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On a Measurement of Deflection of Various Shaped Plates and Its Application to Thermal Stress Problems*

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The deflection of various shaped plates for which it is difficult to obtain the analytic solution is measured by an optical interference method. Several examples of application of the above analogy to thermal stress problems are illustrated, utilizing the analogy between the problems of the bending stresses in clamped plates and those of plane thermoelasticity.

1. Introduction

It is well known that there is an analogy between the problems of plane thermoelasticity with non-traction boundary and those of the deflection of bending in clamped plates⁽¹⁾. This paper is a part of our studies on plane thermoelastic problems with heat generation^{(2)~(10)}. The study of the finding of the deflection of the clamped plate in this paper arose in connection with the analysis of the plane thermoelasticity due to heat generation. Unfortunately, it is difficult to obtain the analytic solution of bending of arbitrary shaped plates. Therefore it is worth while to measure the deflection of various shaped plates with not only clamped boundary condition but also another boundary condition by experimental method. Using an optical interference method, we present results of measuring the deflection of various shaped plates. This measurement procedure provides a convenient and accurate method to make the direct visual observation in full field of the deflection in the arbitrary shaped plates, and it is easy to count the fringes because of very clear fringes pattern. The experimental results show that the method has satisfactory accuracy in determining the deflection of arbitrary shaped plates.

2. Analogy

For a simply connected region with non-traction boundary, thermal stress function χ and

thermal stress component $\sigma_t^\dagger$ (suffix B means boundary) must satisfy the following equations respectively

$$\Delta\Delta\chi = -k\Delta\tau \quad \dots\dots\dots(1)$$

$$\sigma_t = \partial^2\chi/\partial n^2 \quad \dots\dots\dots(2)$$

$$(\chi)_B = 0, \quad (\partial\chi/\partial n)_B = 0 \quad \dots\dots\dots(3)$$

where

τ : temperature

k : $\alpha_t E$ for plane stress, $\alpha_t E/(1-\nu)$ for plane strain problem

α_t : coefficient of thermal expansion

E : Young's modulus

ν : Poisson's ratio

n : coordinate directed normal to edge

$$\Delta : \partial^2/\partial x^2 + \partial^2/\partial y^2$$

The deflection W and bending moment M_n must satisfy the following equations respectively

$$\Delta\Delta W = p/D \quad \dots\dots\dots(4)$$

$$M_n = -D\partial^2 W/\partial n^2 \quad \dots\dots\dots(5)$$

$$(W)_B = 0, \quad (\partial W/\partial n)_B = 0 \quad \dots\dots\dots(6)$$

where

p : analogous plate pressure

D : flexural rigidity of the plate

Therefore the analogy between χ and W is shown as

$$\chi = KW \quad \dots\dots\dots(7)$$

where K is an analogous constant.

The heat conduction equation with heat generation q is

$$\Delta\tau = -q/\lambda_t \quad \dots\dots\dots(8)$$

where λ_t is thermal conductivity. Substituting Eq. (8) into Eq. (1), we get

$$\Delta\Delta\chi = kq/\lambda_t \quad \dots\dots\dots(9)$$

Comparing Eq. (4) with Eq. (9), we obtain

$$\chi = (kqD/p\lambda_t)W \quad \dots\dots\dots(10)$$

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† σ_t is hoop stress component.

$$K = kqD/p\lambda_i \dots\dots\dots(11)$$

Therefore, the thermal stress function χ can be determined by deflection W of plate.

3. Measurement

3.1 Testing apparatus

Using the remodeling of photo-elastic test apparatus as shown in Fig. 1, we have constructed a device for measuring the bending of various shaped plates by an optical interference method. Its essential components include: (A) a source of monochromatic light, such as sodium with wave-length $\lambda=5893\text{\AA}$; (B) starter; (C) condensing lens; (D) diaphragm; (E) convex lens; (F) partial reflector of sufficient size to cover the specimen; (G) quartz optical flat (60 ϕ mm) of sufficient size to cover the specimen; and (I) camera. We used a universal projector to measure the optical interference patterns. Three

Fig. 1 Test-equipment arrangement for optical interference method

Fig. 2 Arrangement of loading setup for concentrated load

types of arrangements of loading setup were used. For a concentrated load one of them is a Maruto ring apparatus as shown in Fig. 2, and the other is a lever type with a spindle on which a steel sphere (3 ϕ mm) is set. For uniformly distributed load, a pressure apparatus as shown in Fig. 3 is used.

