Stress Analysis of Expansion Joints of Pressure Vessels under Internal Pressure

By Shigeo TAKEZONO**, Kazuhiko INOUE***, Humitaka OGASAWARA****, and Yukio KATO

In this paper an elastic analysis of the expansion joints used in pressure vessels and pipe lines under internal pressure is presented. Although these joints are widely used, few investigations on their strength have been published and at present designers are employing the rough formulas of M. W. Kellogg Co. for practical designing. Therefore in this investigation the problem is treated by using a shell theory and a calculating method is established by which the stresses and deformation of these joints can be evaluated. Numerical results of stress distributions and deformation for some dimensions of the expansion joints are shown, and these results are compared with experimental ones. The general solutions for the case of both axial load and internal pressure can be obtained by combining the solutions in this paper with ones in the previous paper.

1. Introduction

In pressure vessels and pipe lines the expansion joints as shown in Fig. 1 are widely used to prevent thermal fracture. The elastic analysis of these joints under axial load was described in the previous paper(1). In this paper the analysis of these joints subject to internal pressure is developed by using a shell theory and the calculating method is established by which the stresses and axial displacement of the joints can be evaluated with enough accuracy for practical purposes. And some numerical results of calculation by this method are compared with experimental ones.

2. Analytical method

Now in analyzing this problem the following are assumed. Since the states of the stresses and deformations of the type A and the type B in Fig. 1 are both symmetric with respect to point A, it is enough to consider only the right parts of point A. The elements AC and BD are toroidal shells and the element CD is an annular plate. The right part of point B may be assumed as a semi-infinite cylindrical shell in

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* Received 3rd October, 1969.
** Associate Professor, Faculty of Engineering, Gifu University, Kagemigahara.
*** Kobe Cast Iron Works, Ltd.
**** Graduate Student, Division for Research of Engineering, Graduate School, Gifu University.
☆ Mitsubishi Electric Co.

Fig. 1 Shape and dimension of expansion joints
the type A and as a rigid body in the type B. Hence both the type A and the type B are analyzed by employing the similar method except for boundary conditions at point B.

2.1 Relations for toroidal shells

The basic differential equation for the axially symmetric problem of a toroidal shell is

\[
d \frac{d}{d\varphi} \left( \frac{1 + \lambda \sin \varphi}{d \varphi} \right) + i \mu \sin \varphi = -i \mu A \cos \varphi
\]

(1)

where \( \lambda = r_i/r_e \), \( \mu = mr_i/r_e h \), \( A = mr_e A/Eh^2 \)

\( r_e \): radius of ring in toroidal shell

\( m = \sqrt{12(1 - \nu^2)} \), \( \nu \): Poisson’s ratio

\( A = (1 + \lambda \sin \varphi) V_0 = (pr_i(2 + \lambda \sin \varphi) \sin \varphi_0)/2 \)

\( V_0 \): value of \( V \) at \( \varphi = \varphi_0 \) (refer to Fig. 2)

\( \gamma \): functions of the solution for toroidal shell

\( p \): internal pressure

Let the homogeneous solutions of Eq. (1) be \( \gamma_1 = \gamma_1 + i\gamma_2 \) and \( \gamma_2 = \gamma_2 + i\gamma_2 \), and the particular solution be \( \gamma_p = A \gamma_p \), and we have

\[ \gamma = C_1 \gamma_1 + C_2 \gamma_2 + C_2 \gamma_2 + C_2 \gamma_2 + A \gamma_p \]

\[ \chi = C_1 \gamma_1 - C_2 \gamma_2 + C_2 \gamma_2 - C_2 \gamma_2 - A \gamma_p \]

(2)

where

\( \gamma \): increment of \( \varphi \) due to deformation

\( \chi = \chi - P(2 + \lambda \sin \varphi) \sin \varphi \cos \varphi \)

\( A = mp(d^2 - d^2)/2Eh^2 \)

\( q = (b^2 - d^2)/(a^2 - d^2) \)

\( \lambda = r_i/a = \lambda d/(1 + \xi) \)

\( \lambda = -r_i/b = -\lambda d/(1 - \xi) \)

