Deflections and Moments Due to a Concentrated Load on a Rack-Shaped Cantilever Plate with Finite Width for Gears*

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In helical gear design, deflections and moments of rack-shaped cantilever with finite width, must be taken into consideration.

This paper contains numerical solutions for deflections and moments of rack-shaped cantilever with finite width. These solutions are obtained with applying calculus of finite differences to the basic differential equation.

For a finite rack-shaped cantilever, deflections and moments are closely related to its pressure angle and profile of its section.

The numerical solutions showed good agreement with the experimental results of rack shaped cantilever simulating a gear tooth.

1. Introduction

The stiffness of a helical gear tooth is necessary to determine the tooth profile correction and the crowning.

The deflections and bending moments of a cantilever plate loaded by a concentrated force have been discussed by MacGregor(2). His solution of the problem was obtained by applying the thin plate theory to a cantilever plate with infinite width and constant thickness, in the case when the load acted at the top free edge of the plate.

An exact solution in terms of improper integrals for deflections and moments due to a transverse concentrated load acting on an arbitrary point of an infinite width cantilever with constant thickness was obtained by Jaramillo(6) and Fujita(14).

The thin-plate theory does not suffice for the problems of helical gear tooth. The first reason is that gear tooth thickness is in the same order as its whole depth. The second is that usual gear face width is finite.

Ishikawa(13) obtained the solution for a rectangular plate, elastically jointed to a supporting body under the condition of uniform load being parallel to the free edge, with a new stress function.

The flexural and the torsional rigidities at the elastically built-in edge were found by Weber and Banaschek(8). Their solution of the rigidity was obtained under the condition of uniform load being parallel to the edge and of infinite width of cantilever plate, making use of strain-energy method.

Most of literatures on tooth deflection deal with the solution based on the thin plate theory or uniform load being parallel to the tip free edge. In other words, they are based on an infinite width of constant thickness or two-dimensional problems. However, it is well known that the tooth face in practice is usually of considerable width and that the load distributes on a contact line which is not parallel to the tip edge in helical gears. The cantilever beam and the thin-plate approach are, therefore, no longer justified for these helical gear design problems.

Holl(1) used the method of finite differences to obtain approximate solutions for moments and deflections in a cantilever thin-plate with finite width.

Wellauer and Seireg(10) gave a semi-empirical solution for a finite cantilever plate under transverse load at any location on its surface. The solution, which was based on the principle of superposition and a proposed “moment-image” method, showed agreement with the results of strainage investigations on cantilever plates simulating a gear tooth.

The author(12) gave semi-theoretical solutions for a thick cantilever-plate problem whose thickness was in the same order as its length, under the
condition that transverse loads act at any location on the surface of a finite width cantilever. This semi-theoretical solutions of deflections were checked experimentally with good agreement.

The theory of a constant thickness plate does not suffice for the problems of gear yet. The rectangular plate of variable thickness was given by Olsson(11).

The flexure of infinite rectangular plates of varying thickness simulating the gear tooth, were discussed by Conway(9). This problem has been solved by Aida and Fujio(10).

Kajita and Naruoka(11) gave a finite element approach to the analysis of bending and vibration of rectangular plates with variable thickness under uniformly distributed load.

In this work, a semi-theoretical solution is given for the variable thickness cantilever plate problem under transverse loads at any location on the plate surface. These solutions, which are based on the work of Holl, and of Weber and Banashek, were found using the method of finite differences and new boundary conditions.

2. Notations

\( w \): deflection
\( E \): modulus of elasticity
\( \nu \): Poisson's ratio
\( D(y) \): plate rigidity
\( T(y) \): plate thickness
\( L \): width of cantilever plate
\( B \): length of cantilever plate
\( \theta \): pressure angle of rack
\( m_0 \): module
\( M_0,M_2 \): bending moment per unit length
\( V_1,V_2 \): modified shearing forces
\( U = \frac{P^2w}{\partial_x^2} + \frac{\partial^2w}{\partial y^2} \)
\( A_1,A_2,A_3 \): derivative of plate rigidity
\( q(x,y) \): load per unit area
\( P(x,y) \): concentrated load
\( p(x,y) \): reference load
\( \lambda_0,\lambda_2 \): net width
\( \alpha = (\lambda_0/\lambda_2)^{\nu}, \beta = (1-\nu)\alpha \)
\( B_a,C_{--N,O} \): pattern names at each load point
\( a,b,...1,m \): positions names of net points
\( w_a,w_b,...w_1,w_m \): deflections of each net point
\( K \): coefficient of deflection

