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Optimum Configuration of Vibration Isolators for Systems with Random Excitations*

By Naofumi FUJIWARA**, Yoshisada MUROTSU**, and Kenji NAKAGAWA***

This paper discusses the optimum configuration of isolators for systems with stationary random excitations. Optimum control theory in frequency domain is applied to obtain an optimum isolating system that minimizes the response of the system to the excitations under certain conditions. A possible method of solving the optimization problems is presented for multi-degree-of-freedom systems and illustrated on a few typical systems. The optimum isolators thus obtained may in general be characterized by active elements rather than conventional spring-dashpot elements.

1. Introduction

A method⁽¹⁾ so far used for designing vibration isolators is to specify the configuration of the system and then to determine the optimum tuning of the parameters assuming sinusoidal excitations. In many vibration problems, however, the excitations are random in nature. Furthermore, such a design method has the disadvantage that the designer's attention is restricted to a limited number of candidate configurations of vibration isolators. Thus, our main interest lies in designing the isolators so that the system with stationary random excitations may have the best performance.

In the previous papers⁽²⁾⁽³⁾, the present authors proposed a design procedure for the optimum configuration of the vibration isolators for general multi-degree-of-freedom systems subjected to random forces and/or random foundation motion. A few examples were presented to illustrate the procedure.

This paper briefly summarizes the optimization technique and is mainly devoted to the application of it to the systems which are of technical interest.

2. Optimization procedure

The procedure is briefly summarized below. For the detail, the readers are referred to the

authors' previous paper⁽³⁾.

Let us consider a multi-degree-of-freedom system consisting of n masses coupled together by springs and dashpots and excited by random external forces and/or a random foundation motion. Here we assume that the random forces and foundation motion are stationary random processes with gaussian distribution. The vibration isolators are inserted between the masses and/or between the mass and the foundation. In designing isolators, we first specify the number and locations of isolators. However, desired characteristics of the isolators are unknown yet. They can be determined through use of a certain criterion that will be discussed below. The equations of motion for the system are

$$\begin{aligned} m_i \ddot{x}_i(t) = & - \sum_{k=0(k \neq i)}^n [c_{ik} \{\dot{x}_i(t) - \dot{x}_k(t)\} \\ & + k_{ik} \{x_i(t) - x_k(t)\}] + \sum_{k=0(k \neq i)}^n f_{ik}(t) + d_i(t) \\ = & - \sum_{k=0}^n \{C_{ik} \dot{x}_k(t) + K_{ik} x_k(t)\} \\ & + \sum_{k=0(k \neq i)}^n f_{ik}(t) + d_i(t) \dots\dots\dots (1) \\ & (i=1, 2, \dots, n) \end{aligned}$$

where

- m_i : absolute displacement of the i th mass
- c_{ik} : damping coefficient between the i th and k th masses
- k_{ik} : spring constant between the i th and k th masses
- C_{ik} : resultant damping force acting on the

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** Research Associate, College of Engineering, University of Osaka Prefecture, Sakai.

*** Professor, College of Engineering, University of Osaka Prefecture.

i th mass due to a unit velocity of the k th mass

K_{ik} : resultant restoring force acting on the i th mass due to a unit displacement of the k th mass

$f_{ik}(t)$: isolating force of the isolators inserted between the i th and k th masses

$d_i(t)$: external forces acting on the i th mass
 $\dot{}$: time derivative

The design problems of the vibration isolators can be classified into the following three types:

(I) to minimize the acceleration of the masses with a limited range on the strokes of the isolators or the relative displacements of the masses,

(II) to minimize the forces transmitted to the foundation with a limited range on the strokes of the isolators,

(III) to minimize the strokes of the isolators with a limited range on the accelerations of the masses or the forces transmitted to the foundation.