3.2 Specimens

Various shaped test specimens (Fig. 4) were made of mild steel ($E=2.1 \times 10^6 \text{ kg/cm}^2$, $\nu=0.3$) and the dimensions of one of them are shown in Fig. 5. Considering the clamped boundary condition, specimens were cut into circular cylinders two or three times larger than the part of measurement. Initial flat surface of specimen to be studied must be polished. It is desirable that the surface of specimens is perfectly flat, but a slight deviation might be allowed because it would easily be eliminated

1: Optical flat	2: Guide ring
3: Specimen	4: Fixed setup of specimen
5: Pressure gage	

Fig. 3 Arrangement of loading setup for uniformly distributed load

Fig. 4 Specimen

in the process of data arrangements. The specimens are polished within a few fringes pattern as initial deviation of surface in our experiment. Then, maximum dimension of initial deviation is within 1 micron.

3.3 Principle of measurement

The fringes as shown in Fig. 6 are obtained by interference between light reflected from surface of specimen and light from an optical flat surface situated a few microns away from it. The phase relationships in the light reflected from the bottom surface of optical flat and

(a) One body

(b) Bonded

Fig. 5 Dimension of specimen

Fig. 6 Formation of the interference pattern between the optical flat and the specimen

polished surface of specimen are such that interference occurs where the total optical path lengths differ by half-length of the wave of light. Since the beam is reflected in this case

Fig. 7 Method of obtaining a magnified profile from the photograph of the interference pattern

(a) Uniformly distributed load
 $p = 0.49 \text{ kg/cm}^2$

(b) $p = 0.12 \text{ kg/cm}^2$

Side length $b = 35 \text{ mm}$, Diameter of hole
 $2r_i = 17.5 \text{ mm}$, Thickness $t = 0.88$

Fig. 8 Clamped square plate with circular hole having free edge under uniformly distributed load

the fringes would occur where two surfaces are separated by

$$d = \frac{(2n-1)\lambda}{4} \dots\dots\dots(12)$$

d : normal distance between two surfaces

n : integer

The method of deriving the profile of the polished surface from the photograph of interference pattern in Fig. 6 is illustrated in Fig. 7. Then the angle between the specimen and the

Fig. 9 Deflection of clamped square plate with circular hole having free edge under uniformly distributed load ($p=0.49 \text{ kg/cm}^2$)

Diameter of specimen $2r_o=40 \text{ mm}$, $2r_i=11 \text{ mm}$,
Eccentricity $e=10.5 \text{ mm}$, $t=1.15 \text{ mm}$

Fig. 10 Clamped circular plate with eccentric circular hole having free edge under a concentrated load ($P=1.14 \text{ kg}$) at the center

optical flat has no effect on the deflection of specimen.

3.4 Result

In our previous paper⁽¹¹⁾, we experimented on various simple shaped specimens, i.e., circular, equilateral triangular, isosceles right triangular, semicircular and circular ones with concentric circular hole, and found that the experimental results almost coincided with the theoretical results. In this paper, various shaped specimens, of which it is difficult to obtain the theoretical solution, are measured in the following cases. (A) Clamped square plate with a free boundary hole under uniformly distributed load (Fig. 8 and Fig. 9). (B) Clamped circular plate with an eccentric circular hole under a concentrated load at the center or eccentric point. The inner boundary is free edge (Fig. 10 and Fig. 11). (C) Clamped equilateral hexagon plate under a concentrated load at the center or uniformly distributed load (Fig. 12 and Fig. 13). (D) Clamped equilateral trapezium plate under a concentrated load at the center or uniformly

