Application of Finite-Element Method to Hydrodynamic Lubrication Problems*  
(Part 2, Finite-Width Bearings)  

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The applications of the finite-element method to infinite-width bearing problems were presented in the first paper.  
In this paper, the finite-element method applied to the partial and the ordinary differential Reynolds equation is studied on finite-width bearing problems. The treatments of the boundary and the connecting condition at nodes in the finite-element method are presented. The finite-element methods for finite journal bearings have more advantages as compared with the finite-difference method and the analytical approximate solution.

1. Introduction

The authors have published the applications(1) of the finite-element method to infinite-width bearing problems.  
In this paper, the treatments of the finite-element method for finite-width bearings will be clarified. There are two methods in the analysis of finite-width bearings. One of them is to solve a Reynolds equation in the form of a partial differential equation, and another method is to solve the Reynolds equation transformed into an ordinary differential equation by Fourier's series. We discuss the application of the finite-element method for each analysis and establish the treatments of the boundary condition and the connecting condition for the state vector at nodes in finite-width bearings.  
The numerical solution by the finite-element method is compared with that by the finite-difference method and the approximate solution obtained by the conventional analytical method. We have more accurate solution by the finite-element method even if a few elements are used.

2. Nomenclature

$D$ : diameter of journal  
$L$ : bearing width  
$R$ : radius of journal  
$U$ : circumferential velocity of journal  
$W$ : load capacity  
$h$ : film thickness  
$l$ : number of components of state vector  
$m$ : number of unknown constants  
$n$ : number of elements  
$p$ : film pressure  
$p_a$ : ambient pressure  
$q$ : oil flow  
$x$ : coordinate in sliding direction, or bearing angle  
$z$ : coordinate in axial direction  
$\alpha$ : unknown constant  
$\varepsilon$ : eccentricity ratio  
$\mu$ : viscosity  
$\phi$ : attitude angle

Matrices

$(A)$ : matrix  
$[A]$ : square matrix  
$[A]^{-1}$ : inverse matrix  
$[A]^T$ : transpose matrix  
$\{A\}$ : column matrix  
$\Gamma A_J$ : row matrix

Subscripts

$e$ : refer to $e$th element  
$\Lambda$ : dimensional quantity

The non-dimensional quantities in this paper are given in Table 1.
3. The finite-element method applied to the partial differential equation

3.1 Variational principle and Reynolds equation

The bearing surface is divided into \( n \) elements as shown in Fig. 1, in which \( p_e \) is the film pressure in the \( e \)th element.

The function \( G^{(s)} \) is defined as follows,

\[
G = -\frac{1}{2} \left( \frac{h^3}{12} \left( \frac{\partial p_e}{\partial x} \right) \right)^2 + \frac{1}{4} \left( \frac{D}{L} \right) \frac{h^3}{12} \left( \frac{\partial p_e}{\partial z} \right)^2 \\
+ \frac{h}{2} \frac{\partial p_e}{\partial x}
\]

Let us consider the condition that \( I_e = \int_{D_e} \int_o G(x, z; p_e, p_e'; p_e') dxdz \) has a stationary value.

\[
\delta I_e = \int_{D_e} \left( G_{pp} - \frac{\partial}{\partial x} G_{pp'} - \frac{\partial}{\partial z} G_{pp'} \right) \delta p_e dxdz \\
= 0
\]

where

\[
p_e' = \frac{\partial p_e}{\partial x}, \quad p_e' = \frac{\partial p_e}{\partial z}
\]

\[
G_{pp} - \frac{\partial}{\partial x} G_{pp'} - \frac{\partial}{\partial z} G_{pp'} = \frac{\partial G}{\partial p_e}, \quad G_{pp'} = \frac{\partial G}{\partial p_e'}
\]

and \( C_e \) is the boundary line of element and consists of \( C_1^e \), \( C_2^e \), \( C_3^e \), \( C_4^e \) and \( C_5^e \); and \( D_e \) is the domain of integration as shown in Fig. 1.

Performing similar procedure for each element and rearranging, we have

\[
\delta I = \sum_{e=1}^{n} \delta I_e = \sum_{e=1}^{n} \left( G_{pp} - \frac{\partial}{\partial x} G_{pp'} - \frac{\partial}{\partial z} G_{pp'} \right) \delta p_e \\
+ \sum_{e=1}^{n} \int_{D_e} \left( \frac{\partial}{\partial x} G_{pp'} - \frac{\partial}{\partial z} G_{pp'} \right) \delta p_e dxdz \\
= 0
\]

We have the following conditions.