First consider the problem (I). This is formulated as an optimum control problem: Under the constraints

$$\langle x_{ij}(t)^2 \rangle \leq M_{ij} \quad (i, j=0, 1, \dots, n; i > j) \quad \dots\dots\dots (2)$$

design the linear vibration isolators to minimize the performance index

$$I = \sum_{i=1}^n \alpha_i^2 \langle \ddot{x}_i(t)^2 \rangle + \sum_{i=\mu+1}^n \sum_{j=0}^{\mu} \lambda_{ij}^2 \langle x_{ij}(t)^2 \rangle \quad \dots\dots\dots (3)$$

where

$x_{ij}(t)$: relative displacement between the i th and j th masses $= x_i(t) - x_j(t)$

α_i^2 : weighting factors,

λ_{ij}^2 : Lagrange's undetermined multipliers,

$\langle \rangle$: time average,

M_{ij} : given constants.

Let us now consider the functional relation of the isolating forces $f_{kl}(s)$ and the excitation forces $d_{\mu}(s)$. Since the candidate isolators being sought are of linear type as assumed, $f_{kl}(s)$ is characterized by the relation

$$f_{kl}(s) = \sum_{\mu=0}^n F_{kl}^{\mu}(s) d_{\mu}(s) \quad (k, l=0, 1, \dots, n; k \neq l) \quad \dots\dots\dots (4)$$

where $F_{kl}^{\mu}(s)$ is a transfer function, which will be determined below. Taking the Laplace transform of Eq. (1) and substituting Eq. (4) into it, we obtain

$$\begin{aligned} \ddot{x}_i(s) &= \sum_{\mu=0}^n \left[G_{i\mu} + \sum_{k=0}^n \sum_{l=0(l \neq k)}^n G_{ik} F_{kl}^{\mu} \right] \\ &\times d_{\mu}(s) \stackrel{\Delta}{=} \sum_{\mu=0}^n H_i^{\mu} d_{\mu}(s) \quad \dots\dots\dots (5) \\ (i &= 1, 2, \dots, n) \end{aligned}$$

$$\begin{aligned} x_{ij}(s) &= \frac{1}{s^2} \sum_{\mu=0}^n \left[(G_{i\mu} - G_{j\mu}) + \sum_{k=0}^n \sum_{l=0(l \neq k)}^n (G_{ik} - G_{jk}) \right. \\ &\times F_{kl}^{\mu} \left. \right] d_{\mu}(s) = \sum_{\mu=0}^n H_{ij}^{\mu} d_{\mu}(s) \quad \dots\dots\dots (6) \\ (i, j &= 0, 1, \dots, n; i \neq j) \end{aligned}$$

where $G_{ik}(s) = \ddot{x}_i(s)/f_{kl}(s) = \ddot{x}_i(s)/d_k(s)$ and $G_{i0}(s) = \ddot{x}_i(s)/\ddot{x}_0(s)$.

We denote the power spectral densities of the accelerations and the relative displacements as $\phi_{\ddot{x}_i}$ and $\phi_{x_{ij}}$. Applying the optimization theorem by S.S.L. Chang⁽⁴⁾, we obtain the condition for I to be minimum:

$$\begin{aligned} \sum_{i=1}^n \alpha_i^2 \frac{\partial \phi_{\ddot{x}_i}}{\partial \bar{F}_{kl}^{\mu}} + \sum_{i=\mu+1}^n \sum_{j=0}^{\mu} \lambda_{ij}^2 \frac{\partial \phi_{x_{ij}}}{\partial \bar{F}_{kl}^{\mu}} &= R_{kl}^{\mu} \quad \dots\dots\dots (7) \\ (k, l, \mu &= 0, 1, \dots, n; k \neq l) \end{aligned}$$

where $R_{kl}^{\mu}(s)$ denotes an arbitrary function which does not possess any pole in the LHP (left-half-plane of the s -plane). The above condition yields a set of linear simultaneous equations for $F_{kl}^{\mu}(s)$. By solving these equations, we can determine the optimum transfer function $F_{kl}^{\mu}(s)$. For details, the readers are referred to the authors' previous paper⁽⁵⁾.