$2r_o=40 \text{ mm}$, $2r_i=11 \text{ mm}$, $e=10.5 \text{ mm}$
Eccentricity of load $e_p=10 \text{ mm}$, $t=1.15 \text{ mm}$

Fig. 11 Clamped circular plate with an eccentric circular hole having free edge under an eccentric concentrated load; ($P=1.45 \text{ kg}$)

$b=25.8 \text{ mm}$, $t=0.658 \text{ mm}$

Fig. 12 Clamped equilateral hexagon plate under a concentrated load at the center. Deflection on the diagonal line

$b=25.08$, $t=0.658 \text{ mm}$

Fig. 13 Clamped equilateral hexagon plate under uniformly distributed load ($p=0.1 \text{ kg/cm}^2$). Deflection on the diagonal line

distributed load (Fig. 14 and Fig. 15). (E) Clamped parallelogram plate under uniformly distributed load (Fig. 16). In each figure, the unit of deflection W is micron [μ].

4. Application

We apply the measured result of deflection of plate to a thermal stress problem. The

Upper side length $b_u=30.35 \text{ mm}$, Lower side length $b_l=44.53 \text{ mm}$, Height $h=29.85 \text{ mm}$, $t=0.638 \text{ mm}$

Fig. 14 Clamped equilateral trapezium plate under a concentrated load ($P=0.326 \text{ kg}$) at the center

$b_u=30.35 \text{ mm}$, $b_l=44.53 \text{ mm}$, $h=29.85 \text{ mm}$, $t=0.638 \text{ mm}$

Fig. 15 Clamped equilateral trapezium plate under uniformly distributed load ($p=0.14 \text{ kg/cm}^2$)

analogical analysis of a steady state axially symmetric thermal stress with constant heat generation q leads from Eqs. (2) and (11) to the following equations

$$\left. \begin{aligned} \sigma_{rr} &= \frac{Dkq}{\lambda_i p} \frac{1}{r} \frac{\partial W}{\partial r} \\ \sigma_{\theta\theta} &= \frac{Dkq}{\lambda_i p} \frac{\partial^2 W}{\partial r^2} \end{aligned} \right\} \dots\dots\dots(13)$$

For a simply connected region, thermal stress components are given by numerical differentiation of deflection of an analogous clamped plate utilizing the above results and analogy between Eqs. (1), (2) and Eqs. (4), (5). In this paper, thermal stress components are calculated using the following numerical differential formulas given by interpolation formula derived from deflection curves obtained by fringe patterns.

(a) Numerical differential formula by equal intervals on the center of standard point is

$$\left(\frac{\partial y}{\partial x}\right)_{x=0} = \frac{1}{s} \left\{ \frac{2}{3}(y_1 - y_{-1}) - \frac{1}{12}(y_2 - y_{-2}) \right\} \dots\dots\dots(14)$$

where y_1 and y_2 are deflections at the distances s and $2s$ to right hand from the center point ($x=0$) respectively and y_{-1} and y_{-2} to left hand. (b) Numerical differential formula by equal intervals on the end of standard point is

$$\left(\frac{\partial y}{\partial x}\right)_{x=0} = \frac{1}{s} \left(-\frac{25}{12}y_0 + 4y_1 - 3y_2 + \frac{4}{3}y_3 - \frac{1}{4}y_4 \right) \dots\dots\dots(15)$$

(c) Numerical differential formula by unequal intervals ξs (where $\xi \leq 1$) on the end of standard point is

$$\left(\frac{\partial y}{\partial x}\right)_{x=0} = \frac{1}{s} \left\{ -\frac{1+2\xi}{\xi(1+\xi)}y_0 + \frac{1+\xi}{\xi}y_1 - \frac{\xi}{1+\xi}y_2 \right\} \dots\dots\dots(16)$$

Applying the above formulas to Fig. 17, thermal stress components with a point heat generation at the center are shown in Fig. 18. For thermal stress problems, a doubly connected region with a hole must be considered the secondary edge boundary condition in addition to primary edge boundary condition. As in our previous paper⁽³⁾, for the secondary edge of boundary, thermal stress function should have to take the form of

$$(\chi)_B = C_{11}x + C_{21}y + C_{31} \dots\dots\dots(17)$$

Now, in the hollow circular cylinder the temperature distribution which has to satisfy the condition of $C_{31}=0^*$, corresponds to the condition $(w)_B = (\partial w / \partial n)_B = 0$, in the bending problems. For illustrative purpose, we consider the thermal stress problem of a hollow circular plate under