(a) The value of \( p_e \) is constant along the boundary line of bearing, because it is a boundary value. This condition is called the boundary condition.

(b) The values of \( p_e \) in adjacent elements are continuous along the boundary line of element. This condition is called the connecting condition.

From these conditions, the first and the second terms of right-hand side of Eq. (3) simultaneously become zero to make \( \delta I = 0 \), because \( \delta p_e \) is arbitrary.

Since the first term is zero, \( G_{pp} \) in adjacent elements becomes continuous along the \( x = \) constant lines i.e. \( C_1^e \) and \( C_5^e \).

And \( G_{pp'} \) becomes continuous along the \( z = \) constant lines i.e. \( C_2^e \) and \( C_4^e \). From Eq. (1), we have

\[
G_{pp} = \frac{G_{pp} = q_e = -\frac{h^3}{12} \frac{\partial p_e}{\partial x} + \frac{h}{2}}{12} \frac{\partial p_e}{\partial x} + \frac{h}{2}
\]

\[
G_{pp'} = \frac{1}{2} \left( \frac{D}{L} \right) q_e = -\frac{1}{2} \left( \frac{D}{L} \right) \frac{h^3}{12} \frac{\partial p_e}{\partial z}
\]

Then the continuity of \( q_e \) between adjacent elements holds along the boundary lines of element \( C_1^e \), \( C_2^e \), \( C_3^e \) and the continuity of \( q_e \) holds along the lines \( C_1^e \), \( C_4^e \). Since the values of \( p_e \) in adjacent elements are continuous along the boundary lines of element, \( p_e \) is continuous along \( C_1^e \) and \( C_5^e \), and \( q_e \) is continuous along \( C_2^e \) and \( C_4^e \). Therefore the continuity of \( q_e \) and \( q_e \) between adjacent elements holds along the boundary line of element \( C_e \).

Since the second term is zero, we have

\[
G_{pp} - \frac{\partial}{\partial x} G_{pp'} - \frac{\partial}{\partial z} G_{pp'} = 0
\]

Substituting Eq. (1) into the above equation,

\[
\frac{\partial}{\partial x} \left( \frac{h^3}{12} \frac{\partial p_e}{\partial x} \right) + \frac{1}{4} \left( \frac{D}{L} \right) \frac{h^3}{12} \frac{\partial p_e}{\partial z}
\]

\[
= \frac{1}{2} \frac{\partial h}{\partial x}
\]

This is a Reynolds equation for finite-width bearings.
From the above mentioned, if the function \( p_e \) satisfying \( \delta J_e = 0 \) can be found for each element under the boundary and connecting conditions described above, then it is the solution of the Reynolds equation and the oil flow simultaneously becomes continuous along the boundary line of element.

3.2 The finite-element method using the film pressure assumed to be linear

The pressure of element is assumed to be linear as follows.

\[
p_e = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}
\]

(7)

where \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \) are unknown constants.

Substituting Eq. (7) into Eq. (1) and integrating, we have

\[
J_e = \frac{1}{2} \tau [I_p] \{ a \} + \tau [I_h] \{ a \}
\]

(8)

where

\[
[I_p] = - \int_{D_e} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & z \\
0 & 0 & 0 & 0 \\
0 & z & 0 & z^2
\end{array} \right) \frac{h^3}{12} ~ dx dz + \frac{1}{4} \left( \frac{D}{L} \right)^2 \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & x \\
0 & 0 & x & x^2
\end{array} \right) ~ dx dz
\]

(9)

\[
[I_h] = \int_{D_e} \frac{h}{2} \left( \begin{array}{c}
0 \\
0 \\
z_e \\
0
\end{array} \right) \frac{h}{2} \right) ~ dx dz
\]

We will determine the constant vector \( \{ a \} \) such as to make \( \delta J_e = 0 \) under the boundary and connecting conditions mentioned above. We introduce the following procedure in order to facilitate the use of these conditions.

We have the boundary and connecting conditions at nodes in the case of infinite-width bearings described in the first paper, but hold these conditions in finite-width bearings along boundary lines of bearing and element in this paper. The state vector will be defined such that the boundary condition may be established along the boundary line of bearing by making the state vector at nodes on this line constant, and such that the connecting condition may hold along the boundary line of element by making the state vector at nodes on this line continuous.