\( r_i = a, \ r_i = b \)

\( p = -p \equiv p \)

\( P = ma \rho r_i^2/2Eh^2 \)

\( \mu = mr_i^2/ah = m \lambda^2 / \lambda_i \)

\( \xi = r_0/r_m \)

(10)

And Eqs. (2) become as follows.

\[ \gamma^{(1)} = C^{(1)} \gamma_1 + C^{(1)} \gamma_2 + C^{(1)} \gamma_2 + C^{(1)} \gamma_2 + A^{(1)} \gamma_p \]

\[ \chi^{(1)} = C^{(1)} \gamma_1 - C^{(1)} \gamma_2 - C^{(1)} \gamma_2 - C^{(1)} \gamma_2 - A^{(1)} \gamma_p \]

(11)

\[ \gamma^{(2)} = C^{(2)} \gamma_1 + C^{(2)} \gamma_2 + C^{(2)} \gamma_2 + C^{(2)} \gamma_2 + A^{(2)} \gamma_p \]

\[ \chi^{(2)} = C^{(2)} \gamma_1 - C^{(2)} \gamma_2 - C^{(2)} \gamma_2 - C^{(2)} \gamma_2 - A^{(2)} \gamma_p \]

(12)

The constants \( C^{(i)} \) and \( C^{(i)} \) \( (i = 1 \sim 4) \) in Eqs. (11) and (12) are determined from the boundary conditions as described later.
2-2 Relations for annular plate

Each component of internal forces (per unit length) and each component of displacements in the annular plate are given by the following equations (refer to Fig. 2).

\[ G_u = -2A_0 \left(1 - \frac{2}{1+\nu}\right) \frac{r - B_0}{r} \]  

(13)

\[ w = C_1 + C_2 r^2 + C_3 r^4 \log r + C_4 \log r - \frac{p}{D} \frac{r^4}{64} \]  

(14)

\[ N_o = 2 A_0 h + B_0 h^2 / r \]  

(15)

\[ N_o = 2 A_0 h - B_0 h^2 / r \]  

(16)

\[ Q = -2d_3 + \frac{pr}{D} \]  

(17)

\[ M_o = d_2 (1 + \nu) - d_3 \left[ 1 + (1 + \nu) \log r \right] + \frac{(1 - \nu) d_1}{r^2} + \frac{(1 + 3 \nu) p r^2}{16 D} \]  

(18)

\[ M_o = d_2 (1 + \nu) - d_3 \left[ 1 + (1 + \nu) \log r \right] + \frac{(1 - \nu) d_1}{r^2} + \frac{(1 + 3 \nu) p r^2}{16 D} \]  

(19)

where

\[ d_1 = C_0, \ d_2 = 2C_0 + C_0, \ d_3 = 2C_0 \]  

(20)

2-3 Relations for cylindrical shell

Each component of internal forces (per unit length) and each component of displacements in the cylindrical shell are given by the following equations (refer to Fig. 2).

\[ \tau^* = m / 2 \Delta h \]  

(21)

\[ Q = \frac{E h^3}{m d} e^{z \tau_1} (K_2 \cos \gamma z - K_1 \sin \gamma z) \]  

(22)

\[ N_o = 0 \]  

(23)

\[ N_o = \frac{E h^3}{m d} e^{z \tau_1} ((K_1 + K_2) \cos \gamma z - (K_1 - K_2) \sin \gamma z + p d) \]  

(24)

\[ M_o = D \gamma e^{z \tau_1} ((K_1 - K_2) \cos \gamma z + (K_1 + K_2) \sin \gamma z) \]  

(25)

\[ M_o = \nu M_o \]  

(26)

\[ \varphi = e^{z \tau_1} (K_1 \cos \gamma z + K_2 \sin \gamma z) \]  

(27)

\[ u = \frac{e^{z \tau_1}}{2 \gamma} \left( (K_1 + K_2) \cos \gamma z - (K_1 - K_2) \sin \gamma z \right) \]  

(28)

where \( K_1 \) and \( K_2 \) are constants.