For the problem (II), the formulation is done in the same way as that of the problem (I) except for substituting $f_{li}(s)$ into $\ddot{x}_i(s)$. $f_{li}(s)$ is the resultant force transmitted from the i th mass to the foundation and is expressed as

$$\begin{aligned} f_{li}(s) &= -f_{i0}(s) - (C_{i0}s + K_{i0})x_{i0}(s) \\ &= - \sum_{\mu=0}^n \left[F_{i0}^{\mu} + (C_{i0}s + K_{i0})(G_{i\mu} - G_{0\mu})/s^2 \right. \\ &\left. + \sum_{k=0}^n \sum_{l=0(l \neq k)}^n (G_{ik} - G_{0k})F_{kl}^{\mu} \right] d_{\mu}(s) \quad \dots\dots\dots (8) \\ (i &= 1, 2, \dots, n) \end{aligned}$$

The formulation for the problem (III) is done in the same way as those of problems (I) and (II).

The preceding arguments lead to the following design procedure:

Step 1 Specify the number and locations of the vibration isolators added to the basic system.

Step 2 Using the specified values of α_i^2 and assuming the values of λ_{ij}^2 , determine $F_{kl}^{\mu}(s)$ according to Eq. (7).

Step 3 Using $F_{kl}^{\mu}(s)$ thus obtained, we calculate the mean square values of $\ddot{x}_i(t)$ and $x_{ij}(t)$. If the constraints Eq. (2) are satisfied, we proceed to the next step. If not, change the values of λ_{ij}^2 and repeat the steps 2 and 3 until the conditions Eq. (2) are satisfied.

Step 4 Using the values of λ_{ij}^2 thus determined, calculate the transfer functions H_{ij}^{μ} and H_i^{μ} .

Step 5 Referring to the relations Eqs. (5)

and (6) and the transfer functions obtained in step 4, realize the isolating force by the feedback of $\ddot{x}_i(t)$ and/or $x_{ij}(t)$ signals.

In the following sections, a few examples will be provided in order to illustrate the application of the procedure.

3. Single-degree-of-freedom system

3.1 System without inherent springs and dashpots

Consider a system subjected to random force excitation as shown in Fig. 1. The type (II) is applicable to the design problem of the optimum isolator for this system. The constraint condition and the performance index for the system are

$$\langle x_{10}(t)^2 \rangle \leq M_{10} \quad (9)$$

$$I = \langle f_{t1}(t)^2 \rangle + \lambda_{10}^2 \langle x_{10}(t)^2 \rangle \quad (10)$$

The equation of motion is

$$m_1 \ddot{x}_1(t) = f_{10}(t) + d_1(t) \quad (11)$$

Also, from Eqs. (8) and (4), we obtain

$$f_{t1}(s) = -f_{10}(s) = -F_{10}^* d_1(s) = -F_{10} d_1(s) \quad (12)$$

$$x_{10}(s) = (1 + F_{10}) d_1(s) / m_1 s^2 \quad (13)$$

Then, calculating the power spectral densities of $\phi_{f_{t1}}$ and $\phi_{x_{10}}$ and substituting them into the optimum condition Eq. (7), we get

$$\frac{\phi_{d_{11}}}{s^2(-s)^2} \{ (s^4 + \lambda^2) F_{10} + \lambda^2 \} = R_1 \quad (14)$$

where $\phi_{d_{11}}$ is the power spectral density of $d_1(t)$ and $\lambda^2 = \lambda_{10}^2 / m_1^2$. If the functional form of $\phi_{d_{11}}$ is specified, the complete form of $F_1(s)$ can be found by the computation procedure given in our previous paper⁽⁵⁾. In the following, we consider two cases of white noise and coloured noise excitations.