The nodes in the \( e \)th element are numbered as shown in Fig. 2. The boundary lines of element are \( C_1^e, C_2^e, C_4^e \) and \( C_3^e \), and the film pressures at nodes are \( p_1^e, p_2^e, p_4^e \) and \( p_3^e \). If \( p_1^e \) and \( p_3^e \) have a constant value \( p_a \) as shown in Fig. 3, the film pressure is constant along the boundary line \( C_1^e \), because the pressure of element is assumed to be linear as in Eq. (7). If \( p_3^e = p_4^e+1 \) and \( p_3^e = p_4^e+1 \) as shown in Fig. 4, then the film pressures in the \( e \)th element and \( (e+1) \)th element coincide along the boundary line \( C_3^e \).

If the state vector is defined by the film pressure at nodes, then the boundary condition (a) and the connecting condition (b) along the boundary lines of bearing and element are rewritten into the following conditions for the state vector at nodes.

(c) The pressure at nodes on the boundary line of bearing is constant (boundary condition).

d) The pressure at nodes on the boundary line of element is continuous (connecting condition).
Thus, the state vector of element is defined as follows.

$$\{\beta_e\} = \begin{bmatrix} p_{1e} \\ p_{2e} \\ p_{3e} \\ p_{4e} \end{bmatrix}$$

Substituting Eq. (7) into the above equation, we have

$$\{\beta_e\} = [B] \{\alpha\}$$

where

$$[B] = \begin{bmatrix} 1 & x_i & z_i & x_i z_i \\ 1 & x_j & z_j & x_j z_j \\ 1 & x_j & z_j & x_j z_j \\ 1 & x_i & z_i & x_i z_i \end{bmatrix}$$

we have \(l=m=4\) where \(l\) is the number of components of state vector and \(m\) is the number of unknown constants. Then \([B]\) becomes a square matrix, and \(\{\alpha\}\) can be expressed by \(\{\beta_e\}\) as follows.

$$\{\alpha\} = [M] \{\beta_e\}$$

where

$$[M] = [B]^{-1}$$

Introducing Eq. (13) into Eq. (8), we have

$$J_e = \frac{1}{2} \gamma \{K_e\} \{\beta_e\} + \gamma \{H_{e}\} \{\beta_e\}$$

where

$$K_e = [M]^T [I_p] [M] \quad \{H_e\} = [I_p] [M]$$

It is found from Eq. (15) that \(J_e\) is a function of \(\{\beta_e\}\) only. From the condition that \(\delta J_e = 0\) i.e. \(\delta J_e/\delta \{\beta_e\} = 0\), we have the following equation by considering that \([K_e]\) is a symmetric matrix.

$$\{K_e\} \{\beta_e\} + \{H_e\} = 0$$

A similar procedure is performed for each element, so that Eq. (17) is obtained in each element. Rearranging Eq. (17) by using the boundary condition (c) and the connecting condition (d), we have

$$[K] \{\beta\} + \{H\} = 0$$

where \(\{\beta\}\) is called the state vector of system, because it consists of the state vectors at all nodes.

Since Eq. (18) is a simple simultaneous equation, the state vector at nodes, namely the film pressure, can be obtained by solving this equation, then the state vector of element is immediately determined. The film pressure in each element is

$$p_e = \gamma 1 x z x z [M] \{\beta_e\}$$

In journal bearings, we have

$$W_e \sin \phi = \int_{p_{e}} \int_{x} p_e \sin x d x d z$$

$$= \gamma I_{e,j} [M] \{\beta_e\}$$

$$W_e \cos \phi = - \int_{p_{e}} \int_{x} p_e \cos x d x d z$$

$$= \gamma I_{e,j} [M] \{\beta_e\}$$

where

$$I_{e,j} = \gamma \int_{p_{e}} \int_{x} 1 x z x z j \sin x d x d z$$

$$I_{e,j} = - \gamma \int_{p_{e}} \int_{x} 1 x z x z j \cos x d x d z$$

Therefore, the load capacity of system is

$$W = \left( \sum_{e=1}^{n} W_e \sin \phi \right)^2 + \left( \sum_{e=1}^{n} W_e \cos \phi \right)^2$$

3.3 The finite-element method using the film pressure assumed to be cubic

The pressure of element is assumed by the following cubic equation.

$$p_e = \gamma 1 x z x z x z + x z + x z$$

where \(\{\alpha\}\) is an unknown constant.

In this section, the boundary condition (a) and the connecting condition (b) will be also rewritten into the conditions for the state vector at nodes.