A,B,...G,H,W: matrix

3. Basic equation

3.1 The basic differential equation

The flexural rigidity of a varying thickness plate being shown in Fig. 1, is given by:

\[
D(y) = \frac{E}{12(1-\nu^2)} \quad T(y)^3 \quad (1)
\]

When a cantilever plate with varying thickness is a rack shaped cantilever, the thickness of plate \( T(y) \) is given by:

\[
T(y) = T_0(1-2\alpha \tan \theta) \quad (2)
\]

The deflection of a plate having a thickness variation which is a function of \( y \) only, is governed by the equation(10,11):

\[
D(y)\frac{p^2w}{\partial_y} + 2\frac{\partial D(y)}{\partial y} \frac{\partial^3w}{\partial y^3} + \frac{\partial^3w}{\partial x^2 \partial y} = q(x,y) \quad (3)
\]

3.2 Boundary conditions

We begin the discussion of boundary conditions with the case of a rectangular plate having a thickness variation, as shown in Fig. 1.

This homogeneous elastic plate of width \( L \) and length \( B \) is clamped horizontally along one longitudinal edge.

(i) Free edge. Along the free edge \( y = B \),

\[
M_y = -D(y) \left( \frac{\partial^3w}{\partial y^3} + \nu \frac{\partial^3w}{\partial x^2 \partial y} \right) = 0 \quad (4)
\]

\[
V_x = -D(y) \left( \frac{\partial^2w}{\partial y^2} + (2-\nu)\frac{\partial^2w}{\partial x \partial y} \right) = 0 \quad (5)
\]

(ii) Both transverse free edges. Along both the transverse free edges \( x = 0, x = L \),

\[
M_y = -D(y) \left( \frac{\partial^3w}{\partial y^3} + \nu \frac{\partial^3w}{\partial x^2 \partial y} \right) = 0 \quad (6)
\]

\[
V_x = -D(y) \left( \frac{\partial^2w}{\partial x \partial y} + 2(1-\nu) \frac{\partial^2w}{\partial x^2} \right) - 2(1-\nu) \frac{\partial D(y)}{\partial y} \frac{\partial^2w}{\partial x \partial y} = 0 \quad (7)
\]

And the forces concentrated at corners of plate must be zero,

\[
M_{2x} = -(1-\nu)D(y) \frac{\partial^2w}{\partial x \partial y} = 0 \quad (8)
\]

(iii) Built-in edge. For the problems of the gear tooth, assuming the built-in edge to be a semi-elastically built-in edge, the rotation is directly

Fig. 1 Labelling of net points
proportional to the moment, along y = 0,
\[
\left( \frac{\partial w}{\partial y} \right)_{\text{edge}} = -k_i M_e.
\]
where
\[ k_i = 18(1-\nu^2) / \pi E T_0^2 \]
This constant \( k_i \) is the flexural rigidity at the built-in edge. This rigidity was given by Weber and Banascheck's solution\(^{(1)}\) which was obtained under condition of uniform load being parallel to the edge and infinite length cantilever plate making use of strain-energy method.

The other boundary condition is the deflection along this edge to be zero,
\[
w_{\text{edge}} = 0
\]

\[ \text{(10)} \]

4. Calculus of finite differences

4-1 The replacement of basic partial differential equation with finite differences equation

Differential equation will be replaced with finite differences equations. The plate is divided into nets of width \( \lambda_x \) and \( \lambda_y \) as shown in Fig. 1. Thirteen net points are respectively named a, b, ..., m, as indicated in Fig. 1.

When Laplace's equation is replaced with finite differences equations, Laplace's equation is transformed as\(^{(2)}\)\(^{(3)}\) : \[
U_a = \left( \frac{\partial^2 w}{\partial y^2} \right)_a
\]
\[ = \frac{1}{\lambda_y^2} \left( \alpha (w_c + w_d) - 2(1+\alpha) w_a + w_b + w_c \right) \]
\[ \text{(11)} \]
\[
\left( \frac{\partial^2 U}{\partial y^2} \right)_a = \frac{1}{\lambda_y^2} \left( \alpha (U_c + U_d) - 2(1+\alpha) U_a \right)
\]

\[ \text{(12)} \]

where
\[ \alpha = (\lambda_y / \lambda_x)^2 \]

