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Forced Vibrations of a Rotor with Rotating Inequality*

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Lateral vibration of a rotating shaft system with a rotating anisotropy is discussed in the present paper. Let the rotating speed of the shaft, the angular velocity of the anisotropy, the natural frequency of the system and the frequency of an external periodic force acting on the system be ω , $\lambda\omega$, p and ω_0 respectively. A resonant phenomenon arises when $\omega_0 \doteq 2\lambda\omega - p$ as well as when $\omega_0 \doteq p$ in such a vibratory system. It can be concluded that forced vibration with the frequency ω_0 or $\omega'_0 (= 2\lambda\omega - \omega_0)$ arises when $\omega_0 \doteq p$ or $\omega_0 = \bar{p} (= 2\lambda\omega - p)$. Furthermore, a certain vibration the cause of which is relatively hard to understand can be defined physically by referring to the above conclusion.

1. Introduction

A rotating shaft system with an unsymmetrical rotor, or one with unsymmetrical shaft stiffness may be regarded as a vibratory shaft system having rotating inequality which rotates with the rotor, i.e., rotates at an angular velocity of the shaft of ω . In the present paper, a more general shaft system which has rotating anisotropy at a rotating velocity of $\lambda\omega$ is treated, where λ is a certain constant, and the lateral vibration appearing in such a shaft system is studied.

Since free vibration with a natural frequency of p appears in free vibration with a frequency of $\bar{p} (= 2\lambda\omega - p)$ in a shaft system having rotating inequality⁽¹⁾⁽⁵⁾, one external periodic force with a frequency of ω_0 brings about two forced vibrations with frequencies of ω_0 and $\omega'_0 (= 2\lambda\omega - \omega_0)$, and hence resonant phenomena are observed under the conditions of $\omega_0 \doteq p$ and $\omega_0 \doteq \bar{p}$ ⁽⁷⁾. Analytical results show that, in the state of resonance at $\omega_0 \doteq \bar{p}$, the predominant vibration is not the forced vibration with a frequency of ω_0 but is always the forced vibration with a frequency of $\omega'_0 (= 2\lambda\omega - \omega_0)$ which is apparently quite different from the frequency ω_0 of the external force, provided the rotating inequality is relatively small.

By using the above conclusion, definite physical meanings can be given to the whirling of synchronous backward precession appearing

in a rotating shaft supported by flexible bearing pedestals, the whirling of forward precession at the so-called secondary critical speed which takes place in a horizontal flat shaft and the whirling motion with a peculiar mode of backward precession caused by a small difference in diameter between balls in ball bearings.

2. Equation of motion and amplitude ratio of forced vibration

When there exists anisotropy with respect to shaft stiffness rotating at an arbitrary angular velocity of $\lambda\omega$ in a vibratory system consisting of a rotor and a light shaft, the system is governed by the following equations of motion⁽⁵⁾:

$$\left. \begin{aligned} m\ddot{x} + \alpha x + \gamma\theta_x - \Delta\alpha(x \cos 2\lambda\omega t + y \sin 2\lambda\omega t) \\ - \Delta\gamma(\theta_x \cos 2\lambda\omega t + \theta_y \sin 2\lambda\omega t) \\ = P \cos(\omega_0 t + \beta_1) \\ m\ddot{y} + \alpha y + \gamma\theta_y - \Delta\alpha(x \sin 2\lambda\omega t - y \cos 2\lambda\omega t) \\ - \Delta\gamma(\theta_x \sin 2\lambda\omega t - \theta_y \cos 2\lambda\omega t) \\ = P \sin(\omega_0 t + \beta_1) \\ I\ddot{\theta}_x + I_p\omega\dot{\theta}_y + \gamma x + \delta\theta_x - \Delta\gamma(x \cos 2\lambda\omega t \\ + y \sin 2\lambda\omega t) - \Delta\delta(\theta_x \cos 2\lambda\omega t \\ + \theta_y \sin 2\lambda\omega t) = M \cos(\omega_0 t + \beta_2) \\ I\ddot{\theta}_y - I_p\omega\dot{\theta}_x + \gamma y + \delta\theta_y - \Delta\gamma(x \sin 2\lambda\omega t \\ - y \cos 2\lambda\omega t) - \Delta\delta(\theta_x \sin 2\lambda\omega t \\ - \theta_y \cos 2\lambda\omega t) = M \sin(\omega_0 t + \beta_2) \end{aligned} \right\}$$

.....(1)

where m is the mass of the rotor, I_p is the polar moment of inertia of the rotor, I is the diametral moment of inertia of the rotor, x and y deflections of the rotor, θ_x and θ_y components of an inclination angle θ of the rotor in the x - and y -directions, α , γ and δ the mean values of spring constants of the shaft, $\Delta\alpha$, $\Delta\gamma$, $\Delta\delta$ inequalities in shaft stiffness, P the magnitude

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of a periodic external force, M the magnitude of a periodic external moment, ω_0 the frequency of disturbances, β_1 and β_2 phase angles, and t time. It is assumed that the anisotropies $\Delta\alpha$, $\Delta\gamma$ and $\Delta\delta$ in stiffness are rather small quantities of the order of ε , i.e.,

$$\Delta\alpha/\alpha \ll 1, \quad \Delta\gamma/\gamma \ll 1, \quad \Delta\delta/\delta \ll 1.$$

Since gyroscopic moment acts on the rotor and there are the so-called gyroscopic terms $I_p\omega\dot{\theta}_y$ and $-I_p\omega\dot{\theta}_x$ in Eq. (1), rectilinear vibration can not take place in the system and all motions of the shaft with regard to lateral vibration become whirling motions⁽⁶⁾. Accordingly, the frequency of vibration is regarded as the angular velocity of the whirling of the shaft and positive frequency represents the whirling motion of forward precession in which the direction of whirling is the same as that of rotation of the shaft, and negative frequency means a backward precession in which the whirling is in the reverse direction⁽⁶⁾.

There exist terms of the rotating anisotropies $\Delta\alpha$, $\Delta\gamma$ and $\Delta\delta$ in the equation of motion shown in Eq. (1), and hence free vibrations with natural frequencies of p and $\bar{p}(=2\lambda\omega-p)$ appear in the system, and one external periodic force with a frequency of ω_0 brings about two forced vibrations with frequencies of ω_0 and $\omega_0' (=2\lambda\omega - \omega_0)$. It follows that free and forced vibrations in the system should be expressed in the form

$$\left. \begin{aligned} x &= A \cos(pt + \beta) + \bar{A} \cos(\bar{p}t - \beta) \\ y &= A \sin(pt + \beta) + \bar{A} \sin(\bar{p}t - \beta) \\ \theta_x &= B \cos(pt + \beta') + \bar{B} \cos(\bar{p}t - \beta') \\ \theta_y &= B \sin(pt + \beta') + \bar{B} \sin(\bar{p}t - \beta') \end{aligned} \right\} \dots\dots\dots (2)$$