The film pressure and the pressure gradient at nodes are \(p_{e}^{k}, \partial p_{e}^{k}/\partial x, \partial p_{e}^{k}/\partial z\) \((k=1, 2, 3, 4)\) as shown in Fig. 5.

Now, the boundary condition is discussed. Let us consider the condition that the film pressure is constant along the \(x\)-constant line, for example \(C_{e}^{k}\) in the element as shown in Fig. 6(a). From Eq. (24) the following equations are obtained on the \(x\)-constant line.

$$p_e = A_{e}^{k} + x A_{e}^{k} + x z A_{e}^{k} + x z A_{e}^{k}$$

$$\frac{\partial p_e}{\partial z} = A_{e}^{k} + 2 z A_{e}^{k} + 3 z^2 A_{e}^{k}$$

Fig. 5 State vector of element
where $A_1^e$, $A_2^e$, $A_3^e$, and $A_4^e$ are constants.

If $p_i^e$ and $p_i^{e+1}$ have a constant value $p_a$, and $\partial p_i^e/\partial z$ and $\partial p_i^{e+1}/\partial z$ are zero, then from Eq. (25) we have

$$
\begin{bmatrix}
1 & z_i^e & z_i^{e+1} \\
0 & 1 & 2z_i^e & 3z_i^{e+1} \\
1 & z_i^e & z_i^{e+1} \\
0 & 1 & 2z_i^e & 3z_i^{e+1}
\end{bmatrix}
\begin{bmatrix}
A_1^e \\
A_2^e \\
A_3^e \\
A_4^e
\end{bmatrix} =
\begin{bmatrix}
p_a \\
0 \\
p_a \\
0
\end{bmatrix}
$$

(26)

Since $z_i^e - z_i^{e+1} = 0$, the following equation is obtained from Eq. (26).

$$
\begin{bmatrix}
A_1^e \\
A_2^e \\
A_3^e \\
A_4^e
\end{bmatrix} =
\begin{bmatrix}
p_a \\
0 \\
0 \\
0
\end{bmatrix}
$$

(27)

Substituting the above equation into Eq. (25), we obtain the following equations along the boundary line $C_i^e$.

Similarly, if $p_i^e = p_i^{e+1} = p_a$ and $\partial p_i^e/\partial x = \partial p_i^{e+1}/\partial x = 0$ in the element as shown in Fig. 6(b), then the film pressure has a constant value $p_a$ along the $x$-constant line, for example $C_i^e$.

Next, the connecting condition is discussed. Let us consider the condition that the film pressure in adjacent elements is continuous along the $x$-constant line, for example $C_i^e$ as shown in Fig. 7(a). From Eq. (25), the following two equations are obtained for the $e$th and the $(e+1)$th elements.

$$
\begin{align}
\begin{bmatrix}
p_i^e \\
\partial p_i^e/\partial z \\
p_i^{e+1} \\
\partial p_i^{e+1}/\partial z
\end{bmatrix} &=
\begin{bmatrix}
1 & z_i^e & z_i^{e+1} \\
0 & 1 & 2z_i^e & 3z_i^{e+1} \\
1 & z_i^e & z_i^{e+1} \\
0 & 1 & 2z_i^e & 3z_i^{e+1}
\end{bmatrix}
\begin{bmatrix}
A_1^e \\
A_2^e \\
A_3^e \\
A_4^e
\end{bmatrix} \\
\frac{\partial p_i^e}{\partial x} &= \frac{\partial p_i^{e+1}}{\partial x}
\end{align}
$$

(28)

$$
\begin{align}
\begin{bmatrix}
p_i^e \\
\partial p_i^e/\partial z \\
p_i^{e+1} \\
\partial p_i^{e+1}/\partial z
\end{bmatrix} &=
\begin{bmatrix}
1 & z_i^e & z_i^{e+1} \\
0 & 1 & 2z_i^e & 3z_i^{e+1} \\
1 & z_i^e & z_i^{e+1} \\
0 & 1 & 2z_i^e & 3z_i^{e+1}
\end{bmatrix}
\begin{bmatrix}
A_1^{e+1} \\
A_2^{e+1} \\
A_3^{e+1} \\
A_4^{e+1}
\end{bmatrix}
\end{align}
$$

(29)

If $p_i^e = p_i^{e+1}$, $p_i^e = p_i^{e+1}$, $\partial p_i^e/\partial z = \partial p_i^{e+1}/\partial z$ and

$$
\frac{\partial p_i^e}{\partial x} = \frac{\partial p_i^{e+1}}{\partial x}
$$

the following equation is obtained from Eqs. (28) and (29).

$$
\begin{bmatrix}
A_1^e \\
A_2^e \\
A_3^e \\
A_4^e
\end{bmatrix} =
\begin{bmatrix}
A_1^{e+1} \\
A_2^{e+1} \\
A_3^{e+1} \\
A_4^{e+1}
\end{bmatrix}
$$

(30)

Therefore the film pressure in adjacent elements is always continuous along the line $C_i^e$.