In like manner with Eq. (11) and Eq. (12), Equation (3) is transformed as:
\[
\lambda_x^2 \left( \frac{\partial^2 U}{\partial x^2} \right)_a + A1 \lambda_x^2 (U_b - U_a) + A2 \lambda_x^2 U_a
\]
\[ + A3 \left( w_c - w_a + w_b \right) = \frac{\lambda_y^2 q(\gamma y)}{D(y)} \]
\[ \text{(13)} \]
where \( A1, A2, A3 \) are derivatives of plate rigidity; these are obtained as:
\[
A1 = \left( \frac{\partial D(y)}{\partial y} \right) \left( \frac{\lambda_y}{D(y)} \right)
\]
\[ A2 = \left( \frac{\partial^2 D(y)}{\partial y^2} \right) \left( \frac{\lambda_y^2}{D(y)} \right) \]
\[ A3 = \left( \frac{\partial^3 D(y)}{\partial y^3} \right) \left( -\frac{(1-\nu)\lambda_y^2 \alpha}{D(y)} \right) \]
\[ \text{(14)} \]

Equation (13) is the difference analogy of Eq. (3).

4-2 The finite differences equations for each boundary condition

The basic difference analogy is defined with deflections of the thirteen net points, as indicated in Eq. (13).

When the center net point a is selected near the free edge or on the free edge, its pattern has same imaginary net points which are defined net points to be outside cantilever surface. For instance, B pattern in Fig. 2 has an imaginary point j.

The deflection \( w_j \) of imaginary point j, must be changed by some net points which are on the cantilever surface. For elimination of this the boundary condition of Eq. (9) shall be used. So, B pattern is defined with deflection of twelve net points, except the imaginary net point j.

The finite differences equations have twelve kinds of pattern according as where the center net point a is selected on the cantilever surface. The twelve kinds of pattern are respectively named B, C, ..., N and O, as indicated in Fig. 2.

The number \( r \) of M pattern, H pattern, and C pattern is affected by the relation between net width and length of cantilever plate.

4-3 The elimination of imaginary point

(imaginary net point f) D pattern, I pattern, and N pattern have deflection of imaginary point f as indicated in Fig. 3 (1). For elimination of deflection \( w_f \), Eq. (4) is expanded
\[
w_f = 2w_b - w_a + (\beta - \alpha)(w_a - 2w_b + w_c)
\]
\[ \text{(15)} \]
where \( \beta = (1-\nu) \alpha \).

That is, deflection \( w_f \) of imaginary point f can be evaluated with \( w_b, w_a, w_c \) and \( w_f \).
(imaginary net points b, f, g, m) E pattern, J pattern and O pattern have deflections of imaginary points b, g, m and f, as indicated in Fig. 3 (ii).

From the boundary condition of Eq. (5),
\[ \frac{\partial U}{\partial y} + (1 - \nu) \frac{\partial}{\partial y} \left( \frac{\partial^2 w}{\partial x^2} \right) = 0 \]  
(16)
is given. This Eq. (16) is translated into difference analogy,

\[ \lambda_2 U_s = \lambda_2 U_e - \beta (w_b - w_c) \]
\[ -2(w_b - w_c) + (w_m - w_d) \]  
(17)

\( U_s \) is obtained from this equation.

Deflections of imaginary net points b, f, g and m can be eliminated at the same time, to make use of Eq. (17) which is a difference analogy of \( U_e \).

( imaginary net point j) B pattern, G pattern and L pattern have deflection of imaginary point j, as indicated in B pattern of Fig. 2. For eli-

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Fig. 4 Diagrams of each pattern
mination of imaginary point \( j \), Eq. (9) becomes (12)

\[
\omega_j = BM_A w_{i_1} + BM_B w_{i_2} + BMC w_{i_3} + \ldots 
\]

where

\[
BM_A = (-2BM + \lambda_3)/(2BM + \lambda_3)
\]

\[
BM_B = 4BM(1 + \alpha \nu)/(2BM + \lambda_3)
\]

\[
BMC = -2BM\alpha(\nu)/(2BM + \lambda_3)
\]

\[
BM = 3T \sqrt{2}/2\pi
\]

That is, deflection \( \omega_j \) of imaginary point \( j \) can be evaluated with \( \omega_{i_1}, \omega_{i_2}, \omega_{i_3}, \) and \( \omega_k \).