$$\left. \begin{aligned} x &= E_P \cos(\omega_0 t + \beta_1) + E_{P'} \cos(\omega_0' t - \beta_1) \\ &\quad + E_M \cos(\omega_0 t + \beta_2) + E_{M'} \cos(\omega_0' t - \beta_2) \\ y &= E_P \sin(\omega_0 t + \beta_1) + E_{P'} \sin(\omega_0' t - \beta_1) \\ &\quad + E_M \sin(\omega_0 t + \beta_2) + E_{M'} \sin(\omega_0' t - \beta_2) \\ \theta_x &= F_P \cos(\omega_0 t + \beta_1) + F_{P'} \cos(\omega_0' t - \beta_1) \\ &\quad + F_M \cos(\omega_0 t + \beta_2) + F_{M'} \cos(\omega_0' t - \beta_2) \\ \theta_y &= F_P \sin(\omega_0 t + \beta_1) + F_{P'} \sin(\omega_0' t - \beta_1) \\ &\quad + F_M \sin(\omega_0 t + \beta_2) + F_{M'} \sin(\omega_0' t - \beta_2) \end{aligned} \right\} \dots\dots\dots (3)$$

where β and β' are phase angles, $E_P, E_{P'}, F_P$ and

$$\left. \begin{aligned} \text{where } H &= H(\omega_0) = \alpha - m\omega_0^2, & H' &= H(\omega_0') = \alpha - m\omega_0'^2, \\ G &= G(\omega_0) = \delta + I_p\omega\omega_0 - I\omega_0^2, & G' &= G(\omega_0') = \delta + I_p\omega\omega_0' - I\omega_0'^2 \end{aligned} \right\} \dots\dots\dots (9)$$

Amplitude ratios between forced vibrations with frequencies ω_0 and $\omega_0' = 2\lambda\omega - \omega_0$ are derived from Eq. (8) in the form:

$$\left. \begin{aligned} \frac{E_{P'}}{E_P} &= \frac{GG'\Delta\alpha - \gamma'(G+G')\Delta\gamma + \gamma^2\Delta\delta - \{\Delta\alpha\Delta\delta - (\Delta\gamma)^2\}\Delta\delta}{Gf(\omega_0') - G'(\Delta\gamma)^2 - H'(\Delta\delta)^2 + 2\gamma\Delta\gamma\Delta\delta} \\ \frac{F_{P'}}{F_P} &= \frac{\gamma G'\Delta\alpha - (GH' + \gamma^2)\Delta\gamma + \gamma H'\Delta\delta - \{\Delta\alpha\Delta\delta - (\Delta\gamma)^2\}\Delta\gamma}{\gamma f(\omega_0') - G'\Delta\alpha\Delta\gamma + \gamma\Delta\alpha\Delta\delta + \gamma(\Delta\gamma)^2 - H'\Delta\gamma\Delta\delta} \\ \frac{E_{M'}}{E_M} &= \frac{\gamma G'\Delta\alpha - (HG' + \gamma^2)\Delta\gamma + \gamma H\Delta\delta - \{\Delta\alpha\Delta\delta - (\Delta\gamma)^2\}\Delta\gamma}{\gamma f(\omega_0') + \gamma(\Delta\gamma)^2 - G'\Delta\alpha\Delta\gamma + \gamma\Delta\alpha\Delta\delta - H'\Delta\gamma\Delta\delta} \\ \frac{F_{M'}}{F_M} &= \frac{\gamma^2\Delta\alpha - \gamma(H+H')\Delta\gamma + HH'\Delta\delta - \{\Delta\alpha\Delta\delta - (\Delta\gamma)^2\}\Delta\alpha}{Hf(\omega_0') - G'(\Delta\alpha)^2 - H'(\Delta\gamma)^2 + 2\gamma\Delta\alpha\Delta\gamma} \end{aligned} \right\} \dots\dots\dots (10)$$

$F_{P'}$ and $E_M, E_{M'}, F_M$ and $F_{M'}$ are the amplitudes of forced vibrations induced by the disturbances $P \cos(\omega_0 t + \beta_1)$, $P \sin(\omega_0 t + \beta_1)$, $M \cos(\omega_0 t + \beta_2)$ and $M \sin(\omega_0 t + \beta_2)$ respectively, and E_P, E_M, F_P and F_M are the amplitudes of forced vibrations with a frequency of ω_0 and $E_{P'}, E_{M'}, F_{P'}$ and $F_{M'}$ are those with a frequency of $\omega_0' = 2\lambda\omega - \omega_0$.

Substituting Eq. (2) into Eq. (1) and eliminating all amplitudes $A, \bar{A}, B, \bar{B}$, one obtains the following frequency equation:

$$\Phi(p) = f(p)f(\bar{p}) + \varphi(p) = 0 \dots\dots\dots (4)$$

where p is a natural frequency, $\bar{p} = 2\lambda\omega - p$, and

$$\left. \begin{aligned} f(p) &= (\alpha - mp^2)(\delta + I_p\omega p - Ip^2) - \gamma^2 \\ &= H(p)G(p) - \gamma^2 \\ \varphi(p) &= -G(p)G(\bar{p})(\Delta\alpha)^2 - \{G(p)H(\bar{p}) \\ &\quad + H(p)G(\bar{p}) + 2\gamma^2\}(\Delta\gamma)^2 \\ &\quad - H(p)H(\bar{p})(\Delta\delta)^2 + 2\gamma\{G(p) + G(\bar{p})\} \\ &\quad \times \Delta\alpha\Delta\gamma - 2\gamma^2\Delta\alpha\Delta\delta \\ &\quad + 2\gamma\{H(p) + H(\bar{p})\}\Delta\gamma\Delta\delta \\ &\quad + \{\Delta\alpha\Delta\delta - (\Delta\gamma)^2\} \end{aligned} \right\} \dots\dots\dots (5)$$

$$H(p) = \alpha - mp^2, \quad G(p) = \delta + I_p\omega p - Ip^2 \dots\dots\dots (6)$$

In the system without rotating anisotropy, i.e., when all inequalities $\Delta\alpha$, $\Delta\gamma$ and $\Delta\delta$ vanish, the frequency equation is reduced to

$$f(p) = (\alpha - mp^2)(\delta + I_p\omega p - Ip^2) - \gamma^2 = 0 \dots\dots\dots (7)$$