(a) White noise excitation

Let us first consider the case of the white noise excitation:

$$\phi_{d_{11}} = S_0 = \text{const.} \quad (15)$$

Then, solving formally Eq. (14), we obtain

$$F_1(s) = \frac{-\lambda^2 + R_1 s^4 / S_0}{(s^2 + \sqrt{2\lambda} s + \lambda)(s^2 - \sqrt{2\lambda} s + \lambda)} \quad (16)$$

Fig. 1 Single-degree-of-freedom system

We see that the roots of the first term in the denominator are in the LHP and those of the second in the RHP. For the system to be stable, all the poles of $F_1(s)$ must be in the LHP and since the mean square value of $f_{t1}(s)$ should be finite, we have

$$F_1(s) = \frac{A_1 s + A_0}{s^2 + \sqrt{2\lambda} s + \lambda} \quad (17)$$

where A_0 and A_1 are undetermined coefficients. Substituting Eq. (17) into Eq. (14) and cancelling out all the poles in the LHP, we obtain

$$A_0 = -\lambda, \quad A_1 = -\sqrt{2\lambda} \quad (18)$$

The transfer functions of the optimum system become

$$\frac{f_{t1}(s)}{d_1(s)} = \frac{\sqrt{2\lambda} s + \lambda}{s^2 + \sqrt{2\lambda} s + \lambda} \quad (19)$$

$$\frac{x_{10}(s)}{d_1(s)} = \frac{1/m_1}{s^2 + \sqrt{2\lambda} s + \lambda} \quad (20)$$

Thus, the power spectral densities $\phi_{f_{t1}}$ and $\phi_{x_{10}}$ are

$$\phi_{f_{t1}} = \left| \frac{\sqrt{2\lambda} s + \lambda}{s^2 + \sqrt{2\lambda} s + \lambda} \right|^2 S_0 \quad (21)$$

$$\phi_{x_{10}} = \left| \frac{1/m_1}{s^2 + \sqrt{2\lambda} s + \lambda} \right|^2 S_0 \quad (22)$$

Using the well-known relation between the mean square value $\langle y(t)^2 \rangle$ and the power spectral density ϕ_y of a quantity $y(t)$,

$$\langle y(t)^2 \rangle = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \phi_y ds \quad (23)$$

the mean square values are calculated as follows:

$$\langle f_{t1}(t)^2 \rangle = \frac{3S_0}{2\sqrt{2m_1}} \sqrt{\lambda_{10}} \quad (24)$$

$$\langle x_{10}(t)^2 \rangle = \frac{S_0}{2\sqrt{2m_1}} \left(\frac{1}{\sqrt{\lambda_{10}}} \right)^3 \quad (25)$$

From Eqs. (24) and (25), we get the plots of $\langle f_{t1}(t)^2 \rangle$ and $\langle x_{10}(t)^2 \rangle$ against λ_{10} as shown in Fig. 2.

Fig. 2 Performance of the optimum isolator with white noise excitation

Next, let us determine the value of λ_{10} satisfying the constraint condition. As seen from Fig. 2, the minimum of $\langle f_{t1}(t)^2 \rangle$ is attained at the maximum value of $\langle x_{10}(t)^2 \rangle$. Thus, the solution of the equation $\langle x_{10}(t)^2 \rangle = M_{10}$ is the optimum value λ_{10M} of λ_{10} . By using λ_{10M} thus determined and Eq. (17), Eq. (12) is reduced to

$$f_{10M}(s) = \frac{-\sqrt{2\lambda_M} s - \lambda_M}{s^2 + \sqrt{2\lambda_M} s + \lambda_M} d_1(s) \quad (26)$$

where $\lambda_M = \lambda_{10M}/m_1$.

Let us now consider the realization of the isolator providing the above isolating force. Since a random exciting force is generally not measurable, the feedback of the measurable variables is the only way to realize the isolating force. This can be done as follows.