2-4 Boundary conditions

2-4.1 Case of type A

Due to the symmetry with respect to point \( A (\phi = \pi/2, \ \psi = 0 \) in shell I), \( \eta^{(1)} \) and \( Q^{(1)} \) vanish at this point, i.e.

\[ \eta^{(1)} = 0, \ Q^{(1)} = 0 \ (or \ \phi^{(1)} = 0) \]  

(29)

From the properties of \( \eta \)-functions (26),

\[ \eta_{1R}^{(1)} = 1, \ \eta_{1L}^{(1)} = \eta_{2R}^{(1)} = \eta_{2L}^{(1)} = \eta_{ov}^{(1)} \]  

(30)

at point A. Accordingly from Eqs. (29) and (11)

\[ C_i^{(1)} = 0, \ C_i^{(1)} = 0 \]  

(31)

The conditions of continuity between the toroidal shell (shell II) and the cylindrical shell (shell IV) are as follows. Namely at point B (\( \phi = \pi/2 \) in shell II, \( \psi = 0 \) in shell IV),

\[ \begin{align*}
(a) & \quad (N_{o2}^{(1)})_{\psi=0} = (N_{o2}^{(1)})_{\psi=0} = 0 \\
(b) & \quad (M_{e2}^{(1)})_{\psi=0} = (M_{e2}^{(1)})_{\psi=0} = 0 \\
(c) & \quad (M_{e3}^{(1)})_{\psi=0} = (M_{e3}^{(1)})_{\psi=0} = 0 \\
(d) & \quad (Q_{e2}^{(1)})_{\psi=0} = (Q_{e2}^{(1)})_{\psi=0} \\
(e) & \quad (Q_{e3}^{(1)})_{\psi=0} = (Q_{e3}^{(1)})_{\psi=0} \\
\end{align*} \]  

(32)

It is seen from Eqs. (4) and (23) that the condition (a) in Eqs. (32) has been maintained. Since at point B

\[ \begin{align*}
\eta_{1R}^{(1)} &= 1, \quad \eta_{1L}^{(1)} = \eta_{2R}^{(1)} = \eta_{2L}^{(1)} = \eta_{ov}^{(1)} \\
\gamma_{1R}^{(1)} = 0, \quad \gamma_{1L}^{(1)} = 0, \quad \gamma_{2R}^{(1)} = 0, \quad \gamma_{2L}^{(1)} = 0, \quad \gamma_{ov}^{(1)} = 0 \\
\end{align*} \]  

(33)

by substituting Eqs. (12) and the relations for the toroidal shells and the cylindrical shell into Eqs. (32), the constants \( C_i^{(1)} (i = 1, 2, 3, 4) \) become as follows.

\[ \begin{align*}
C_i^{(1)} &= -K_i, \quad C_i^{(1)} = K_i, \quad C_i^{(1)} = -\omega (K_i + K_2) \\
C_i^{(1)} &= \lambda^{(1)} \left[ \nu A^{(1)} - P^{(1)} \right] [(1 - 2\nu) + (1 - \nu) \lambda^{(1)}] \\
&\quad - \frac{m}{h} \left( K_i + K_2 + \frac{pd}{2\gamma} \right) \\
\end{align*} \]  

(34)

where

\[ \omega = \gamma_{1R} (1 + \gamma_{1R}^{(1)}) = \sqrt{\frac{1}{2} - 1 - \xi} \sqrt{\frac{\lambda^{(1)}}{\lambda^{(1)}} - \lambda^{(1)}} - \lambda^{(1)} \gamma_{1R}^{(1)} \]  

From Eqs. (34), Eqs. (11) and (12) become

\[ \begin{align*}
\eta_{1R}^{(1)} &= X_1 \eta_{2R}^{(1)} + X_2 \eta_{2L}^{(1)} + \eta_{ov}^{(1)} \\
\gamma_{1R}^{(1)} &= X_1 \gamma_{2R}^{(1)} - X_2 \gamma_{2L}^{(1)} - \gamma_{ov}^{(1)} \\
\eta_{1L}^{(1)} &= X_1 \eta_{2L}^{(1)} + \omega (\eta_{2R}^{(1)} - \eta_{2L}^{(1)}) \\
\gamma_{1L}^{(1)} &= X_1 \gamma_{2L}^{(1)} + \omega (\gamma_{2R}^{(1)} - \gamma_{2L}^{(1)}) \\
\gamma_{ov}^{(1)} &= -N_{ov}^{(1)} - q_{ov}^{(1)} \\
\end{align*} \]  