Inserting Eq. (3) in Eq. (1), the amplitudes of forced vibrations are written in the form

$$\left. \begin{aligned} E_P &= P\{Gf(\omega_0') - G'(\Delta\gamma)^2 - H'(\Delta\delta)^2 \\ &\quad + 2\gamma\Delta\gamma\Delta\delta\}/\Phi(\omega_0) \\ E_{P'} &= P[GG'\Delta\alpha - \gamma(G+G')\Delta\gamma + \gamma^2\Delta\delta \\ &\quad - \{\Delta\alpha\Delta\delta - (\Delta\gamma)^2\}\Delta\delta]/\Phi(\omega_0) \\ F_P &= P\{\gamma f(\omega_0') - G'\Delta\alpha\Delta\gamma + \gamma\Delta\alpha\Delta\delta \\ &\quad + \gamma(\Delta\gamma)^2 - H'\Delta\gamma\Delta\delta\}/\Phi(\omega_0) \\ F_{P'} &= P[\gamma G\Delta\alpha - (GH' + \gamma^2)\Delta\gamma + \gamma H'\Delta\delta \\ &\quad - \{\Delta\alpha\Delta\delta - (\Delta\gamma)^2\}\Delta\gamma]/\Phi(\omega_0) \\ E_M &= M\{\gamma f(\omega_0') + \gamma(\Delta\gamma)^2 - G'\Delta\alpha\Delta\gamma \\ &\quad + \gamma\Delta\alpha\Delta\delta - H'\Delta\gamma\Delta\delta\}/\Phi(\omega_0) \\ E_{M'} &= M[\gamma G'\Delta\alpha - (HG' + \gamma^2)\Delta\gamma + \gamma H\Delta\delta \\ &\quad - \{\Delta\alpha\Delta\delta - (\Delta\gamma)^2\}\Delta\gamma]/\Phi(\omega_0) \\ F_M &= M\{Hf(\omega_0') - G'(\Delta\alpha)^2 - H'(\Delta\gamma)^2 \\ &\quad + 2\gamma\Delta\alpha\Delta\gamma\}/\Phi(\omega_0) \\ F_{M'} &= M[\gamma^2\Delta\alpha - \gamma(H+H')\Delta\gamma + HH'\Delta\delta \\ &\quad - \{\Delta\alpha\Delta\delta - (\Delta\gamma)^2\}\Delta\alpha]/\Phi(\omega_0) \end{aligned} \right\} \dots\dots\dots (8)$$

A rotating shaft carrying an unsymmetrical rotor is also regarded as a vibratory system with the rotating anisotropy, and is governed by the following equations of motion:

$$\left. \begin{aligned} m\ddot{x} + \alpha x + \gamma \theta_x &= P \cos(\omega_0 t + \beta_1) \\ m\ddot{y} + \alpha y + \gamma \theta_y &= P \sin(\omega_0 t + \beta_1) \\ I\ddot{\theta}_x + I_p \omega \dot{\theta}_y - \Delta I(d/dt)(\dot{\theta}_x \cos 2\omega t + \dot{\theta}_y \sin 2\omega t) \\ &+ \gamma x + \delta \theta_x = M \cos(\omega_0 t + \beta_2) \\ I\ddot{\theta}_y - I_p \omega \dot{\theta}_x - \Delta I(d/dt)(\dot{\theta}_x \sin 2\omega t - \dot{\theta}_y \cos 2\omega t) \\ &+ \gamma y + \delta \theta_y = M \sin(\omega_0 t + \beta_2) \end{aligned} \right\} \dots\dots\dots (1 \cdot a)$$

where

$$I = (1/2)(I_1 + I_2), \quad I = (1/2)(I_1 - I_2),$$

and I_1 and I_2 are unequal diametral moments of inertia of the rotor, and hence I is the mean value of unequal diametral moments of inertia and ΔI represents asymmetry of the rotor.

For a vibratory shaft system with an unsymmetrical rotor, the following relations corresponding to Eqs. (5), (8), (10) are obtained in like manner⁽⁷⁾.

$$\varphi(p) = -(\Delta I)^2(\alpha - mp^2)(\alpha - m\bar{p}^2)p^2\bar{p}^2 \dots (5 \cdot a)$$

$$\left. \begin{aligned} E_P &= P \{Gf(\omega_0') - (\Delta I)^2 H' \omega_0^2 \omega_0'^2\} / \Phi(\omega_0) \\ E_{P'} &= P \Delta I \gamma^2 \omega_0 \omega_0' / \Phi(\omega_0) \\ F_P &= -P \gamma f(\omega_0') / \Phi(\omega_0) \\ F_{P'} &= -P \Delta I \gamma H' \omega_0 \omega_0' / \Phi(\omega_0) \\ E_M &= -M \gamma f(\omega_0') / \Phi(\omega_0) \\ E_{M'} &= -M \Delta I \gamma H' \omega_0 \omega_0' / \Phi(\omega_0) \\ F_M &= M H f(\omega_0') / \Phi(\omega_0) \\ F_{M'} &= M \Delta I H H' \omega_0 \omega_0' / \Phi(\omega_0) \end{aligned} \right\} (8 \cdot a)$$

$$\left. \begin{aligned} \frac{E_{P'}}{E_P} &= \frac{\Delta I \gamma^2 \omega_0 \omega_0'}{Gf(\omega_0') - (\Delta I)^2 H' \omega_0^2 \omega_0'^2} \\ \frac{E_{M'}}{E_M} &= \frac{\Delta I H \omega_0 \omega_0'}{f(\omega_0')} \\ \frac{F_{P'}}{F_P} &= \frac{\Delta I H' \omega_0 \omega_0'}{f(\omega_0')} \\ \frac{F_{M'}}{F_M} &= \frac{\Delta I H' \omega_0 \omega_0'}{f(\omega_0')} \end{aligned} \right\} \dots\dots\dots (10 \cdot a)$$

3. Characteristics of vibratory system with rotating inequality

When the frequency ω_0 of disturbances comes near one of the roots of $f(p)=0$, i.e., a natural frequency p , one obtains $f(\omega_0) \doteq 0$. Furthermore when ω_0 becomes nearly equal to $\bar{p}$, the relation $\omega_0 \doteq \bar{p} = 2\lambda\omega - p$ leads to $2\lambda\omega - \omega_0 =$

$\omega_0' \doteq p$, and hence $f(\omega_0') \doteq 0$. Since $\varphi(p)$ in Eq. (5) is a small quantity of the order of ε^2 , the relation $f(\omega_0') \doteq 0$ or $f(\omega_0) \doteq 0$ gives $\Phi(\omega_0) = f(\omega_0) \times f(\omega_0') + \varphi(\omega_0) \doteq 0$. Accordingly, it can be seen from Eq. (8) that, when $f(\omega_0') \doteq 0$ or $f(\omega_0) \doteq 0$, all amplitudes of forced vibrations in Eq. (8) take large values and the system shows a resonant phenomenon because of the small quantity of $\Phi(\omega_0)$ of the order of ε^2 .

In the state of resonance at $\omega_0 \doteq p$, $f(\omega_0')$ in denominators of Eq. (10) is not small in value and hence all amplitude ratios in Eq. (10) take small values of the order of $\Delta\alpha$ etc. i.e., of the order of ε . It follows that the forced vibration with the same frequency ω_0 as that of disturbances become far larger than those with the frequency $\omega_0' = 2\lambda\omega - \omega_0$ in the state of resonance at $\omega_0 \doteq p$. On the other hand, in the state of resonance at $\omega_0 \doteq \bar{p}$, $f(\omega_0')$ becomes nearly equal to zero and all denominators in Eq. (10) take small values of the order of ε^2 , and hence one has large amplitude ratios of the order of ε^{-1} . This leads to the following general rule which can be applied to the vibratory shaft system with anisotropy rotating at an angular velocity of $\lambda\omega$: "In the state of resonance at $\omega_0 \doteq \bar{p}$, the forced vibration with a frequency of $\omega_0' (= 2\lambda\omega - \omega_0)$ which is apparently independent of the frequency ω_0 of external periodic force arises remarkably".