Eliminating $d_1(s)$ from Fqs. (20) and (26), we obtain

$$f_{10}(s) = -m_1(\sqrt{2\lambda_M} s + \lambda_M)x_1(s) \quad (27)$$

Taking the inverse Laplace transform of Eq. (27) yields

$$f_{10}(t) = -c_1 \dot{x}_1(t) - k_1 x_1(t) \quad (28)$$

where $c_1 = m_1 \sqrt{2\lambda_M} = \sqrt{2\lambda_{10M} m_1}$ and $k_1 = m_1 \lambda_M = \lambda_{10M}$. Thus, it is seen that the optimum vibration isolator can be mechanized by a spring and a dashpot as shown in Fig. 3.

(b) Coloured noise excitation

Consider the case of the coloured noise excitation:

$$\phi_{d11} = \frac{\omega_0^2}{-s^2 + \omega_0^2} \quad (29)$$

where ω_0 is constant. Substituting Eq. (29) into Eq. (14) and solving formally the equation, we have

$$F_1(s) = \frac{-\lambda^2 + R_1 s^4 (-s^2 + \omega_0^2)/\omega_0^2}{(s^2 + \sqrt{2\lambda} s + \lambda)(s^2 - \sqrt{2\lambda} s + \lambda)} \quad (30)$$

Using the same procedure as the case of the white noise excitation, we obtain

$$F_1(s) = \frac{A_2 s^2 + A_1 s + A_0}{s^2 + \sqrt{2\lambda} s + \lambda} \quad (31)$$

Fig. 3 Configuration of the optimum isolator with white noise excitation

where

$$\left. \begin{aligned} \lambda &= \lambda_{10}/m_1 \\ A_0 &= -\lambda \\ A_1 &= -\sqrt{2\lambda} \\ A_2 &= \frac{-\sqrt{2\lambda} \omega_0 - \lambda}{\omega_0^2 + \sqrt{2\lambda} \omega_0 + \lambda} \end{aligned} \right\} \quad (32)$$

The optimum transfer functions thus obtained are

$$\frac{f_{t1}(s)}{d_1(s)} = -\frac{A_2 s^2 + A_1 s + A_0}{s^2 + \sqrt{2\lambda} s + \lambda} \quad (33)$$

$$\frac{x_{10}(s)}{d_1(s)} = \frac{(1 - A_2)/m_1}{s^2 + \sqrt{2\lambda} s + \lambda} \quad (34)$$

Eliminating $d_1(s)$ from the above equations, we get

$$f_{t1}(s) = \frac{m_1}{A_2 - 1} (A_2 s^2 + A_1 s + A_0) x_{10}(s) \quad (35)$$

Taking the inverse Laplace transform of Eq. (35) yields

$$f_{10}(t) = \frac{m_1}{1 - A_2} \{A_2 \ddot{x}_1(t) + A_1 \dot{x}_1(t) + A_0 x_1(t)\} \quad (36)$$

where $f_{t1}(t) = -f_{10}(t)$ and $x_{10}(t) = x_1(t)$.

As seen from Eq. (36), the optimum isolating force can not be realized by using the spring-dashpot element only. The optimum isolator can be mechanized by the active element as shown in Fig. 4. Figure 5 shows the plots of $\langle f_{t1}(t)^2 \rangle$ and $\langle x_{10}(t)^2 \rangle$ against λ for constant ω_0 . As seen from Fig. 5, when ω_0 is greater than 20π rad/sec, the performance is nearly equal to that for $\omega_0 = \infty$, i.e., the white noise excitation.

