can be used as the measure of damping of the swirl. However, in the vicinity of the entrance section, the damping of the swirl is more rapid than the exponential decay.

The variation of temperature in the axial direction is given in Fig. 6 to illustrate the comparison between the temperature profile in the flow with swirl and that in the flow without swirl. The temperature gradient near the pipe wall is slightly larger in the range $z>0.01$ when the swirl is present.

Figure 7 shows the variation of pressure in the axial direction. If the flow were fully developed immediately from the entrance section, the pressure distribution would be

$$p(r,0)-p(r,z)=8z$$

The pressure distribution shown in Fig. 7 is seen to approach a straight line with the same gradient for all values of $K$ considered now. Then it is of practical importance to introduce a dimensionless additional pressure drop $C_\infty$ in the entrance region defined by

$$C_\infty=\lim_{z\to\infty} \left[ p(0,0)-p(0,z)-8z \right]$$

It should be noted here that the pressure at the center of the entrance section is taken as the reference. This choice of reference pressure means that $C_\infty$ can directly be compared with the additional pressure drop for the case of $K=0$. The values of $C_\infty$ numerically computed at $z=0.4$ are tabulated in Table 3(a). $C_\infty$ is seen to decrease as $K$ increases, and the value of $C_\infty$ for $K=1.0$ is only 74% of the one for $K=0$. In Table 3(b), values of the additional pressure drop obtained by previous investigators are compared with

![Fig. 4 Axial distribution of tangential velocity component for $K=1.0$](image1)

![Fig. 5 Axial variation of damping coefficients of swirl](image2)
the result of the present analysis. The results of Hornbeck (13) and Sparrow, Lin & Lundgren (19) are closest to the present result. Moreover, a few experimental studies (11,16) have been reported concerning the laminar flow in the entrance region of a circular pipe. Riemann (17), among others, measured very accurately the pressure drop in the entrance region and obtained \( C_w = 0.624 \pm 0.006 \) which agrees well with the present result.

Figure 8 shows the distribution of local Nusselt number \( N_u_z \) defined by Eq. (22). This distribution of \( N_u_z \) for \( K = 0 \) agrees quite well with that obtained by Kaye (18). It is worth mentioning that the value of \( N_u_z \) in the neighbourhood of the entrance section is a little smaller than that for \( K = 0 \), while it becomes a little larger in the range \( z > 0.01 \) owing to the effect of the swirl. This tendency becomes more significant as \( K \) increases. For instance the value of \( N_u_z \) for \( K = 1.5 \) is 5% larger than the case of \( K = 0 \) at \( z = 0.05 \). Accordingly, it can be said that the existence of swirl at the entrance section causes an appreciable decrease of the additional pressure drop in the entrance region and at the same time an increase in the local Nusselt number by a few per-cent at least in the range \( 0.01 < z < 0.1 \).

Concerning a problem of vortex breakdown in a circular pipe, Bosse (19) has recently reported an

**Table 3** Dimensionless additional pressure drop

<table>
<thead>
<tr>
<th>( K )</th>
<th>( C_w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.627</td>
</tr>
<tr>
<td>0.5</td>
<td>0.586</td>
</tr>
<tr>
<td>1.5</td>
<td>0.466</td>
</tr>
<tr>
<td>2.0</td>
<td>0.366</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method of solution</th>
<th>( C_w (K = 0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>present result</td>
<td>0.627</td>
</tr>
<tr>
<td>Hornbeck (13)</td>
<td>0.634</td>
</tr>
<tr>
<td>Langhaar (17)</td>
<td>0.64</td>
</tr>
<tr>
<td>Campbell &amp; Slattery (8)</td>
<td>0.59</td>
</tr>
<tr>
<td>Sparrow, Lin &amp; Lundgren (19)</td>
<td>0.62</td>
</tr>
<tr>
<td>Christiansen &amp; Lemmon (14)</td>
<td>0.637</td>
</tr>
<tr>
<td>Tomita (16)</td>
<td>0.61</td>
</tr>
</tbody>
</table>

Fig. 8 Distribution of local Nusselt number for \( K = 0, 1.0 \) and 1.5. The prandtl number is 0.72
analytical study on the swirling flow which has the same velocity profile at the entrance section as that assumed in this paper. Although his analysis is restricted to the region near the pipe axis, it is shown that the vortex breakdown will occur when $K = 1.916$. In this connection it is interesting to note that the application of the present finite difference scheme to the case of $K = 2.0$ failed to give a converging solution. Once the vortex breakdown occurred, the boundary layer approximation would not be able to describe the flow phenomena concerned and the full Navier-Stokes equations or their reasonable forms of approximation must be used.

It would be of great practical importance, in view of the flow in a vortex tube, to analyze characteristics of the swirling laminar flow in the entrance region of a circular pipe by means of a suitable approximation of the Navier-Stokes equations, especially to establish conditions for the occurrence of vortex breakdown. However, this remains as a future study.

8. Conclusions

In this paper, the laminar swirling flow in the entrance region of a circular pipe is theoretically considered on the basis of the laminar boundary layer theory. The results obtained can be summarized as follows:

1) A finite difference scheme developed by Leigh and Terrel for the calculation of two-dimensional laminar boundary layers is successfully extended to the case of simultaneous equations which describe a rotationally symmetric flow with a tangential velocity component.

2) The entrance length increases as the swirling component of velocity increases. For example, the entrance length for $K = 1.0$ is larger by 27% than that for $K = 0$.

3) The additional pressure drop in the entrance region which is referred to the pressure at the center of the entrance section decreases appreciably in the presence of swirl. For example, the additional pressure drop for $K = 1.0$ is about 74% of the value for $K = 0$.