(35)

where

\[ \begin{align*}
X_1 &= C_0^{(1)} / A, \quad X_2 = C_4^{(1)} / A, \quad X_3 = K_1 / A, \quad X_4 = K_2 / A \\
N &= \lambda^{(1)} \left[ \nu q + \frac{P^{(1)}}{A} \left( (1 - 2\nu) + (1 - \nu) \lambda^{(1)} \right) \right] \\
&\quad + \frac{m}{h} \frac{pd}{EA} \\
\end{align*} \]  

(36)
Next the conditions of continuity between the toroidal shells (shells I and II) and the annular plate (shell III) become as follows. Namely at points C (ψ=0 in shell I, r=a in shell III) and D (ψ=0 in shell II, r=b in shell III),

\[
\begin{align*}
(a) & \quad (N_0)_{r=a} = (N_0)_{\psi=0, r=a}, \\
(b) & \quad (u^{(0)}, v^{(0)})_{r=a} = (u^{(0)})_{\psi=0}, \\
(c) & \quad (M_0)_{r=a} = (M_0)_{\psi=0, r=a}, \\
(d) & \quad (Q^{(0)}, u^{(0)})_{r=a} = (Q^{(0)})_{\psi=0}, \\
(e) & \quad (\partial u^{(0)}/\partial r)_{r=a} = (\partial u^{(0)}/\partial \psi)_{\psi=0}.
\end{align*}
\]
\[
\frac{\phi^{(1)}}{A} = Y_1 \gamma_{2n}^{(1)} + Y_3 \gamma_{2m}^{(1)} + \eta_{4f}^{(1)}, \quad \frac{X_3^{(1)}}{A} = Y_1 \gamma_{2f}^{(1)} - Y_2 \gamma_{2m}^{(1)} - \eta_{6h}^{(1)}
\]
\[
\frac{\phi^{(1)}}{A} = Y_3 \gamma_{1f}^{(1)} + Y_3 \gamma_{2m}^{(1)} - T \gamma_{2h}^{(1)} - q \gamma_{4f}^{(1)}
\]
\[
\frac{X_3^{(1)}}{A} = -Y_3 \gamma_{1h}^{(1)} + Y_3 \gamma_{2f}^{(1)} + T \gamma_{2f}^{(1)} + q \gamma_{6h}^{(1)}
\]

where

\[Y_1 = C_4^{(1)}/A, \quad Y_1 = C_4^{(1)}/A, \quad Y_3 = C_4^{(1)}/A, \quad Y_4 = C_4^{(1)}/A\]

\[T = \lambda^{(1)} \left[ 1 + \frac{P_0}{A} \right] \left[ (1-2\nu) + (1-\nu) \lambda^{(1)} \right] \]

For deciding the remaining unknown constants \(Y_i (i=1 \sim 4)\), \(A_0\), \(B_0\) and \(d_i (i=1 \sim 3)\), by substituting the relations for the toroidal shells and the annular plate into Eqs. (36) and transforming in a similar way to the case of the type A, the next simultaneous equations with four unknowns are obtained.

\[\sum_{j=1}^{4} b_{ij} Y_j = b_i (i=1 \sim 4)\]

where

\[b_{11} = a_{11}, \quad b_{12} = a_{12}, \quad b_{13} = -\alpha (\gamma_{1h}^{(1)})_0 + K (\gamma_{2m}^{(1)})_0, \quad b_{14} = \alpha (\gamma_{2f}^{(1)})_0 - K (\gamma_{2f}^{(1)})_0\]

\[b_1 = a_1 + (N-T) (\alpha (\gamma_{1h}^{(1)})_0 - K (\gamma_{2m}^{(1)})_0), \quad b_{21} = a_{21}, \quad b_{22} = a_{22}, \quad b_{23} = -C (\gamma_{1h}^{(1)})_0 - F (\gamma_{1h}^{(1)})_0\]