Similarly, if $p_i^e = p_i^{e+1}$, $p_i^e = p_i^{e+1}$, $\partial p_i^e/\partial x = \partial p_i^{e+1}/\partial x$, and $\partial p_i^e/\partial x = \partial p_i^{e+1}/\partial x$, then the film pressure in adjacent elements is continuous along $z$-constant line, for example $C_i^e$ as shown in Fig. 7(b).

Thus, the boundary condition (a) and the connecting condition (b) along the boundary lines of bearing and element are rewritten into the following conditions for the state vector at nodes. These conditions at nodes are somewhat different from those of the finite-element method using the film pressure assumed to be linear, because the condition in this section should be discussed depending on whether $x$-constant line or $z$-constant line.

(e) When the node is located on the $x$-constant boundary line of bearing, the pressure at node is constant and $\partial p_i/\partial z$ at node is zero. Also, when the node is located on the $z$-constant line, the pressure at node is constant and $\partial p_i/\partial x$ at node is zero (boundary condition).

![Fig. 6 Boundary condition](a)

![Fig. 7 Connecting condition](b)
The values of \( p_e \) and \( \partial p_e / \partial z \) at node in adjacent elements which join along the \( x = \text{constant} \) line are continuous. Also, \( p_e \) and \( \partial p_e / \partial x \) at node in adjacent elements which join along the \( z = \text{constant} \) line are continuous (connecting condition).

Evaluating the unknown constant \( \{ a \} \) in Eq. (24) by using the variational principles under these boundary and connecting conditions, the values of \( p_e \) and \( \partial p_e / \partial z \) become continuous along the \( x = \text{constant} \) line but \( \partial p_e / \partial x \) is discontinuous. Because the connecting condition \( (f) \) is the condition for the continuity of \( p_e \) along the boundary line of element and is not one for the continuity of pressure gradient. If the pressure \( p_e \) is continuous, then the continuity of \( \partial p_e / \partial z \) is always satisfied along the \( x = \text{constant} \) line but the continuity of \( \partial p_e / \partial x \) is not always satisfied along this line. In smooth bearings such as journal bearings, the discontinuity of \( \partial p_e / \partial x \) means the discontinuity of oil flow \( q_e \). However the continuity of oil flow has been proved in the section of variational principle described above. This contradiction is caused by the assumption that the pressure of element is approximately expressed by a cubic equation as in Eq. (24) while the true pressure should be expressed by complicated functions. In the finite-element method, the unknown constant is calculated such that the Reynolds equation and the continuity of oil flow may be approximately satisfied under the assumption of Eq. (24); then \( \partial p_e / \partial x \) is unavoidably discontinuous along the \( x = \text{constant} \) line. This fact is true also along the \( z = \text{constant} \) line. However, as the number of elements increases, Eq. (24) comes closer to the true pressure distribution, and the Reynolds and the continuity of oil flow could be more approximately satisfied. Therefore accurate results are obtained by increasing the number of elements.

Now, in smooth bearings such as journal bearings, the pressure gradient \( \partial p_e / \partial x \) is not continuous anywhere along the \( x = \text{constant} \) line but at least continuous at node, if \( p_e \), \( \partial p_e / \partial x \) and \( \partial p_e / \partial z \) at node in adjacent elements are continuous as shown in Fig. 8. Namely the oil flow is always continuous at node. In journal bearings, the connecting condition \( (f) \) is simplified as follows, so that the numerical calculations become easy.

The values of \( p_e \), \( \partial p_e / \partial x \) and \( \partial p_e / \partial z \) at node are continuous (connecting condition).

For above reasons, the boundary condition \( (e) \) and the connecting condition \( (g) \) are used here. Of course, the connecting condition \( (f) \) must be used in step bearings, because the continuity of pressure gradient does not mean the continuity of oil flow.