3.2 System with inherent springs and dashpots

Let us design the optimum isolator for the system with the inherent spring and dashpot as shown in Fig. 6. The equation of the motion for the system is

$$m_1 \ddot{x}_1(t) = -C_{11} \dot{x}_1(t) - K_{11} x_1(t) + f_{10}(t) + d_1(t) \quad (37)$$

Fig. 4 Configuration of the optimum isolator with coloured noise excitation

where $C_{11}=c_{10}$ and $K_{11}=k_{10}$. The transmitted force to the foundation becomes

$$f_{t1}(t) = -f_{10}(t) + C_{11}\dot{x}_1(t) + K_{11}x_1(t) \quad (38)$$

The performance index is

$$I = \langle f_{t1}(t)^2 \rangle + \lambda_{10}^2 \langle x_{10}(t)^2 \rangle \quad (39)$$

Then, we have the following condition for I to be minimum:

$$\phi_{d11} \{ (m_1^2 s^4 + \lambda_{10}^2) F_1 - m_1 C_{11} s^3 - m_1 K_{11} s^2 + \lambda_{10}^2 \} / D \bar{D} = R_1 \quad (40)$$

where $D(s) = m_1 s^2 + C_{11} s + K_{11}$ and $f_{10}(s) = F_1 d_1(s)$.

In the case of the white noise excitation $\phi_{d11} = S_0$, we find

$$F_1(s) = \frac{A_1 s + A_0}{s^2 + \sqrt{2\lambda_{10}/m_1} s + \lambda_{10}/m_1} \quad (41)$$

where

$$\left. \begin{aligned} A_0 &= K_{11}/m_1 - \lambda_{10}/m_1 \\ A_1 &= C_{11}/m_1 - \sqrt{2\lambda_{10}/m_1} \end{aligned} \right\} \quad (42)$$

Fig. 5 Performance of the optimum isolator with coloured noise excitation ($m_1=1$)

Fig. 6 System with inherent spring and dashpot

Thus, the optimum transfer functions are

$$\frac{x_{10}(s)}{d_1(s)} = \frac{1}{m_1 s^2 + \sqrt{2\lambda_{10} m_1} s + \lambda_{10}} \quad (43)$$

$$\frac{f_{10}(s)}{d_1(s)} = \frac{A_1 s + A_0}{s^2 + \sqrt{2\lambda_{10}/m_1} s + \lambda_{10}/m_1} \quad (44)$$

Eliminating $d_1(s)$ from the above equations, we obtain

$$f_{10}(s) = -(\sqrt{2\lambda_{10} m_1} - C_{11}) s x_{10}(s) - (\lambda_{10} - K_{11}) x_{10}(s) \quad (45)$$

As seen from Eqs. (27) and (45), the optimum isolating force for the system with the inherent spring and dashpot is equal to the force, i.e., the force caused by the inherent spring and dashpot as subtracted from the optimum isolating force for the system without them. Therefore, in designing the isolators for a system with inherent springs and dashpots, it is sufficient to design the isolators for a system without them.

4. Two-degree-of-freedom system

Consider the design of the optimum isolators for a two-degree-of-freedom system subjected to random foundation motion as shown in Fig. 7. The design problem now becomes a type (I). Let us specify the constraint conditions and the performance index as follows:

$$\langle x_{10}(t)^2 \rangle \leq M_{10} \text{ and } \langle x_{21}(t)^2 \rangle \leq M_{21} \quad (46)$$

$$I = \alpha_1^2 \langle \dot{x}_1(t)^2 \rangle + \alpha_2^2 \langle \dot{x}_2(t)^2 \rangle + \lambda_{10}^2 \times \langle x_{10}(t)^2 \rangle + \lambda_{21}^2 \langle x_{21}(t)^2 \rangle \quad (47)$$