The above general rule can be also applied to the rotating shaft system with an unsymmetrical rotor, to which the relations shown in Eqs. (1·a), (5·a) and (10·a) apply⁽⁷⁾.

When the source of disturbance exists within a rotating shaft system, the frequency ω_0 of the disturbance is usually proportional to the angular velocity ω of the shaft, and hence one has

$$\omega_0 = \kappa\omega \dots\dots\dots (11)$$

where κ is a certain constant. The condition of resonance at $\omega_0 = \bar{p} (= 2\lambda\omega - p)$ and Eq. (11) give the relations

$$p = (2\lambda - \kappa)\omega = \nu\omega, \quad \omega = p/\nu \dots\dots\dots (12)$$

where

$$\nu = 2\lambda - \kappa \dots\dots\dots (13)$$

Insertion of Eq. (12) in Eq. (7) leads to

$$f(\nu\omega) = (\alpha - m\nu^2\omega^2) \{\delta + \nu(I_p - \nu I)\omega^2\} - \gamma^2 = 0 \dots\dots\dots (14)$$

which can be solved analytically and gives

$$\omega^2 = p^2/\nu^2 = \frac{\{\alpha(I_p - \nu I) - \nu\delta m\} \pm \sqrt{\{\alpha(I_p - \nu I) + \nu\delta m\}^2 - 4\nu\gamma^2 m(I_p - \nu I)}}{2\nu^2 m(I_p - \nu I)} \dots\dots\dots (15)$$

The rotation speed ω given by the above equation furnishes one with the so-called "resonant point" at which the frequency ω_0 of the disturbance becomes equal to $\bar{p} (= 2\lambda\omega - p)$.

When ω_0 is proportional to ω , the general rule mentioned above can be rewritten as follows: "In the neighborhood of the resonant point given by Eq. (12) or (15), the system presents a resonant

phenomenon because the frequency ω_0 of the disturbance becomes nearly equal to $\bar{p}$, and the forced vibration with a frequency of

$$\omega_0' = 2\lambda\omega - \omega_0 = 2\lambda\omega - \kappa\omega = \nu\omega \doteq p \quad \dots\dots\dots(16)$$

which is apparently independent of ω_0 arises remarkably."

In the following several sections, definite physical meanings will be given to a certain forced vibration by use of the general rule obtained in this section.

4. Forced vibration of synchronous backward precession caused by flexible bearing pedestal⁽²⁾⁽³⁾

The authors have pointed out that forced vibration of synchronous backward precession with a frequency of $-\omega$, whose absolute value is equal to the angular velocity of a shaft, takes place when the circular shaft is supported by bearing on the pedestals which are allowed to deflect slightly⁽²⁾. In such a system, the pedestals hardly deform in the longitudinal direction (y -direction) in which they are subjected to tension or compression, while they can slightly deflect in the lateral direction (x -direction) in which bending moment acts on them. By taking the deflection of the pedestals into account, one has a slightly larger spring constant of the shaft in y -direction than one in x -direction, and hence the equations of motion of this system

can be written in the form⁽²⁾⁽⁸⁾:

$$\left. \begin{aligned} m\ddot{x} + (\alpha - \Delta\alpha)x + (\gamma - \Delta\gamma)\theta_x \\ = me\omega^2 \cos(\omega t + \beta_1) \\ m\ddot{y} + (\alpha + \Delta\alpha)y + (\gamma + \Delta\gamma)\theta_y \\ = me\omega^2 \sin(\omega t + \beta_1) \\ I\ddot{\theta}_x + I_p\omega\dot{\theta}_y + (\gamma - \Delta\gamma)x + (\delta - \Delta\delta)\theta_x \\ = (I_p - I)\tau\omega^2 \cos(\omega t + \beta_2) \\ I\ddot{\theta}_y - I_p\omega\dot{\theta}_x + (\gamma + \Delta\gamma)y + (\delta + \Delta\delta)\theta_y \\ = (I_p - I)\tau\omega^2 \sin(\omega t + \beta_2) \end{aligned} \right\} \dots\dots\dots(17)$$

In Eq. (17), the eccentricity of the rotor, i.e., static unbalance is represented by e , and τ is a small angle which the rotating axis of the rotor makes with the axis of principal moment of inertia I_p , i.e., dynamic unbalance.

Since the anisotropy in bearing pedestals does not rotate and hence the system governed by Eq. (17) can be regarded as one of vibratory shaft systems with rotating inequality which rotates at $\lambda\omega = 0$, it follows that

$$\lambda = 0 \quad \dots\dots\dots(18)$$

and

$$\bar{p} = 2\lambda\omega - p = -p \quad \dots\dots\dots(19)$$

Disturbances induced by the unbalances e and τ have a frequency of ω , as is seen from Eq. (17). Thus

$$\omega_0 = \kappa\omega = \omega, \quad \therefore \kappa = 1 \quad \dots\dots\dots(20)$$

Hence from Eqs. (18), (20)

$$\nu = 2\lambda - \kappa = 0 - 1 = -1 \quad \dots\dots\dots(13 \cdot a)$$

Substituting $\nu = -1$ into Eq. (15), one gets the resonant point

$$\frac{\omega_{b1}^2}{\omega_{b2}^2} = \frac{\{\alpha(I_p + I) + \delta m\} \pm \sqrt{\{\alpha(I_p + I) - \delta m\}^2 + 4\gamma^2 m(I_p + I)}}{2m(I_p + I)} \quad \dots\dots\dots(15 \cdot a)$$

at which $\omega_0 (= \omega)$ becomes equal to $\bar{p} (= -p)$, i.e., the relation

$$\omega = p/\nu = -p \quad \dots\dots\dots(12 \cdot a)$$

applies. Insertion of $\nu = -1$ into Eq. (16) gives the frequency

$$\omega_0' = \nu\omega = -\omega \quad \dots\dots\dots(16 \cdot a)$$

of the forced vibration which arises remarkably in the state of resonance $\omega_0 \doteq \bar{p}$.

The above discussion leads to the following conclusion: "When bearing pedestals deflect slightly, the forced vibration of synchronous backward precession having a frequency of $-\omega$ which is induced by unbalances e and τ of the rotor arises in the neighborhood of the resonant point ω_{b1} or ω_{b2} where the angular velocity ω becomes equal to $-p$."

An example of p - ω diagram obtained from Eq. (7) is illustrated in Fig. 1. As shown in Fig. 1, the resonant points ω_{b1} and ω_{b2} given by Eq. (15·a) can be obtained graphically from the intersecting points B_1 and B_2 where the straight line $p = -\omega$ intersects the curves p_3 and p_4 and the relations $\omega_0 = \omega = \bar{p}_4 = -p_4$ and $\omega_0 = \omega = \bar{p}_3 = -p_3$ come into existence.