Thus, the state vector of element is defined as follows.

\[
\{ \beta_e \} = \begin{bmatrix} p_e^1 \\ \frac{\partial p_e^1}{\partial x} \\ p_e^2 \\ \frac{\partial p_e^2}{\partial x} \\ \vdots \end{bmatrix}
\]

(31)

In this section, \( l = 12 \) where \( l \) is the number of components of state vector. Substituting Eq. (24) into the above equation, we have

\[
\{ \beta_e \} = (B) \{ a \}
\]

(32)

where the matrix \( (B) \) is the \( l \) by \( m \) and \( (B) \) can not be inverted when \( m \neq l \). In such a case, the following reduction is introduced as described in the first paper.

\[
\{ a \} = [M] \{ \beta_e \}
\]

(33)

where

\[
[M] = \begin{bmatrix} [B_e]^{-1} & -[B_e]^{-1}(B_s) \\ 0 & 1 \end{bmatrix}
\]

(34)

In the above equation, the matrices \( [B_e] \) and \( (B_s) \) are the \( l \) by \( l \) and the \( l \) by \( (m-l) \) sub-matrices of \( (B) \) respectively, and \( \{ a \} \) is the \( (m-l) \) sub-matrix of \( \{ a \} \).

Introducing Eqs. (24) and (33) into Eq. (1), we have

\[
J_v = \frac{1}{2} \{ \beta_e \}^T [K_L] \{ \beta_e \} + \{ \beta_e \}^T [H_L] \{ \beta_e \}
\]

(35)

where

\[
[K_L] = [M]^T[I_e][M]
\]

\[
[H_L] = [I_{e, L}][M]
\]

(36)

\[
[K_L] = \begin{bmatrix} p_e^1 & \xi_{e, 1} & \xi_{e, 2} \\ \xi_{e, 1} & \delta_{e, 1} & \delta_{e, 2} \\ \xi_{e, 2} & \delta_{e, 1} & \delta_{e, 2} \end{bmatrix}
\]

\[
[H_L] = \begin{bmatrix} \gamma_{L, 1} & \gamma_{L, 2} \\ \gamma_{L, 2} & \gamma_{L, 3} \end{bmatrix}
\]

Fig. 8 Connecting condition
From $\delta J_e=0$, we have

$$[K_e]\{\beta_e\} + \{H_e\} = 0 \quad \cdots \quad (37)$$

Eliminating $\{\alpha_e\}$ in Eq. (37) by the procedure described in the first paper, we have

$$[K_e]\{\beta_e\} + \{H_e\} = 0 \quad \cdots \quad (38)$$

where

$$[K_e] = [K_{aa}] - (K_{ab}) [K_{ba}]^{-1} (K_{ba}) \right)$$

$$[H_e] = (H_a) - (K_{ab}) [K_{ba}]^{-1} (H_a) \right)$$

In the above equation, the matrices $[K_{aa}]$, $[K_{ab}]$, $[K_{ba}]$ and $[K_{sa}]$ are the $l$ by $l$, the $l$ by $(m-l)$, the $(m-l)$ by $l$ and the $(m-l)$ by $(m-l)$ sub-matrices of $[K_e]$ respectively. Also the matrices $[H_a]$ and $[H_a]$ are the $l$ by $l$ and $(m-l)$ by $l$ sub-matrices of $[H_e]$ respectively.

Similar procedure is performed for each element, so that Eq. (38) is obtained in each element. Rearranging Eq. (38) by using the boundary condition (e) and the connecting condition (g), we have

$$[K] \{\beta\} + \{H\} = 0 \quad \cdots \quad (40)$$

The state vector of system is obtained by solving the above equation and then the film pressure in each element and the load capacity can be calculated similarly to the above section. The details are not discussed here.

3.4 Discussions

In order to examine the effects of the finite-element method, the pressure distribution and the load capacity of finite journal bearings calculated by the finite-element method are compared with the numerical results by the finite-difference method and the approximate solution of Aoki$^{(3)}$ derived analytically by the conventional method.

The geometry of journal bearing is shown in Fig. 9, in which the film thickness $h=1+\varepsilon \cos \alpha$. The boundary condition is the half Sommerfeld condition.

Table 2 shows the results of the finite-element method using the film pressure assumed to be linear. The number of unknown constants $m$ is equal to 4 in this case. From this table, we see that the finite-element method is less accurate as compared with the finite-difference method, because the film pressure changes abruptly when the eccentricity ratio is 0.8. Thus unavoidably the accuracy of the finite-element method using the film pressure assumed to be linear is somewhat decreased. Of course, good results can be obtained by increasing the number of elements.