For elimination of any deflection at any imaginary net points, deflections at imaginary points can be changed with deflections at some net points which are on the surface of cantilever.

For this elimination, the boundary conditions of Eqs. (4) ~ (10) will be used.

4-4 Diagrams of each pattern

\( \alpha \)C pattern has not an imaginary point. Every deflection at every net point of \( \alpha \)C pattern must satisfy the relation of Eq. (13). The diagram of \( \alpha \)C pattern in Fig. 4 is defined the same as Eq. (13).

Every net point is defined with pattern name and position name. For instance, the name of center point of \( \alpha \)C pattern is \( \alpha \alpha \). That is, \( \alpha \alpha \) means \( 6\alpha^2 + 8x + 6 - 2(1 + \alpha)A2 - 2A3 \).

4-5 Plate rigidity

Equation (13) is the difference analogy of Eq. (3). When basic equation is replaced by Eq. (13), the reference load \( p \) is obtained from the concentrated load \( P \). (i) \( p = P/\lambda_3 \lambda_3 \) when loaded point is on the plate. (ii) \( p = 2P/\lambda_3 \lambda_3 \) when loaded point is on the free edge. (iii) \( p = 4P/\lambda_3 \lambda_3 \) when loaded point is on the corner.

When the concentrated load is modified to the reference load per unit area, it must be considered that the plate rigidity of a rack shaped cantilever is variable.

In the case that the loaded point of concentrated load is selected on the tip free edge, the reference load per unit area acts on the plate surface as indicated in Fig. 5 (ii).

So, reference plate rigidity on the tip free edge is defined as the mean value of the plate rigidity all over the plate surface where reference load acts, as indicated in Fig. 5 (ii).

The reference plate rigidity on the tip free edge \( D_{top} \) can be evaluated as

\[
D_{top} = \frac{ET_{top}^3}{12(1 - \nu^2)} \frac{T_{top}^4}{4\tan \beta \lambda_y} \left( \left( \frac{T}{T_{top}} \right)^4 - 1 \right)
\]

The relationship between the deflection when a concentrated load acts on the tip free edge and when a concentrated load acts on the inner plate surface, must satisfy the reciprocal theory (15) (17).

When a concentrated load acts on inner plate surface, the plate rigidity is defined since the reciprocal theory is satisfied with the relationship between the deflection when a concentrated load acts on the tip free edge and when a concentrated load acts on inner plate surface. This reference plate rigidity \( D' \) is defined,

\[
D'(y) = \xi D(y) \tag{20}
\]

where \( \xi \) is the coefficient for modification when the reciprocal theory is satisfied between deflections. This coefficient \( \xi \) can be evaluated from numerical solutions, as described later.

4-6 Deflections of a rack shaped cantilever with finite width

A plate is divided into unit rectangular meshes of \( \lambda x \) and \( \lambda y \) which are \( \Psi \times \Phi \) pieces, as shown in Fig. 6. Every net point on the plate has a pair of coordinates associated with the number pair \((\phi, \psi)\). Deflection of every net point \((\phi, \psi)\) is defined \( W_{\phi, \psi} \).

All deflections on the same vertical line are defined as

\[
W_{\phi, \psi} = \{w_{\phi, \psi}, \ldots, w_{\phi, \psi}, \ldots, \}
\]

For computing, it is convenient to put all the coefficients in Fig. 4 into some classes.

For instance, let us discuss the coefficients of patterns whose center net point is on the side free edge. All the coefficients on the side free edge are defined as one group \( A \) matrix,

\[
A = \begin{bmatrix}
L_a & L_b & L_f & M_d & M_a & M_b & M_f \\
M_j & M_d & M_a & M_b & M_f & N_j & N_d & N_a & N_b & O_j & O_d & O_a
\end{bmatrix}
\]

\( \ldots \)
Equation (22) is defined simply as,

\[
A = \begin{bmatrix}
L_a & L_b & L_f \\
M_d & M_a & M_b & M_f \\
N_j & N_d & N_a & N_b \\
O_j & O_d & O_a \\
\end{bmatrix}
\]  