Long side length=39.50 mm, Short side length=30.375 mm, $t=0.646$ mm, Long diagonal length=54.19 mm, Short diagonal length=43.78 mm

Fig. 16 Clamped parallelogram plate under uniformly distributed load ($p=0.09$ kg/cm²).

$2r_0=45$ mm, $t=0.869$ mm

Fig. 17 Clamped circular plate under concentrated load ($P=1.5$ kg) at the center

Fig. 18 Thermal stress distribution of circular plate under a point heat generation at the center

* In authors' paper⁽³⁾, for an axially symmetric problem we get $C_{11}=C_{21}=0$. The condition of $C_{31}=0$ gives $-(4D_{00}+KB_{00})/4D_{03}=0$.

From above consideration, it follows that

$$T_i = \frac{q}{4\lambda_i} (b^2 - a^2) \left\{ 1 - \frac{\ln(b/a)}{2} \frac{(b^2 - a^2)(b^2 + a^2) - 4a^2b^2 \ln(b/a)}{(b^2 - a^2)^2 - 4a^2b^2 [\ln(b/a)]^2} \right\}$$

where T_i =inner temperature, q =constant heat generation, a =inner radius of circular plate, b =outer radius of circular plate, and outer temperature is zero.

constant heat generation. In this case, observing the optical interference pattern of a hollow circular plate with clamped boundary edges under uniformly distributed load as shown in Fig. 19, we find the deflection in Fig. 20 and the thermal stress distribution in Fig. 21. It can

$2r_o = 39.4 \text{ mm}$, $2r_i = 10.7 \text{ mm}$, $t = 0.83 \text{ mm}$

Fig. 19 Clamped circular plate with a concentric circular hole having fixed edge under uniformly distributed load ($p = 1 \text{ kg/cm}^2$)

Fig. 20 Deflection of clamped circular plate with a concentric circular hole having fixed edge under uniformly distributed load ($p = 1 \text{ kg/cm}^2$)

Fig. 21 Thermal stress distribution of concentric circular plate with a concentric circular hole under uniformly distributed heat generation

be seen that thermal stress distributions of experimental method coincide with theoretical results for both a simply connected region and a doubly connected one.

5. Conclusions

(1) In our previous paper⁽¹¹⁾, we found that the optical interference technique as measurement method of small deflection of plates gave accurate results comparing with the theoretical results in simple shaped plates for which theoretical results can be easily obtained. Then, in this paper the small deflections of various shaped plates for which it is difficult to obtain the theoretical results, are measured under a concentrated point load and a uniformly distributed load. Application to thermal stress problems is shown briefly.

(2) Errors of this technique are considered: (A) flatness of optical flat, (B) spherical aberration of optical lens, (C) personal error of measurement of fringe, (D) non-uniformity of thickness of plate. However, errors are found negligible from the comparison between the theoretical and experimental results in our previous paper⁽¹¹⁾.

(3) As for application to thermal stress problems, this method gives inferior results in accuracy. Because it is necessary to differentiate numerically the experimental values twice. However, it is seen that thermal stress components given by this technique coincide broadly with theoretical results.

(4) It is not essential for the surface to be perfectly flat, because the true deflection is given by the difference between measured deflection and initial deviation of specimen. However, it is desirable that the surface be perfectly flat and that the angle between the optical flat and the specimen surface be zero, because only in this case equi-interference patterns coincide with equi-deflection pattern. As it is difficult to polish the surface of specimens to perfect flatness, we polish the surface of specimen so as to produce a few fringes pattern for initial deviation of surface.

(5) It is seen that this primitive procedure provides a convenient and accurate visual method for direct full field observation of deflection in an arbitrary shaped plate for which it is difficult to obtain the theoretical solution.

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