The results of the finite-element method using the film pressure assumed to be cubic are shown in Fig. 10, in which the line of $m=12$ indicates that the pressure is assumed by the cubic equation omitting the last term in Eq. (24) and the line of $m=13$ comes from Eq. (24). The results of the finite-element method using the film pressure assumed to be linear i.e. $m=4$ are also shown for comparison. The dashed line is the solution of Aoki and the symbol $\bigcirc$ refers to the numerical results by the finite-difference method. As shown in this figure, the finite-element method indicates very accurate results even by 9 elements (trisected in $x$- and $z$-directions respectively). However 144 elements are needed in the finite-difference method. The relation between the load capacity and the number of elements is shown in Fig. 11, from which the solution of the finite-element method of $m=13$ and $n=9$ is known to almost

![Fig. 9 Geometry of journal bearing](image)

![Fig. 10 Pressure of journal bearing](image)

<table>
<thead>
<tr>
<th>The number of elements</th>
<th>Aoki's solution</th>
<th>Finite-difference method</th>
<th>Finite-element method $(m=4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.050</td>
<td>2.652</td>
<td>4.088</td>
</tr>
<tr>
<td>9</td>
<td>2.562</td>
<td>2.002</td>
<td>2.652</td>
</tr>
<tr>
<td>16</td>
<td>6.135</td>
<td>3.814</td>
<td>3.179</td>
</tr>
<tr>
<td>25</td>
<td>4.695</td>
<td>4.091</td>
<td>4.722</td>
</tr>
<tr>
<td>36</td>
<td>5.234</td>
<td>4.722</td>
<td>4.722</td>
</tr>
</tbody>
</table>

where $L/D=1$ and $\varepsilon=0.8$
agree with the solution of Aoki, but it is necessary that the number of elements be more than 324 in order to obtain the same degree of accuracy by the finite-difference method.

In infinite-width bearings described in the first paper, the finite-difference method requires several times more elements than those of the finite-element method in order to obtain the same degree of accuracy. However it is necessary to use several times more elements in infinite-width bearings. In the finite-element method, the film pressure is defined in each element, then the integration to obtain the load capacity can be evaluated easily and exactly. In the finite-difference method, the integration is performed numerically because the pressure is not defined in element, therefore the accuracy of integration becomes very inferior. Thus, accurate results cannot be drawn from the finite-difference method unless a large number of elements are used. Since the integration is double in infinite-width bearings, a pronounced difference appears as compared with infinite-width bearings.

Then the finite-element method for infinite-width bearings is able to give good results as compared with the finite-difference method.

4. The finite-element method applied to the ordinary differential equation

In hydrodynamic lubrication, it is ordinary that the film thickness changes in sliding direction only.

4.1 Variational principle and Reynolds equation

The film pressure is assumed as follows by considering that the pressure is zero on both sides of bearing.

\[
 p = \sum_{k=0}^{\infty} p_k \sin(2k+1)\pi z \tag{41}
\]

Substituting Eq. (41) into the Reynolds equation, we have

\[
 \frac{d}{dx} \left( \frac{h^3}{12} \frac{dp_k}{dx} \right) - \lambda_k \frac{h^3}{12} p_k = \frac{2}{\pi(2k+1)} \frac{dh}{dx} \tag{42}
\]

where

\[
 \lambda_k^2 = \pi^2 \left( \frac{D}{L} \right)^2 \frac{(2k+1)^2}{4}
\]

Now, the function \( G \) is defined as follows.

\[
 G = -\frac{1}{2} \left[ \frac{h^3}{12} \left( \frac{dp_k}{dx} \right)^2 + \lambda_k \frac{h^3}{12} p_k^2 \right] + \frac{2}{\pi(2k+1)} \frac{h \frac{dp_k}{dx}}{dx} \tag{43}
\]

If the pressure \( p_k \) can be found such that \( J_\infty = \int G(x; p_k, p_k') dx \) may have a stationary value, it satisfies Eq. (42). Since Eq. (42) is an ordinary differential equation, the pressure \( p_k \) can be obtained by the finite-element method for infinite-width bearings described in the first paper.