The intersecting point C where the straight line $p = \omega$ intersects the curve p_2 gives the so-called major critical speed ω_c . In the neighborhood of ω_c , namely, in the state of resonance at $\omega_0 = \omega \doteq p$, the forced vibration of synchronous forward precession with a frequency of $\omega_0 = +\omega$ arises more markedly than that with a frequency of $\omega_0' = -\omega$.

An example of experimental results is shown in Fig. 2 where the peaks of synchronous backward and forward precessions appear in the neighborhood of ω_{b1} , ω_{b2} and ω_c in Fig. 1 respectively.

Amplitudes and amplitude ratios of forced vibrations of synchronous backward precession can be derived by putting $\lambda = 0$, $\omega_0 = \omega$, $P = me\omega^2$ and $M = (I_p - I)\tau\omega^2$ in Eqs. (8) and (10), respectively.

ω_{b1}, ω_{b2} : critical speeds of synchronous backward precession, ω_c : major critical speed
Rotor: $mg=6.927$ kg, $I=0.5796$ kg cm sec², $I_p \approx 2I$
Shaft stiffness: $\alpha=2.505 \times 10^2$ kg/cm, $\gamma=-2.605 \times 10^3$ kg/rad, $\delta=5.58 \times 10^4$ kg cm/rad

Fig. 1 $P_i-\omega$, $-P_i-\omega$ diagrams and the positions of the critical speeds of synchronous backward precession

ω_{b1}, ω_{b2} : critical speeds of synchronous backward precession, ω_c : major critical speed

Rotor: $mg=6.927$ kg, $I=0.5796$ kg cm sec², $I_p \approx 2I$
Shaft: length=507.8 mm, diameter=11.74φ
Spring constant: $\alpha=2.505 \times 10^2$ kg/cm, $\gamma=-2.605 \times 10^3$ kg/rad, $\delta=5.58 \times 10^4$ kg cm/rad
Stiffness inequalities: $\Delta\alpha=0.195 \times 10^2$ kg/cm, $\Delta\gamma=-0.115 \times 10^3$ kg/rad, $\Delta\delta=0.05 \times 10^4$ kg cm/rad

The rotor is mounted on the shaft at the location $a:b=2.56:7.44$ where a and b are distances from both shaft ends to the center of the rotor. The bearing pedestals with small flexibility in x -direction are used in this experiment. The spring constant of the pedestals at their tops is about 1.15×10^3 kg/cm. Amplitudes shown in the figure are those of motion of deflection of the shaft.

Fig. 2 Whirling of synchronous backward precession of frequency $-\omega$

5. The secondary critical speed⁽⁴⁾⁽⁹⁾⁽¹⁰⁾

It has been known that, in a horizontal shaft with small inequalities $\Delta\alpha$, $\Delta\gamma$ and $\Delta\delta$ in shaft stiffness like a horizontal flat shaft, a phenomenon called secondary critical speed caused by the gravity of the rotor shows itself in the neighborhood of $\omega=\omega_c/2$ ⁽⁹⁾⁽¹⁰⁾. It is not easy, however, to know the reason why only the forced vibration of forward precession appears at the secondary critical speed. Furthermore, it has not been explained yet why the gravity of the rotor acting in the vertical direction (y -direction) causes the whirling motion of forward precession in xy -plane at the secondary critical speed, even in a rotating shaft system in which the spring constant γ vanishes and deflection and inclination of the rotor do not couple each other and hence the deflection is not affected by the gyroscopic moment.

The use of the general rule mentioned above makes clear these questions. Since the inequalities $\Delta\alpha$, $\Delta\gamma$ and $\Delta\delta$ in shaft stiffness rotate with the shaft and hence one has

$$\lambda\omega=\omega, \quad \therefore \lambda=1 \dots \dots \dots (21)$$

Let the external forces on the right hand side of the first and second equations in Eq. (1), i.e., the external forces in x - and y -directions be P_x and P_y respectively.

$$P_x=P\cos(\omega_0 t+\beta_1), \quad P_y=P\sin(\omega_0 t+\beta_1) \dots (22)$$

By putting $P=mg$ (g =gravitational acceleration), $\beta_1=-\pi/2$ and $\omega_0=0$, one has

$$P_x=0, \quad P=-mg \dots \dots \dots (23)$$

which represents the gravity of the rotor acting vertically in y -direction. Accordingly, the gravity can be considered to be an external force with a frequency of $\omega_0=0$. Thus one has

$$\omega_0=\kappa\omega=0, \quad \therefore \kappa=0 \dots \dots \dots (24)$$

From Eqs. (21), (24), one obtains

$$\nu=2\lambda-\kappa=2-0=2 \dots \dots \dots (13 \cdot b)$$

$$\omega_0'=2\lambda\omega-\kappa\omega=\nu\omega=2\omega \dots \dots \dots (16 \cdot b)$$

The resonant point is derived from Eq. (12) in the form:

$$\omega=p/\nu=p/2 \dots \dots \dots (12 \cdot b)$$

Consequently, the phenomenon of the secondary critical speed can be explained as follows: "In the neighborhood of the resonant point $\omega=p/2$ at which the frequency $\omega_0(=0)$ of external force, i.e., the gravity becomes equal to $\bar{p}=2\lambda\omega-p=2\omega-p$, namely, in the neighborhood of the secondary critical speed, the forced vibration of forward precession having the frequency of $\omega_0'=\nu\omega=+2\omega$ takes place remarkably."

By substituting $\nu=2$ in Eqs. (14), (15), one

obtains the secondary critical speed analytically. As shown in Fig. 3, intersecting points S_1 and S_2 at which the straight line $\omega = p/2$ ($p = +2\omega$) intersects the curves p_1 and p_2 furnish graphically the secondary critical speeds ω_{s1} and ω_{s2} . At rotating speeds ω_{s1}' and ω_{s2}' determined by intersecting points S_1' and S_2' at which the straight line $p = -2\omega$ crosses curves p_3 and p_4 , the phenomenon of the secondary critical speed does not occur, because the whirling motion of backward precession with a frequency of -2ω cannot appear in the neighborhood of the secondary critical speed.

By putting $\lambda = 1$, $\omega_0 = 0$, $P = mg$ and $M = 0$ in Eqs. (8) and (10), one obtains amplitudes and amplitude ratios in the neighborhood of the secondary critical speed.

Since the relation $p = \omega_c$ applies in the rotating shaft system in which the spring constant γ becomes equal to zero, the resonant point of the secondary critical speed is given by $\omega = p/2 = \omega_c/2$.

which is a rotating speed of half the major critical speed ω_c .

It is easily seen from Eq. (10·a) that, when rotating inequality arises in an unsymmetrical rotor, the secondary critical speed does not appear because all numerators containing $\omega_0 (= 0)$ vanish.

6. The whirling of backward precession of shaft supported by ball bearing⁽¹²⁾

Balls in a ball bearing rotate around the

Fig. 3

center of the inner ring at the following angular velocity ω_1 which is determined by the condition that the rolling contact is kept between balls and inner and outer rings⁽¹¹⁾:

$$\omega_1 = \frac{D}{2(D+d)} \omega = \alpha_1 \omega, \quad (\alpha_1 < 1/2) \dots \dots \dots (25)$$

where D and d are the diameters of the inner ring and the balls respectively. It has been reported by one of the authors that a difference in diameter between balls results in the anisotropy of shaft stiffness rotating at ω_1 , and this anisotropy causes the whirling of backward precession with a frequency of $(2\alpha_1 - 1)\omega$ in cooperation with unbalances e and τ of the rotor⁽¹²⁾.