With the same expression of simplification to be shown between Eq. (22) and Eq. (23), all the coefficients of Fig. 4 are defined each as matrix,

\[
B = \begin{bmatrix}
H_k & H_e & H_m \\
I_k & I_e & I_m \\
J_k & J_e & J_m \\
G_b & G_h \\
\end{bmatrix}
\]  

\[
C = \begin{bmatrix}
G_e & G_m \\
I_k & I_h \\
J_l & J_k \\
G_l \\
\end{bmatrix}
\]  

\[
D = \begin{bmatrix}
G_m & G_l \\
I_k & I_n \\
J_l & J_n \\
O_i \\
\end{bmatrix}
\]  

\[
E = \begin{bmatrix}
H_d & H_a & H_b & H_f \\
I_j & I_d & I_a & I_b \\
J_j & J_d & J_a \\
B_c & B_g \\
\end{bmatrix}
\]  

\[
F = \begin{bmatrix}
C_i & C_c & C_g \\
D_i & D_c & D_g \\
E_i & E_c \\
\end{bmatrix}
\]  

\[
G = \begin{bmatrix}
L_h & M_h & N_h & O_h \\
B_a & B_b & B_f \\
C_a & C_b & C_f \\
D_j & D_d & D_a & D_b \\
E_j & E_d & E_a \\
\end{bmatrix}
\]  

\[
H = \begin{bmatrix}
L_h & M_h & N_h & O_h \\
B_a & B_b & B_f \\
C_a & C_b & C_f \\
D_j & D_d & D_a & D_b \\
E_j & E_d & E_a \\
\end{bmatrix}
\]  

The deflections of every net point (\(\phi, \psi\)) must satisfy all relations among all coefficients in Fig. 4, under the condition that transverse loads act at any location. Thus, all the coefficients on the plate must satisfy Eq. (31)

\[
\begin{bmatrix}
A & D & G \\
B & E & F \\
C & F & H & F \\
C & F & H & C \\
C & F & H & C \\
C & F & H & F \\
C & F & E & B \\
G & D & A \\
\end{bmatrix}
\begin{bmatrix}
W_0 \\
W_1 \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
W_{n-1} \\
W_n \\
\end{bmatrix} = \frac{\lambda^2 p(x,y)}{D'}
\]  

Fig. 7. Deflections of 14.5° pressure angle rack shaped cantilever
5. Numerical solutions of deflections

5.1 Deflections and pressure angle

The deflections under the condition of same pressure angle rack-shaped cantilever, are defined with the coefficient of deflections, to make use of module $m_n$. That is,

$$K_{ps} = \frac{w_{ps}}{(P/m_n)} \quad (32)$$

where $K_{ps}$ is the coefficient of deflections, and $m_n$ is a normal module.

The coefficients of 14.5 degrees pressure angle rack shaped cantilever with finite width are shown in Fig. 7.

The coefficients of 20 degrees pressure angle rack shaped cantilever with finite width are shown in Fig. 8.

When pressure angle is varied, the deflections of module 11 rack shaped cantilever are shown in Fig. 9, under the condition that a concentrated load is 720 kg.

5.2 The coefficient for deflections

The deflections on the y-axis are shown in Fig. 10. When a load acts on the free edge, deflections are calculated with use of plate rigidity which is defined by Eq. (19). When a load acts on 18.6 mm point, the deflections on the free edge (S point) must coincide with the deflections of S point. Under this condition, the coefficient for modification $\xi$ is obtained to be 0.57 from Eq. (20).

The relation between the pressure angle and the coefficient for modification is shown in Fig. 11.

6. Distribution of bending moments

6.1 Bending moments

Bending moment $M_y$ on the plate is obtained as

$$M_y = -D(y) \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (33)$$

where $D(y)$ is the plate rigidity at the point where bending moment is to be calculated.

The finite differences analogy of Eq. (33) is

$$M_y = -D(y) \left( \frac{w_{y} - 2w_{y} + w_{y}}{\lambda y^2} \right)$$

Fig. 8 Deflections of 20° pressure angle rack shaped cantilever

(i) $L/B = 0.2$

(ii) $L/B = 0.3$
6.2 Distribution of bending moments along the built-in edge

Distributions of bending moments along the built-in edge are significant for gear design. These distributions can be obtained from boundary conditions Eq. (9) and Eq. (18), that is,

\[ M_y = \frac{1}{k_1} \left( 1 - BMA \right) \frac{w_b}{2k_y} \]

Equation (34) and (35) can be simplified as,

\[ M_y = \gamma P \]

where \( \gamma \) is the coefficient of moment distribution.