In this problem, we do not consider the constraint condition of the relative displacement $x_{20}(t)$. From Eq. (1), we obtain the equations of motion

$$m_1 \ddot{x}_1(t) = f_{10}(t) - f_{21}(t) \quad (48)$$

$$m_2 \ddot{x}_2(t) = f_{21}(t) \quad (49)$$

Also, from Eq. (4), we get

$$f_{10}(s) = F_1 \ddot{x}_0(s) \quad (50)$$

$$f_{21}(s) = F_2 \ddot{x}_0(s) \quad (51)$$

Fig. 7 Two-degree-of-freedom system

where $\ddot{x}_0(s) = d_0(s)$. Using Eqs. (48)~(51), the transfer functions of the system are

$$\ddot{x}_1(s)/\ddot{x}_0(s) = F_1 - \mu F_2 \quad (52)$$

$$\ddot{x}_2(s)/\ddot{x}_0(s) = F_2 \quad (53)$$

$$x_{10}(s)/\ddot{x}_0(s) = (F_1 - \mu F_2 - 1)/s^2 \quad (54)$$

$$x_{21}(s)/\ddot{x}_0(s) = \{(1 + \mu)F_2 - F_1\}/s^2 \quad (55)$$

Calculating the power spectral densities for the system and substituting them into Eq. (7), we obtain the following optimum conditions:

$$A_{11}\phi_{\ddot{x}_0}F_1 + A_{12}\phi_{\ddot{x}_0}F_2 + \Theta_1\phi_{\ddot{x}_0} = R_1 \quad (56)$$

$$A_{21}\phi_{\ddot{x}_0}F_1 + A_{22}\phi_{\ddot{x}_0}F_2 + \Theta_2\phi_{\ddot{x}_0} = R_2 \quad (57)$$

where

$$A_{11} = \alpha_1^2 + (\lambda_{10}^2 + \lambda_{21}^2)/s^4$$

$$A_{12} = A_{21} = -[\mu\alpha_1^2 + \{\mu\lambda_{10}^2 + (1 + \mu)\lambda_{21}^2\}/s^4]$$

$$A_{22} = \alpha_1^2\mu^2 + \alpha_2^2 + \{\lambda_{10}^2\mu^2 + \lambda_{21}^2(1 + \mu)^2\}/s^4$$

$$\Theta_1 = -\lambda_{10}^2/s^4$$

$$\Theta_2 = \mu\lambda_{10}^2/s^4$$

In the case of the white noise excitation $\phi_{\ddot{x}_0} = S_0$, we find

$$F_1(s) = (A_3s^3 + A_2s^2 + A_1s + A_0)/D(s) \quad (58)$$

$$F_2(s) = (B_3s^3 + B_2s^2 + B_1s + B_0)/D(s) \quad (59)$$

$$D(s) = (s - r_1)(s - r_2)(s - r_3)(s - r_4)$$

where $A_0 \sim A_3$ and $B_0 \sim B_3$ are undetermined coefficients and r_1, r_2, r_3 and r_4 are the LHP roots of equation $A_{11}A_{22} - A_{12}A_{21} = 0$. Substituting Eqs. (58) and (59) into Eqs. (56) and (57) and cancelling out all the poles in the LHP, we have

$$B_0 = r_1r_2r_3r_4 \triangleq \gamma_0$$

$$B_1 = -\{r_1r_2(r_3 + r_4) + r_3r_4(r_1 + r_2)\} \triangleq \gamma_1$$

$$B_2 = (r_1 + r_2)(r_3 + r_4) + \frac{r_1^4r_3r_4 - r_3^4r_1r_2}{r_1^4 - r_3^4}$$

$$B_3 = -\frac{r_1^4(r_3 + r_4) - r_3^4(r_1 + r_2)}{r_1^4 - r_3^4}$$

$$A_0 = (1 + \mu)B_0$$

$$A_1 = (1 + \mu)B_1$$

Fig. 8. RMS values of the responses versus $\rho (\eta_1 = \eta_2 = 1)$

$$A_2 = -\frac{\lambda_{10}^2(r_1r_2 - r_3r_4)}{\alpha_1^2(r_1^4 - r_3^4)} + (1 + \mu)B_2$$

$$A_3 = \frac{\lambda_{10}^2\{(r_1 + r_2) - (r_3 + r_4)\}}{\alpha_1^2(r_1^4 - r_3^4)} + (1 + \mu)B_3$$