The general rule previously mentioned serves to explain this vibratory phenomenon. In this case, the angular velocity of inequality $\lambda\omega$ is given by

$$\lambda\omega = \omega_1 = \alpha_1\omega, \quad \therefore \lambda = \alpha_1 \dots \dots \dots (26)$$

hence

$$\bar{p} = 2\lambda\omega - p = 2\alpha_1\omega - p \dots \dots \dots (27)$$

The frequency ω_0 of disturbance induced by unbalances e and τ is obviously equal to the rotating speed ω of the shaft.

$$\omega_0' = \kappa\omega = \omega, \quad \therefore \kappa = 1 \dots \dots \dots (28)$$

From Eqs. (26) and (28), one has

$$\nu = 2\lambda - \kappa = 2\alpha_1 - 1, \quad (\nu < 0) \dots \dots \dots (13 \cdot c)$$

$$\omega_0' = 2\lambda\omega - \kappa\omega = \nu\omega = (2\alpha_1 - 1)\omega,$$

$$(\omega_0' < 0) \dots \dots \dots (16 \cdot c)$$

Accordingly, the resonant point is given by

$$\omega = \frac{1}{\nu} p = \frac{1}{2\alpha_1 - 1} p \dots \dots \dots (12 \cdot c)$$

As seen in Eq. (25), the value of α_1 is always smaller than $1/2$, and hence $\nu = 2\alpha_1 - 1$ and $\omega_0' = \nu\omega$ take negative values. It follows that the vibration discussed in this section is always the whirling of backward precession. In the experiments previously reported⁽¹²⁾, a self-aligning double-row ball bearing with $\alpha_1 = 1/2.65$ was used, and hence one got $\omega_0' = \nu\omega = (2\alpha_1 - 1)\omega = -(1/4.1)\omega$.

The forced vibration studied in this section can be explained as follows: "In the neighborhood of the resonant point $\omega = p/\nu = p/(2\alpha_1 - 1)$ at which the frequency $\omega_0 (= \omega)$ of disturbances induced by unbalances e and τ of the rotor becomes equal to $\bar{p} = 2\alpha_1\omega - p$, the whirling of backward precession with a frequency of $\omega_0' = (2\alpha_1 - 1)\omega (< 0)$ shows itself remarkably."

By substituting $\nu = (2\alpha_1 - 1)$ in Eqs. (14) and (15), one obtains the resonant point analytically. As shown in Fig. 3, intersecting points E_1 and E_2 at which the straight line $\omega = p/\nu$ [$p = \nu\omega = (2\alpha_1 - 1)\omega$] intersects the curves p_4 and p_3 determine

Table 1

	Angular velocity of inequality $\lambda\omega$	Frequency of disturbance $\omega_0=\kappa\omega$	ν $=2\lambda-\kappa$	Frequency of forced vibration $\omega_0'=\nu\omega$	Resonant point $\omega=p/\nu$
Synchronous backward precession	$\lambda\omega=0, \therefore \lambda=0$	$\omega_0=\kappa\omega=\omega, \therefore \kappa=1$	$\nu=0-1=-1$	$\omega_0'=-\omega$	$\omega=-p$
Secondary critical speed	$\lambda\omega=\omega, \therefore \lambda=1$	$\omega_0=\kappa\omega=0, \therefore \kappa=0$	$\nu=2-0=2$	$\omega_0'=2\omega$	$\omega=p/2$
Backward preces- sion due to ball bearing	$\lambda\omega=\alpha_1\omega, \therefore \lambda=\alpha_1$ ($\approx 1/2.65$) ($\approx 1/2.65$)	$\omega_0=\kappa\omega=\omega, \therefore \kappa=1$	$\nu=2\alpha_1-1$ ($\approx -1/4.1$)	$\omega_0'=(2\alpha_1-1)\omega$ ($\approx -1/4.1\omega$)	$\omega=p/(2\alpha_1-1)$ ($\approx -4.1p$)

p =natural frequency

the resonant points ω_{e1} and ω_{e2} .

By putting $\lambda=1$, $\omega_0=\omega$, $P=m\omega^2$ and $M=(I_p-I)\tau\omega^2$ in Eqs. (8) and (10), one obtains amplitudes and amplitude ratios of vibrations with a frequency of $\omega_0'=(2\alpha_1-1)\omega$.

7. Conclusions

Conclusions reached in the present paper may be summarized as follows:

(1) In a rotating shaft system having an anisotropy which rotates at $\lambda\omega$, the system presents resonant phenomena when the frequency ω_0 of disturbances becomes nearly equal to $\bar{p}$ ($=2\lambda\omega-p$) as well as when $\omega_0=p$.

(2) In such a system, one external force with a frequency of ω_0 causes two forced vibration with frequencies of ω_0 and $\omega_0'=2\lambda\omega-\omega_0$.

(3) In the state of resonance at $\omega_0 \doteq p$, the forced vibration with a frequency of ω_0 arises more markedly than that with frequency of ω_0' .

(4) In the state of resonance at $\omega_0 \doteq \bar{p}$, the forced vibration with a frequency of $\omega_0' = 2\lambda\omega - \omega_0$ arises more markedly than that with a frequency of ω_0 .

(5) When the frequency ω_0 of disturbance is proportional to the angular velocity ω of the shaft, the resonant point, the frequency ω_0' of

forced vibration can be determined analytically even for a general vibratory system with $\gamma \neq 0$.

(6) The use of the general rule mentioned in (4) makes it possible to give definite physical meanings to a certain vibration the cause of which is rather difficult to understand.

Dimensions of vibrations treated in the present paper are shown in Table 1.

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Discussion

A. TAMURA (Tokyo Institute of Technology):

(1) It seems that the difference in ball diameter does not cause the change of the shaft stiffness when the amplitude of the lateral vibration of the shaft is rather small. On the other hand, there would exist the anisotropy of angular clearance in ball bearing, when the amplitude becomes somewhat large. Therefore, it is considered that the appearance of anisotropy of the shaft stiffness depends on the fitting condition of the outer race of ball bearing. Then I ask you whether or not the small difference among the diameters always results in the vibration discussed in the present paper.

(2) Since both contact angles between a ball and the inner and the outer races are not identical because of contact between the ball and the cage in self-aligning double-row ball bearing, the radial load of the bearing would cause a somewhat large force to act on the cage which is constrained in its configuration by balls. Accordingly it is expected that, even if there exists no difference in ball diameter, the rotating anisotropy of the shaft stiffness with angular velocity $\alpha_1\omega$ is induced by the irregularity of shape in the cage, especially for the vertical shaft in which there is no backlash in bearing. In the shaft system supported by ball bearings

of deep grooved type or angular contact type, in which the contact angle between ball and inner ring is equal to that between ball and outer ring, it is considered that the above fact is not true.