The deflections all over the plate surface have been computed from Eq. (31). So this coefficient \( \gamma \) can be evaluated for these deflections.

7. Numerical solutions of bending moment

7.1 Bending moments of a flat plate cantilever

When the pressure angle is zero, a rack shaped cantilever becomes a flat plate cantilever.
In the boundary condition of flat plate cantilever on built-in edge, the flexural rigidity $k_1$ is affected by the ratio of plate thickness $T$ to length $B$ and the ratio of width $L$ to length $B$. So the coefficient of moment distribution is affected by these ratios.

The moment coefficients of a thick plate cantilever whose thickness is in the same order as its length, and of a thin plate cantilever are shown in Fig. 12.

In Fig. 12, numerical solutions of a thin plate cantilever are calculated under the condition that boundary condition on built-in edge is $\partial w/\partial y=0$. Numerical solutions of $L/B=3.0$ thin plate are in satisfactory agreement with Jaramillo's analytical solutions.

The moment coefficients of a thick plate cantilever are very different from those of a thin plate, as observed in Fig. 12.

The numerical solutions of a thick plate cantilever are shown in Fig. 13.

7.2 The bending moments of rack shaped cantilever

The numerical solutions of a 20 degrees pres-
sure angle rack shaped (whole depth $2.25 m_a$, tooth thickness on pitch line $\pi m_a/2$) cantilever, are shown in Fig. 14. These solutions can be obtained from Eq. (35) by use of Eq. (31) and (32).

In solutions of a rack shaped cantilever with $L/B=3.0$, when a single load acts on the center of its width, its curve becomes gentle and of very small order near both the transverse free edges.

For this reason, the moment distributions of a finite width cantilever whose ratio of plate width $L$ to whole depth $B$ is more than three, can be used as the distributions of the moments of a rack shaped cantilever with $L/B=3.0$.

The correlation between the moment distributions along the built-in edge and the pressure angle of cantilever, is shown in Fig. 15.

8. Experiment for deflections

8.1 Equipment for measuring cantilever flexibility

The equipment for measuring rack shaped cantilever flexibility and test rack shaped cantilever are illustrated in Fig. 16.

Test rack shaped cantilever and cantilever for measuring apparatus base are cut out and ground from a solid piece of hardened SKS2 (tool steel) whose dimensions are $150 \text{ mm} \times 200 \text{ mm} \times 240 \text{ mm}$, as shown in Fig. 16. Its hardness is 55 $H_R$. There is 1 mm clearance between base and cantilever for measuring apparatus base, because the effect of deflections with surface contact must be avoided.

Test rack-shaped cantilevers with $L/B=8, 4, 3, 2$ and 1 were provided.

The data of rack-profile are: module 11, pressure angle 20 degrees, whole depth $2.25 m_a$ and tooth thickness $\pi m_a/2$, that is, it is JIS standard rack profile. The bottom land of rack shaped cantilever was ground with radius 3 mm whose radius was considered not to influence on the deflections.

There are $3^o$ counter bores on surface, because the concentrated load is applied in the perpendicular direction of elastic surface.

The deflections are measured with electric micro-indicators. The measured positions are determined with block gages, as shown in Fig. 16.

The electric micro-indicators required for measuring are set parallel in the direction of cantilever width.

8.2 The direction of measuring deflections

As shown in Fig. 16, the deflections which must be measured in the direction perpendicular to an elastic plate, are measured upon the oblique surface of compression side. So strictly speaking, deflections must be measured upon the bit measuring surfaces which are cut on compression side in parallel to the elastic plate.

When a large number of these bit measuring surfaces are cut on compression side surface, the rack shaped cantilever does not show normal deflections. For comparison between the deflections which are measured upon the oblique surfaces and parallel surfaces, an experiment was performed as shown in Fig. 17.

Within the limits of this experiment, it is clear that the error due to this measuring surface is at most two or three per cent.

So the deflections of a rack shaped cantilever have been.
measured upon the oblique surface.

8.3 Distribution of deflection in the direction of cantilever width

The experimental results which were obtained with the equipment shown in Fig. 16 and in the way described above, are shown by the dotted line in Fig. 18.

When a concentrated load acts on 1.5 mm inner line from tip free edge, as shown in Fig. 15, the experimental results are obtained under the next best condition that a concentrated load acts on the tip free edge. So under the condition that a concentrated load acts on free edge, the experimental results which were shown in Fig. 17, are extrapolated by the method whose details will be described in Fig. 19.