4) The local Nusselt number increases by a few per-cent in the range $z > 0.01$ compared with the case of no swirl when the temperature of the pipe wall is constant and the Prandtl number is taken as 0.72. For example, the increase in $N_u z$ for $K = 1.5$ at $z = 0.05$ is about 5% of $N_u z$ for $K = 0$ at the same location.

5) Damping coefficients of the swirl $m_1$ and $m_2$ show an exponential decay in the range $z > 0.15$ and $z > 0.08$, respectively. The damping factor $-(1/m) (dm/dz)$ increases a little as $K$ increases, but it can practically be said that it takes a constant value of 10.7 for all $K$ in the range $0.5 \leq K \leq 1.5$. The damping of the swirl near the entrance section is more rapid than an exponential decay.

Appendix

Since the velocity and temperature profiles at the entrance section are $v = K r$, $w = 1$ and $T = 1$ in dimensionless variables, the boundary conditions for $\psi_m$, $\chi_m$, $\theta$ etc. become
References


Discussions

H. Ito (Tohoku University):

(1) Concerning the velocity profiles at the entrance section, it is understood that an ordinary vortex such as seen in the draining pipe which authors mention in the introduction consists of a forced vortex in the central core and a free vortex surrounding it. In what situation the inlet velocity profile of $f(r)=1, g(r)=Kr$ would be realized practically? For example, even if swirl generating vanes are used, the wake of the boss located in the central part will have an effect on the velocity profile. Moreover, if the flow comes from a rotating pipe, the axial velocity profile will have already developed to a certain extent.

S. Ishizawa (Hitachi co., Ltd.):

(2) In this paper, an additional pressure drop defined by

$$C_m = \lim \{ p(0, 0) - p(0, z) - 8 \}$$

is numerically examined. However, the static pressure variation in the radial direction owing to the centrifugal force is generally large for the flow with swirl, so that characteristic properties of the flow considered in this paper cannot be judged to be fully understood by this quantity alone. Since, from a physical point of view, the most important thing is how the energy loss of flow increases along the flow, it is desirable to compute the mean total pressure at cross sections of the pipe and to examine the variation of this quantity in the axial direction, together with $C_m$.

(3) In this paper, only the case of a rigid rotation at the entrance section is calculated. In practice, however, there are many cases in which the vortex is a free vortex or a combination of a rigid body vortex core and a free vortex. Therefore, it may be advisable to examine such cases, too.

Authors' closure

(1) As was pointed out by the discusser, swirling flows encountered most frequently in practice are understood to consist of a forced vortex in the central core and a free vortex outside it. However, a flow at the entrance of a stationary circular pipe which is connected to a reservoir rotating at a constant speed will have the same inlet velocity profile as assumed in this analysis. Moreover, among the data on turbulent swirling flows obtained by Murakami et al.* and Senoo & Nagata**, there can be seen a few cases in which a forced vortex occupies the major part of the pipe. The finite difference scheme presented in this paper may also be applied to vortex flows other than the forced vortex.

(2) The main object of this paper was to find the effect of swirl on the pressure drop in the entrance region, especially to make direct comparison with the case of no swirl, so that the pressure at the center of the entrance section was used to define the additional pressure drop. However, as was pointed out by the

discusser, it will be necessary to examine the variation
of the mean total pressure in the axial direction in
order to clearly understand the physical properties of
the flow phenomena concerned.

The dimensionless total pressure at any point of
the pipe can be written as

\[ E(r, z) = P + \frac{1}{2}(u^2 + v^2 + w^2), \quad p = \int_0^r \frac{v^2}{r} \, dr \]

Since \( u \) is much smaller than the other velocity
components, the dimensionless mean total pressure \( \bar{E}(z) \)
becomes

\[ \bar{E}(z) = 2\int_0^1 E(r, z) r \, dr \]

\[ = P(z) + 2\int_0^1 r \, dr \int_0^r \frac{v^2}{r} \, dr + \int_0^1 (v^2 + w^2) r \, dr \]

The loss of the mean total pressure \( \bar{E}(0) - \bar{E}(z) \) is
plotted in Appendix-Fig. 1 for \( K=1.0 \), together with
\( P(0) - P(z) \) for reference. In this connection, the
additional energy loss in the entrance region is im-
portant. Since \( v \rightarrow 0, w \rightarrow 2(1 - r^2) \) and
\( P(z) \rightarrow -8z + C_m \) as \( z \rightarrow \infty \), it is possible to define the additional
energy loss \( \Delta E_m \) in the form

\[ \Delta E_m = \lim_{z \to \infty} [\bar{E}(0) - \bar{E}(z) - 8z] \]

Utilizing the relation \( v = Kr \) and \( w = 1 \) at \( z = 0 \), one
obtains

\[ \Delta E_m = C_m + \left( \frac{K^2}{2} - \frac{1}{6} \right) \]

Append.-Table 1 shows the relation between \( K \) and
\( \Delta E_m \) and it can be seen that \( \Delta E_m \) increases as \( K \)
increases, while \( C_m \) decreases as \( K \) increases as shown
in Appendix.-Fig. 2.

In the case of swirling flow considered in this
paper, the mean total pressure at the entrance section
is larger by \( K^2/2 \) than in the case of no swirl, so that
one can define this quantity as the excess energy.
Append.-Fig. 2 shows \( C_m, \Delta E_m \) and the difference in
additional energy loss \( \Delta E_m - (\Delta E_m)_{K=0} \) as functions of
the excess energy. It is worth mentioning that the
above quantities can be expressed as linear functions
of the excess energy and the same kind of relationship
is expected to hold for other types of vortices other
than the forced vortex.

(3) Since this question is almost same as that of
H. Ito, the authors' closure (1) may be referred to.