Then, the transfer functions of the optimum system become

$$\ddot{x}_1(s)/\ddot{x}_0(s) = (C_3s^3 + C_2s^2 + \gamma_1s + \gamma_0)/D(s) \quad (60)$$

$$\ddot{x}_2(s)/\ddot{x}_0(s) = (B_3s^3 + B_2s^2 + \gamma_1s + \gamma_0)/D(s) \quad (61)$$

$$x_{10}(s)/\ddot{x}_0(s) = -\{s^2 + (\gamma_3 - C_3)s + (\gamma_2 - C_2)\}/D(s) \quad (62)$$

$$x_{21}(s)/\ddot{x}_0(s) = -\{(C_3 - B_3)s + (C_2 - B_2)\}/D(s) \quad (63)$$

where

$$C_3 = A_3 - \mu B_3$$

$$C_2 = A_2 - \mu B_2$$

$$\gamma_3 = -(r_1 + r_2 + r_3 + r_4)$$

$$\gamma_2 = (r_1 + r_2)(r_3 + r_4) + r_1r_2 + r_3r_4$$

Referring to the values of $A_0 \sim A_3$ and $B_0 \sim B_3$ in Eq. (60) to Eq. (63), we see that the transfer

Fig. 9. RMS values of the responses versus $\eta_1 (\rho = \eta_2 = 1)$

Fig. 10. RMS values of the responses versus $\eta_2 (\rho = \eta_1 = 1)$

functions are dependent on α_1^2 , α_2^2 , λ_{10}^2 and λ_{21}^2 and independent of μ . Then, the mean square values of accelerations and relative displacements are calculated as functions of α_1^2 , α_2^2 , λ_{10}^2 and λ_{21}^2 . The performance index can be re-written as follows:

$$I = \langle \ddot{x}_2(t)^2 \rangle + \rho^2 \langle \ddot{x}_1(t)^2 \rangle + \eta_1^2 \langle x_{10}(t)^2 \rangle + \eta_2^2 \langle x_{21}(t)^2 \rangle \dots (64)$$

(a) RMS values of the relative displacements versus η_1 and η_2 ($\rho=1$)

(b) RMS values of the accelerations versus η_1 and η_2 ($\rho=1$)

Fig. 11

where $\rho^2 = \alpha_1^2 / \alpha_2^2$, $\eta_1^2 = \lambda_{10}^2 / \alpha_2^2$ and $\eta_2^2 = \lambda_{21}^2 / \alpha_2^2$. The plots of typical rms values of accelerations and relative displacements versus ρ with $\eta_1 = \eta_2 = 1$, versus η_1 with $\rho = \eta_2 = 1$ and versus η_2 with $\rho = \eta_1 = 1$ are given in Fig. 8, Fig. 9 and Fig. 10 respectively. Fig. 11 (a) and (b) are the plots of the rms values of accelerations and relative displacements against typical η_1 and η_2 when the weighting factor $\rho=1$. It is found from Fig. 11 (b) that $\sqrt{\langle \ddot{x}_2(t)^2 \rangle}$ is smaller than $\sqrt{\langle \ddot{x}_1(t)^2 \rangle}$ for all the cases calculated.

An inspection of Eqs. (60) through (63) leads us to the optimum isolators by conventional passive elements such as spring and dashpots, as discussed in the previous paper⁽²⁾. The isolators can be implemented by active elements consisting of feedback transducers and actuators.

5. Conclusion

The authors' optimization procedure has been briefly summarized for designing optimum linear vibration isolators under random excitations. It is applied to single-degree-of-freedom systems with white or coloured noise excitation and a two-degree-of-freedom system with white noise foundation motion. It is concluded that the optimum isolator for a single-degree-of-freedom system subjected to white noise excitation can be mechanized by the conventional spring-dashpot element, but that the mechanization of optimum isolators for the other systems can only be realized by use of active elements.

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