4.2 Analysis

The pressure of element is assumed by the following equation.

\[
 p_k = \mathbb{1} x \ x^2 \ x^3 \ \ldots \ x^{m-1} \mathbb{1}(x) \tag{44}
\]

The state vector is defined by the film pressure when the Reynolds equation is an ordinary differential equation in such infinite-width bearings described in the first paper. The procedure of calculating \( p_k \) is not described here, because it is the same as in the first paper. The film pressure \( p \) is evaluated from Eq. (41) and the load capacity is calculated from the following equations if the pressure \( p_k \) is obtained.

\[
 W \sin \phi = \sum_{k=0}^{\infty} p_k \sin x \ dx \ dz \tag{45}
\]

\[
 W \cos \phi = -\sum_{k=0}^{\infty} p_k \cos x \ dx \ dz \tag{46}
\]

\[
 W = \left( (W \sin \phi)^2 + (W \cos \phi)^2 \right)^{1/2} \tag{47}
\]
4.3 Discussions

The pressure distribution of finite journal bearings is shown in Fig. 12, in which the half Sommerfeld condition is used. In this figure, the dashed line is the solution of Aoki and the symbol O refers to the numerical solution by the finite-difference method using 144 elements. This figure shows that the finite-element method using an element gives good results when \( m = 7 \) in Eq. (44). In this case, the treatments of the finite-element method become identical with those of Rayleigh-Ritz method, because the bearing surface is undivided. The relation between the load capacity and the number of unknown constants is shown in Fig. 13, from which the case of \( k = 3 \) does not differ from the case of \( k = 0 \). Therefore it is enough to use the solution of \( k = 0 \), because the convergence of series of Eq. (41) is very good. And good results can be obtained in an element.

The finite-element method has the advantage that accurate results can be obtained even by using a few elements. Of course, the finite-element method applied to the partial differential equation described in the above section should be used for the bearing whose film thickness changes in axial direction also, because the Reynolds equation can not be transformed into an ordinary differential equation such as Eq. (42).

5. Conclusions

The bearing performance of finite journal bearings is evaluated by the finite-element method, and compared with a numerical solution by the finite-difference method and an analytical solution derived from the conventional method. Conclusions are as follows:

1. In the finite-element method applied to the Reynolds equation in the form of a partial differential equation, the boundary condition and the connecting condition along the boundary line of element need to be rewritten into the conditions for the state vector at nodes.

2. Very accurate results can be obtained by the finite-element method using a few elements. However in the finite-difference method, the number of elements is several tens times larger than that in the finite-element method in order to obtain the same degree of accuracy.

3. In the finite-element method applied to the Reynolds equation transformed into an ordinary differential equation, the procedure is nearly similar to that for infinite-width bearings. However this method is applicable when the film thickness changes in sliding direction only. When the film thickness changes in axial direction also, the finite-element method should be applied to a partial differential equation.

In this paper, calculated examples for finite journal bearings have been discussed, and the finite-element method is found applicable to finite-width bearings which have different film shapes.

The authors wish to thank Professor A. Okumura of Waseda University for his helpful advice.

References

Discussion

Y. Shimotsuma (Kansai University): (1) While a large number of elements are needed in the application to stress analyses, the authors could obtain the solution almost agreeing with the solution of Aoki by using only $3 \times 3 = 9$ divisions. This clearly indicates the usefulness of the application of this method to hydrodynamic lubrication. I ask if there are any foreign references in this field.

H. Tahara (Tokyo Shibaura Electric Co., Ltd.) (2) I ask about the difference of computation time between the finite-element method and the finite-difference method, which are programmed suitably so that the same degree of accuracy may be obtained, and their processes of calculations.

Authors’ closure

(1) Although the authors replied on the day of lecture that “The papers on the finite-element method applied to hydrodynamic lubrication are few so far as authors know”, M.M. Reddi published a paper “Finite Element Solution of the Incompressible Lubrication Problem” in Trans. ASME, Ser. F, Vol. 91, No. 3 (1969–7), p. 524.

(2) For example, the calculation processes of the finite-element method applied to Eq. (6) and the finite-difference method (successive over-relaxation method) are given by the flow charts as shown in Append.-Figs. 1 and 2. Using the computer “HITAC-5020E” in Computer Center of University of Tokyo for numerical examples in Fig. 11, the computation time by the finite-element method is 6 seconds when $n=9$ and $m=12$, and 8 seconds when $n=9$ and $m=13$.

Append.-Fig. 1 Flow chart of finite-element method

Append.-Fig. 2 Flow chart of finite-difference method
The finite-difference method requires 4 seconds in order to obtain the same degree of accuracy.

In the authors' program of the finite-element method, various checks are performed in computations. If the parts of checks are omitted, real computation time shall be shorter than that of the above.