In the direction of cantilever width, there are some differences between experimental and theoretical positions where a concentrated load acts, as shown in Fig. 18.

When these differences are considered, the theoretical results agree well with the experimental values.

8.4 The deflection curves in the direction of cantilever height

The experimental results along the center section of rack shaped cantilever with $L/B=3.0$, are shown in Fig. 19.

When a 720 kg concentrated load acts on 1.5 mm, 6.2 mm, 12.4 mm and 18.6 mm points from the free edge, the experimental results are shown by the lines. The deflections at a point where a concentrated load acts on, are shown by the dotted line.

When a concentrated load acts on the free edge, from the condition that the reciprocal theory must be satisfied among these curves, results are extrapolated on free edge from every curve. These are obtained as shown by the broken line.

Comparison of theoretical value and experimental results, in direction of cantilever height, is made in Fig. 20.

The theoretical curve almost coincides with the experimental results in the region where the height of a rack shaped cantilever contacts each other as gear tooth.
9. Experiment for moment distribution

9-1 Stresses along the built-in edge

Two test models cut from solid pieces of steel were used in the investigation.

The thick cantilever plate is 17.5 mm (thickness) x 25.0 mm (height) x 50 mm (width) with a 3 mm fillet at the fixed edge. The other rack shaped cantilever is shown in Fig. 16.

Electric-resistance strain gages with 1 mm active length were used to measure the strain. The strain gages were placed 5 mm apart along the root of the cantilever on the 21 mm line from the free edge. Stresses in the y-directions on the surface are defined by

\[ \sigma_y = -\frac{6}{T} \delta P \]  

Fig. 19 Profiles of deflections on vertical line with 74.4 mm width (experimental results)

Fig. 19 Profiles of deflections on vertical line with 74.4 mm width (concentrated load)

where \( T \) is plate thickness at the considered section and \( \gamma \) are the coefficients of moment distributions which were obtained from Eq. (36).

Comparison with the value to be obtained from Eq. (37) and experimental results with strain gage measurements was performed.

9-2 Stresses along the built-in edge of thick cantilever plate

Under the condition that a 1000 kg concentrated load acts on free edge, the stresses along the built-in edge are shown in Fig. 21.

The theoretical values which are obtained from Eq. (37) by use of plate thickness 17.5 mm and data of Fig. 13 are shown by the line in Fig. 21.

The experimental results with strain gages are shown by dots. These experimental results were corrected with the conditions that a concentrated load acted on 1.5 mm inner line from free edge and that the strain gages were not placed on the root of cantilever.

The theoretical curves agree well with the experimental values.

9-3 Stresses along the built-in edge of rack shaped cantilever

When a 1000 kg concentrated load acts on free edge, the experimental results with strain gages

Fig. 20 The comparison of theoretical values and experimental results, in direction of cantilever height

Fig. 21 Moment distribution of thick cantilever plate along the built-in edge, when a 10^8 kg single load acts on free edge (L/B=2.0)
are shown by dots in Fig. 22.

The theoretical curves are evaluated with Eq. (37), where γ are used with the values in Fig. 14 and where the plate is 24.56 mm thick that strain gages were placed, as shown by the lines, in Fig. 22.

These experimental results were corrected with the condition that a single load acted on 1.5 mm inner line from the free edge.

The theoretical curves are in satisfactory agreement with the experimental results.

It appears to be the proof that this finite differences analogy is a very useful method that deflections and moments of rack shaped cantilever can be obtained.

10. Conclusions

In this work, a semi-theoretical solution is sought for the rack shaped cantilever plate problem, under transverse loads at any location on the plate surface.

On nature of deflections, it has been proved that the values of calculated solutions using new boundary conditions are in close agreement with those obtained by experiments.

That is, on compliance which is significant in gear profile design, the coincidence of calculated solutions and experimental results is considered satisfactory.

On moment distributions along the built-in edge, it has been proved that the theoretical curves agree well with the experimental values in strain gage measurements.

It has been made clear that the method of finite differences is suitable to the finite width rack shaped cantilever plate problem and is used conveniently without any trouble.

Acknowledgment

The author wishes to thank Prof. J. Ishikawa and Associate Prof. K. Hayashi of Tokyo Institute of Technology for many valuable suggestions and encouragements to accomplish the present work.