\[b_{24} = C (\gamma_{2f}^{(1)})_0 + F (\gamma_{2f}^{(1)})_0, \quad b_2 = a_2 + (N-T) (C (\gamma_{1h}^{(1)})_0 + F (\gamma_{1h}^{(1)})_0)\]

\[b_{31} = a_{31}, \quad b_{32} = a_{32}, \quad b_{33} = M (\gamma_{1h}^{(1)})_0 + I (\gamma_{1h}^{(1)})_0, \quad b_{34} = M (\gamma_{2f}^{(1)})_0 + I (\gamma_{2f}^{(1)})_0\]

\[b_{41} = -b_3 - (N-T) (M (\gamma_{1h}^{(1)})_0 + I (\gamma_{1h}^{(1)})_0), \quad b_{42} = a_{42}, \quad b_{43} = -\alpha (\gamma_{1h}^{(1)})_0 + K (\gamma_{1h}^{(1)})_0\]

\[b_{44} = -\alpha (\gamma_{2m}^{(1)})_0 + K (\gamma_{2m}^{(1)})_0, \quad b_4 = a_4 + (N-T) (\alpha (\gamma_{2f}^{(1)})_0 - K (\gamma_{2f}^{(1)})_0)\]

Fig. 3 Examples of stress distributions (Type A)
Solving the above Eqs. (41), the values of $Y_i$ for each parameter are known.

2-5 Stresses and axial displacement

By substituting the values of the constants decided in the previous article into Eqs. (3) ~ (7), Eqs. (15) ~ (19) and Eqs. (22) ~ (26), the stresses in the toroidal shell, the annular plate and the cylindrical shell may be obtained respectively. In these equations $\mathcal{G}$ and $\mathcal{Z}_v$ take different values in the type A or the type B, and Eqs. (35) or (40) should be employed for the type A or the type B respectively.

Next putting the axial displacements of the toroidal shells I, II and the annular plate as $\delta_1$, $\delta_2$ and $\delta_3$ respectively, we have Eq. (42) for the total displacement of the expansion joints.

$$\delta = 2(\delta_1 + \delta_2 + \delta_3)$$

Equation (42) is transformed to next equation.

$$\frac{\delta E}{E} = \frac{m(2-\lambda_2) (2\pi + \lambda_2)}{\lambda_1 \lambda_2 \lambda^2} \left( \frac{\delta_1}{A_{R_1}} + \frac{\delta_2}{A_{R_2}} + \frac{\delta_3}{A_{R_3}} \right)$$

where

$$\frac{\delta_1}{A_{R_1}} = \int_0^{\pi/2} \frac{G(\lambda)}{A} \sin \psi \, d\psi$$
$$\frac{\delta_2}{A_{R_2}} = \int_0^{\pi/2} \frac{G(\lambda)}{A} \sin \psi \, d\psi$$
$$\frac{\delta_3}{A_{R_3}} = \int_0^{\pi/2} \frac{G(\lambda)}{A} \sin \psi \, d\psi$$

3. Numerical examples

Several examples of the distributions of stresses calculated from the expressions in Chapter 2 are shown in Figs. 3 and 4. From these figures the following facts are recognized:

1) the meridional membrane stress $\sigma_{mv}$ is very small compared with the others and vanishes at point B,

2) the circumferential membrane stress $\sigma_{mv}$ is fairly large in shells I and II,

3) the shapes of the distributions of the bending stresses $\sigma_{br}$ and $\sigma_{br}$ are similar and at point A the ratio of $\sigma_{br}$ to $\sigma_{br}$ is 1:2, v,

4) $\sigma_{br}$ is the largest of four stresses and the positions of the maximum values of $\sigma_{br}$ deviate from points A

Fig. 4 Examples of stress distributions (Type B)
and B as the value of \( \lambda_1 \) becomes small. In the type A \( \sigma_{y} \) and \( \sigma_{x} \) in the cylinder rapidly reduce to the states of the stresses in a long cylinder under an inner pressure as the distance from point B increases. In the type B \( \sigma_{y} \) becomes fairly large at point B as \( \lambda_1 \) becomes small.