Editorial director:

(3) In the flat shaft system, since the change of spring constant is represented by $\alpha + \Delta\alpha \sin 2\omega t$ and the equation of motion becomes of Mathieu's type of $m\ddot{x} + (\alpha + \Delta\alpha \sin 2\omega t)x = 0$, the self excited vibration can appear at $2\omega = \omega_c$. Further the gravitational force results in the inertia force

$$\frac{m^2 g e^2}{I} \sin^2 \omega t = \frac{m^2 g e^2}{2I} (1 - \cos 2\omega t)$$

in the vertical direction, and hence the system resonates at $2\omega = \omega_c$. The magnitude of the effect of the gravitational force is generally small, i.e., $(e/k)^2$, and the amplitude of the forced vibration of the flat shaft has the magnitude of the first order of the small quantity, i.e. of $\Delta\alpha/\alpha$. As the amplitude of the forced vibration of the flat shaft is a small quantity of the first order of $\Delta\alpha/\alpha$, and the magnitude of the effect of the gravitational force is generally as small as the second order of $(e/k)^2$, it can be usually considered that the effect of the flat shaft results in the critical speed at $\omega = \omega_c/2$. I wish to learn the relations between the above discussion and the analytical treatment developed in the section of secondary critical speed where the forced vibration of the shaft induced by the gravitational force is discussed (e : eccentricity, k : radius of gyration).

Authors' closure

(1) Ball bearings are grouped into two classes, one being that of ball bearing with the inequality of shaft stiffness caused by the anisotropy of angular clearance, e.g., deep grooved single-row ball bearing, and the other being that without the inequality, e.g., self-aligning double-row ball bearing. When the former is used, there is anisotropy of the shaft stiffness with angular velocity ω_1 of Eq. (25) which is induced by the difference in ball diameter, and the inequality of the shaft stiffness caused by the angular clearance of ball bearing which does not rotate. When the latter is adopted, there exists only the anisotropy of the shaft stiffness with rotating speed ω_1 , the magnitude of which depends on the difference in ball diameter and further the value of radial clearance, the degree of fit among the journal, the inner and outer rings and the bearing box. Through the authors'

experiments using a self-aligning double-row ball bearing with 10ϕ bore, it can be seen that the difference in ball diameter, apart from the question of the magnitude of amplitude, always results in the forced vibration with frequency $(2\alpha_1 - 1)\omega = -(1/4.1)\omega$. In another paper, it has been reported that this sort of vibration takes place in the spinning spindle supported by ball bearings.

(2) Though the effect of the irregularity of shape in the cage is not treated, it can be expected that the axial displacement of ball is checked mainly by the groove in the inner ring. When the groove in the inner ring does not perfectly constrain balls, it is considered that

Append.-Fig. 1

G : gravitational center e : eccentricity
 y : coordinate of G

Append.-Fig. 2

the situation becomes as discussed. In the shaft system supported by deep grooved single-row ball bearings, the circumstance above discussed does not take place.

(3) In the following authors' closure, the vibration of the deflection in the rotating shaft system, where the spring constant γ vanishes and hence the deflection and the inclination of the rotor do not couple each other, is treated.

I (i) The rotating flat shaft system can be governed by Mathieu's equation, provided that the motion of the rotor is constrained only in one direction (y -direction) as shown in Append.-Fig. 1 (which is quoted from Fig. 8.1 in Den Hartog, Mechanical vibrations, 4th Ed., 1956, p. 337). When the center of the rotor M can move freely in xy -plane, the equation of motion is represented by Eqs. (xii), (xiii) shown later, and the motions in x - and y -directions couple each other through the term of the rotating anisotropy of the shaft stiffness with angular velocity ω .

The motion in y -direction of Append.-Fig. 1 is governed by the following Mathieus' equation, as pointed out:

$$m\ddot{y} + (\alpha + \Delta\alpha \sin 2\omega t)y = 0 \quad \dots\dots\dots (i)$$

Accordingly in the neighborhoods of (a) $2\omega \doteq 2\omega_c$, i.e. $\omega \doteq \omega_c = \sqrt{\alpha/m}$, (b) $2\omega \doteq \omega_c$, i.e. $\omega \doteq 1/2 \cdot \omega_c$ and (c) $2\omega \doteq 2/3 \cdot \omega_c$, i.e. $\omega \doteq 1/3 \cdot \omega_c$, ..., rectilinear self excited vibrations of (a) $y = Ae^{n_1 t} \sin(\omega t + \beta)$, (b) $y = Ae^{n_2 t} \sin(2\omega t + \beta)$ and (c) $y = Ae^{n_3 t} \sin(3\omega t + \beta)$, ... take place respectively, where n_1 is a small quantity of order $\Delta\alpha/\alpha$, n_2 of order $(\Delta\alpha/\alpha)^2$, ... Consequently occurrence of self excited vibration appearing in the neighborhood of the secondary critical speed $\omega = 1/2 \cdot \omega_c$, the negative damping coefficient n_2 which is small as $(\Delta\alpha/\alpha)^2$, can be checked by damping force of order $(\Delta\alpha/\alpha)^2$.

(ii) When the gravitational force acts on the horizontal shaft system governed by Eq. (i), the equation of motion is given by

$$m\ddot{y} + (\alpha + \Delta\alpha \sin 2\omega t)y = -mg \quad \dots\dots\dots (ii)$$

where the positive direction of y is taken in the upper sense. Putting

$$y = Y - \frac{mg}{\alpha} = Y - \frac{g}{\omega_c^2}, \quad \dots\dots\dots (iii)$$

we have

$$m\ddot{Y} + (\alpha + \Delta\alpha \sin 2\omega t)Y = \frac{\Delta\alpha}{\alpha} mg \sin 2\omega t \quad \dots\dots\dots (iv)$$

in which there exists a term of an external periodic force of order $\Delta\alpha/\alpha$ with frequency 2ω in the right hand side. Accordingly the system resonates in the neighborhood of the secondary critical speed $\omega = 1/2 \cdot \omega_c$, and hence the rectilinear

forced vibration having a frequency 2ω takes place in the vertical direction (y -direction). The external force of order $\Delta\alpha/\alpha$ in the right hand side of Eq. (iv) is induced by the fact that a statical deflection of the flat shaft caused by the gravitational force changes its magnitude twice for each one revolution of the shaft.