References

Discussion

S. Takanashi (Tohoku University):

(1) Much interest has been aroused in this report, in which the theoretical value almost coincided with experimental results (for instance, Fig. 18). This report initiates the study showing that load distributions on the contact line between helical gear teeth can be evaluated and that the deflection in direction of meshing line of helical gear can be obtained.

However it is observed that there are some differences between theoretical values and experimental results near the built-in edge (bottom land of gear tooth) in Fig. 20. The deflections which are extrapolated from experimental results as shown by dots in Fig. 20, are estimated to be at least one or two micron-millimeters.

Also, I observed myself that there were some deflections on the built-in edge, with my thick plate experiment in which the cantilever thickness was in the same order as its length.

In view of these experimental results, I have a doubt about the boundary condition that the deflection along the built-in edge is zero.

How do you think of this point?

H. Kugimiya (Sumitomo Metal Industries, Ltd.):

(2) On line 9, left side of page 10, it is stated “The moment distributions of a finite width cantilever whose ratio of plate width $L$ to whole depth $B$ is more than three, can be used as the distributions of rack shaped cantilever with $L/B=3.0$."

Please show the distributions of a rack shaped cantilever with $L/B=4.0$.

(3) The strongest relation was observed between an increase in pressure angle and an increase in the maximum value of moment distributions, in Fig. 15.

When the bending strength of gear teeth is evaluated, the increase in pressure angle is unfavorable from the point of view that maximum value of moment distributions must be decreased, and is favourable from the point of view that plate thickness on the built-in edge must be increased.

That is, there is an alternative correlation in the increasing of pressure angle.

From these correlations, can you define the best pressure angle in gear design?

Brugger* concluded from the experiments that the increase in pressure angle is not favourable for the bending strength of gear tooth, when pressure angle has been over 26 degrees.

Do you make this point clear in the experimental result from this report?

(4) When a concentrated load acts on the tip free edge, in Fig. 19, the extrapolation as shown in Append. Fig. 1 must be used. In this extrapolation, the theoretical curve coincides better with experimental results.

(5) On line 7, left side of page 13, it is stated “these experimental results were corrected with the condition that a concentrated load acts on 1.5 mm inner line from the free edge.”

In the experiments with rack shaped cantilever, were these experimental results not corrected with the position where the strain gages were placed?

Author’s closure

(1) How the boundary conditions on the built-in edge are selected, is an important problem for the deflections of thick plate and rack-shaped cantilevers.

I think that the boundary conditions of elastically built-in edge must be used for thick and rack shaped cantilever plate problems with the thickness in the same order as its length.

However, for the reason as described later, the boundary conditions of semi-elastically built-in edge
Append.-Fig. 2  Moment distribution of 20° pressure angle rack shaped cantilever along built-in edge

are used in this rack shaped cantilever report:

(i) From the point of view of numerical calculation: For the thick plate, numerical solutions have been obtained under the condition that boundary conditions of elastically built-in edge were used.

However for the rack shaped cantilever, numerical solution showed that the deflections on elastically built-in edge were very small, that regularity of deflections in small order on elastically built-in edge was observed occasionally to be distracted and that deflections were observed occasionally to be minus, under the condition that boundary conditions of elastically built-in edge were used in the same manner as applied for thick cantilever plate.

(ii) From the point of view of experiments: On the built-in edge, the deflection of rack shaped cantilever was estimated to be 0.5μ-mm. This value is very small in comparison with the value of thick plate (1.5μ-mm).

(2) The Append.-Fig. 2 indicates the moment distribution of rack shaped cantilever with \( \frac{L}{B} = 4.0 \).

(3) I wish to thank Dr. Kugimiya for this interesting suggestion. This problem must be made clear from point of view of load distribution on contact line, as well as from point of view of bending strength when a single load acts.

Now the load distribution on contact line is being studied, with completing of this study this problem will be made clear.

(4) I think so, but the dots in Fig. 19 are experimental results. For instance, when a 720 kg single load acts on 18.6 mm point, deflection at 1.5 mm from free edge is 3.2μ-mm. These results cannot be neglected. For this reason, the extrapolation as shown in Fig. 19 was used.

(5) In the experiment with rack shaped cantilever, the stresses on the 20 degrees oblique surface are defined with Eq.(37). For this reason, these experimental results in Fig. 22 were not corrected with the condition that strain gages were placed.

If this correction is used, experimental results will increase 18 per cent and coincide better with theoretical curves.