Table 1 shows examples of the axial displacement in each part of expansion joints calculated from the relations in Chapter 2. It is found from Table 1 that \( \delta \) is large compared with the others and this is more striking as the value of \( \xi \) becomes larger, and that the axial displacements of the type A are larger than ones of the type B connected to rigid body.

4. Comparison with experimental results

Finally we compare theoretical results with

<table>
<thead>
<tr>
<th>Type</th>
<th>2E(\delta_{y}/rP )</th>
<th>2E(\delta_{x}/rP )</th>
<th>2E(\lambda_{y}/rP )</th>
<th>E(\delta_{y}/rP )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a)</td>
<td>2.681 x 10^4</td>
<td>1.536 x 10^4</td>
<td>3.644 x 10^4</td>
<td>7.861 x 10^4</td>
</tr>
<tr>
<td>(b)</td>
<td>1.546 x 10^4</td>
<td>1.406 x 10^4</td>
<td>2.110 x 10^4</td>
<td>5.062 x 10^4</td>
</tr>
<tr>
<td>(c)</td>
<td>7.371 x 10^4</td>
<td>3.460 x 10^4</td>
<td>3.217 x 10^4</td>
<td>4.330 x 10^4</td>
</tr>
<tr>
<td>(d)</td>
<td>3.917 x 10^4</td>
<td>2.990 x 10^4</td>
<td>1.620 x 10^4</td>
<td>2.310 x 10^4</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a)</td>
<td>2.307 x 10^4</td>
<td>6.039 x 10^4</td>
<td>2.913 x 10^4</td>
<td>5.824 x 10^4</td>
</tr>
<tr>
<td>(b)</td>
<td>1.192 x 10^4</td>
<td>5.415 x 10^4</td>
<td>1.465 x 10^4</td>
<td>3.207 x 10^4</td>
</tr>
<tr>
<td>(c)</td>
<td>6.923 x 10^4</td>
<td>1.642 x 10^4</td>
<td>2.910 x 10^4</td>
<td>3.767 x 10^4</td>
</tr>
<tr>
<td>(d)</td>
<td>3.558 x 10^4</td>
<td>1.548 x 10^4</td>
<td>1.378 x 10^4</td>
<td>1.889 x 10^4</td>
</tr>
</tbody>
</table>

(a), (b) etc. in this column correspond to (a), (b) etc. in Figs. 3 and 4.

Fig. 5 One example of test pieces

Fig. 6 Schematic view of experiments

Fig. 7 Comparison between theoretical and experimental values
experimental ones. We make experiments on three kinds of expansion joints, one of which is shown in Fig. 5. Figure 6 shows the schematic view of experiments. We fit both flanges of the expansion joint with the end plates as shown in Figs. 5 and 6, and load the internal pressure with an air compressor. We measured the values of the strains of the expansion joints by sticking four sheets of strain gauge (foil gauge with 1 mm gauge length) in $\varphi$ and $\theta$ directions on the circle through point A, and measured the values of the axial displacement with two dial gauges fixed symmetrically with respect to the center axis of expansion joints.

Figure 7 shows a comparison between the theoretical and the experimental values in the relation between pressure load and external surface strains at point A and in the relation between pressure load and axial displacement. In this comparison we take account of the thickness of strain gauges. Though the theoretical values are a little different from the experimental values in part, such a difference is unavoidable, because it is very difficult to have an ideal geometry of the test pieces of shells.

5. Conclusions

In this paper we described the method to calculate with enough accuracy the stresses and the axial displacement in expansion joints of pressure vessels under the internal pressure by applying the elastic theories of plates and shells. And we confirmed the validity of the theory by comparing with experimental results. Though we made the design charts showing the maximum stresses and the axial displacement for a wide range of geometric parameters, we omitted them for want of space and limited ourselves to showing the numerical results for some dimensions of expansion joints. The computer used in the calculations is HIPAC 103 and the computing time for a set of parameters is about two minutes. By combining the solutions of this investigation with the solutions of axial loading shown in the previous paper$^{(1)}$, the general solutions for the case of both axial and pressure loads may be obtained.

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References