(iii) The horizontal shaft system with an eccentricity e as shown in Append.-Fig. 2 will be treated. In Append.-Fig. 2, directions of maximum stiffness and eccentricity are denoted by angles φ and $\varphi + \zeta$ separately. Let the polar moment of inertia about the gravitational center be I_p , and put $I_p = mk^2$ (k : radius of gyration), and we have

$$I_p \ddot{\varphi} = -mge \cos(\varphi + \zeta) \quad \dots\dots\dots (v)$$

which leads to

$$\varphi = \omega t + \beta + \frac{mge}{I_p \omega^2} \cos(\omega t + \zeta + \beta) + \varepsilon^2(0) \quad \dots\dots\dots (vi)$$

where $\varepsilon^2(0)$ is a small term of order e^2 , β is a phase angle determined by initial conditions, ω the mean value of the rotating speed of the shaft. Putting $\beta = \pi/4$, the equation of motion is represented, in place of Eq. (ii), as follows:

$$m\ddot{y}_G + (\alpha + \Delta\alpha \sin 2\varphi)y = -mg \quad \dots\dots\dots (vii)$$

The deflection y_G of the gravitational center is furnished from Append.-Fig. 2 as follows:

$$y_G = y + e \cos(\varphi + \zeta) \quad \dots\dots\dots (viii)$$

Inserting Eqs. (vi), (viii) into Eq. (vii), and neglecting small quantities higher than the third order of e and $\Delta\alpha$, we have

$$\begin{aligned} m\ddot{y} + \{\alpha + \Delta\alpha \sin(2\omega t + \pi/2)\}y \\ + \Delta\alpha \frac{mge}{I_p \omega^2} \{\cos(3\omega t + 3\pi/4 + \zeta) \\ + \cos(\omega t + \pi/4 - \zeta)\}y = -mg \\ - \frac{m^2 g e^2}{2I_p} - \frac{m^2 g e^2}{2I_p} \cos(2\omega t + \pi/2 + 2\zeta) \quad \dots\dots\dots (ix) \end{aligned}$$

Since the gravity mg acting on the gravitational center G results in periodical change of peripheral speed, and hence an inertia force with frequency 2ω appears in the third term of the right hand side in Eq. (ix), as pointed out, thus a resonance in the neighborhood of the secondary critical speed $\omega = 1/2 \cdot \omega_c$ is expected. The magnitude of this inertia force is, however, a small quantity of the second order of $\Delta\alpha$ and e , so its effects are not considered in the present paper. When the smaller terms than the second order are rejected, Eq. (ix) reduces to Eq. (ii). So far as the shaft system constrained its motion only in y -direction as shown in Append.-Figs. 1, 2, is concerned, the forced vibration in the neighborhood of $\omega = 1/2 \cdot \omega_c$ caused by the term of $-mg$ in Eqs. (ii), (ix), i.e. the vibration induced by the external force $\Delta\alpha/\alpha mg \sin 2\omega t$

in Eq. (iv) is treated in section 5.

II(i) The flat shaft is usually not constrained in its motion only in y -direction, as shown in Append.-Figs. 1, 2, but the center M of the rotor can freely move in xy plane. In Append.-Fig. 3, directions of x' and y' are those of principal axes of moment of inertia of cross section of shaft, and the coordinate system $O-x'y'$ rotates with the rotating speed ω of the shaft provided that there is no torsion in the shaft. Elastic forces of the shaft $F_{x'}$, $F_{y'}$, F_x , F_y in x' , y' , x , y -directions are represented by

$$\left. \begin{aligned} F_{x'} &= -(\alpha - \Delta\alpha)x' \\ F_{y'} &= -(\alpha + \Delta\alpha)y' \\ F_x &= F_{x'} \cos \omega t - F_{y'} \sin \omega t \\ F_y &= F_{x'} \sin \omega t + F_{y'} \cos \omega t \\ x' &= x \cos \omega t + y \sin \omega t \\ y' &= -x \sin \omega t + y \cos \omega t \end{aligned} \right\} \dots\dots\dots (x)$$

so we have

$$\left. \begin{aligned} F_x &= -\alpha x + \Delta\alpha(x \cos 2\omega t + y \sin 2\omega t) \\ F_y &= -\alpha y + \Delta\alpha(x \sin 2\omega t - y \cos 2\omega t) \end{aligned} \right\} \dots\dots\dots (xi)$$

Thus we have the following equation of motion in place of Mathieus' Eq. (i):

$$\left. \begin{aligned} m\ddot{x} + \alpha x - \Delta\alpha(x \cos 2\omega t + y \sin 2\omega t) &= 0 \\ m\ddot{y} + \alpha y - \Delta\alpha(x \sin 2\omega t - y \cos 2\omega t) &= 0 \end{aligned} \right\} \dots\dots\dots (xii)$$

In the system governed by Eq. (xii), only two free vibrations of frequencies p and $2\omega - p$ appear and the self excited vibration can take place only in the neighborhood of $\omega = \omega_c$ and it cannot occur in the neighborhood of the secondary

Append.-Fig. 3

critical speed $\omega = 1/2 \cdot \omega_c$.

(ii) When the gravitational force acts on the horizontal shaft, the equation of motion is represented as follows⁽⁵⁾:

$$\left. \begin{aligned} m\ddot{x} + \alpha x - \Delta\alpha(x \cos 2\omega t + y \sin 2\omega t) &= 0 \\ m\ddot{y} + \alpha y - \Delta\alpha(x \sin 2\omega t - y \cos 2\omega t) &= -mg \end{aligned} \right\} \dots\dots\dots (xiii)$$

Particular solution of the differential equations (xii) corresponding to the term $-mg$ is furnished by

$$\left. \begin{aligned} x &= \frac{mg\Delta\alpha}{\alpha(\alpha - 4m\omega^2) - (\Delta\alpha)^2} \cos(2\omega t + \pi/2) \\ y &= \frac{mg\Delta\alpha}{\alpha(\alpha - 4m\omega^2) - (\Delta\alpha)^2} \sin(2\omega t + \pi/2) \\ &\quad - \frac{mg(\alpha - 4m\omega^2)}{\alpha(\alpha - 4m\omega^2) - (\Delta\alpha)^2} \end{aligned} \right\} \dots\dots\dots (xiv)$$

As seen from the above equation, in the neighborhood of the secondary critical speed

$$\left. \begin{aligned} \omega &= \frac{1}{2} \sqrt{\frac{\alpha}{m} \left\{ 1 - \left(\frac{\Delta\alpha}{\alpha} \right)^2 \right\}} \\ &= \frac{1}{2} \omega_c \sqrt{1 - \left(\frac{\Delta\alpha}{\alpha} \right)^2} \doteq \frac{\omega_c}{2} \end{aligned} \right\} \dots\dots\dots (xv)$$

which is obtained by putting the denominator of Eq. (xiv) equal to zero, the peak of whirling motion of forced vibration with frequency 2ω and mode of forward precession appears. In section 5 of the present paper, such a forced vibration is discussed. The magnitude of the constant term $mg(\alpha - 4m\omega^2)/\{\alpha(\alpha - 4m\omega^2) - (\Delta\alpha)^2\}$ in Eq. (xiv) is still small even at the resonant point given by Eq. (xv), because the constant term contains $(\alpha - 4m\omega^2)$ in its numerator and hence it takes a value of order $(\Delta\alpha/\alpha)^2$ at the resonant point. On the other hand, the numerator of the amplitude of forced vibration of 2ω is a value of order $(\Delta\alpha/\alpha)$ at the resonant point.

When there exists an eccentricity e in the rotor, Eq. (xiii) would contain further terms, as shown in Eq. (ix). They are, however, smaller than the second order of $\Delta\alpha$, e , and hence their effects are not treated in the present paper.