539.384.2 : 539.313 : 539.373

A Numerical Approach to Finite Elastic-Plastic Deflections of Circular Plates*

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An elastic-plastic analysis of large deflection of axisymmetrically loaded circular plates is presented in this paper. Under the Kirchhoff-Love hypothesis, the incremental theory of plasticity together with the von Mises yield condition and the associated flow rule is adopted, and isotropic workhardening materials as well as elastic-perfectly plastic materials are treated. This is not an inherent limitation of the method. A numerical procedure with the finite difference approximation and the iteration technique is employed for the solution of the derived incremental basic equations of a two-point boundary value problem. Several results for various kinds of boundary condition and some values of plate thickness are given including the solutions of residual stress and strain for the first cycle of a loading-unloading process.

1. Introduction

There are, in short, two trends of procedure in the analysis of plastically deformed solids. One is the bounding solution such as the limit analysis^{(1)~(5)}, and the other is the solution of elastic-plastic deformation taking account of the most realistic constitutive relation if possible^{(6)~(12)}. The former, in which the plastic law, the equilibrium equation and the mode of deformation are approximated as simple as possible, often leads to an unreasonable result which differs from the actual behavior. The latter which has become quickly popular due to the remarkable development of the high speed digital computer, is closely related to the fact that a complicated constitutive relation which agrees with experimental results have been able to be applied to any practical problem⁽¹³⁾⁽¹⁴⁾.

In the present paper, an axisymmetric elastic-plastic analysis of circular plates under the coupling effect of the membrane force and the bending moment is reported from a viewpoint of the latter. The analysis includes several interesting results which have not been obtained and which differ somewhat from the ones accord-

ing to the preceding procedures conducted by other investigators.

Under the assumptions of small strain and small finite deflection, the incremental equilibrium equations and kinematic relations in the plane stress state are combined with the Prandtl-Reuss stress-strain law to derive simultaneous second order ordinary differential equations, which are integrated as a two-point boundary value problem. According to the present analyzing method, the deformation in the unloading process and the local unloading process occurring when a structure is subjected to loading are readily calculated without any difficulties.

2. Notations

- a : radius of plate
- E : Young's modulus
- E_T : tangential modulus of uniaxial stress-strain curve
- e_r, e_θ : strains at arbitrary point of plate
- e_i : equivalent strain
- G : shear modulus of elasticity
- h : plate thickness
- M_r, M_θ : bending moments per unit length (stress couples)
- N_r, N_θ : membrane forces per unit length (stress resultants)
- $\hat{Q}$: transverse shear force per unit length acting in the circumferential section
- q : uniformly distributed pressure

* Received 15th November, 1968.

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u : radial displacement
 w : deflection
 $\varepsilon_r, \varepsilon_\theta$: strains on the middle surface
 κ_r, κ_θ : changes of curvature
 ν : Poisson's ratio
 σ_r, σ_θ : stresses
 σ_i : equivalent stress
 $\bar{E}_T$: non-dimensional tangent modulus
 $\bar{\varepsilon}_r = (a^2/h^2)\varepsilon_r, \bar{\varepsilon}_\theta = (a^2/h^2)\varepsilon_\theta$
 $\bar{\varepsilon}_i = (a^2/h^2)\varepsilon_i$
 $\bar{h} = h/a$
 $\bar{M}_r, \bar{M}_\theta$: non-dimensional stress couples
 $\bar{N}_r, \bar{N}_\theta$: non-dimensional stress resultants
 $\bar{Q}$: non-dimensional pressure increment
 $\bar{u} = (a/h^2)u$
 W : non-dimensional deflection
 x : non-dimensional radius
 Z : non-dimensional plate thickness
 z : distance from the middle surface
 $\bar{\varepsilon}_r = (a^2/h^2)\varepsilon_r, \bar{\varepsilon}_\theta = (a^2/h^2)\varepsilon_\theta$
 $\bar{\kappa}_r, \bar{\kappa}_\theta$: modified curvature changes
 $\bar{\sigma}_r, \bar{\sigma}_\theta$: non-dimensional stresses
 $\bar{\sigma}_i$: non-dimensional equivalent stress
 In the above list, the quantities with subscripts r and θ denote the components in the radial and circumferential directions, respectively.

3. Theoretical foundations

3.1 Stress-strain relations

The general form of the Prandtl-Reuss stress-strain law of isotropic hardening materials can be written as⁽¹⁵⁾

$$\Delta e_{ij}' = s_{ij} \Delta \lambda + \frac{\Delta s_{ij}}{2G} \quad (1)$$

$$= \frac{3}{2} \frac{\Delta \sigma_i}{\sigma_i} \frac{1}{H'} s_{ij} + \frac{\Delta s_{ij}}{2G} \quad (2)$$

where

$$\Delta \lambda = \frac{3}{2} \frac{\Delta \sigma_i}{\sigma_i} \frac{1}{H'} \quad (3)$$

$$H' = \frac{\Delta \sigma_i}{\Delta e_p} \quad (4)$$

$\Delta e_{ij}'$: deviatoric strain increment

s_{ij} : deviatoric stress

Δe_p : equivalent plastic strain increment

Then, we can write, if the tangential modulus of the uniaxial tensile stress-strain curve is denoted by E_T , as

$$H' = \frac{\Delta \sigma_i}{\Delta e_p} = \frac{1}{\frac{1}{E_T} - \frac{1}{E}} \quad (5)$$

In what follows, these relations are specialized for a finite axisymmetric problem of thin circular plates.

3.1.1 For work-hardening materials

The stress-strain law in the plane stress

state is obtained in the polar coordinate system as follows:

$$\mathbf{e} = \mathbf{A} \mathbf{s} \quad (6)^*$$

where

$$\mathbf{e} = \begin{bmatrix} \Delta e_r \\ \Delta e_\theta \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} R & C \\ C & T \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} \Delta \sigma_r \\ \Delta \sigma_\theta \end{bmatrix} \quad (7)$$

$$\left. \begin{aligned} R &= \frac{1}{E} + \left(\frac{1}{E_T} - \frac{1}{E} \right) \left(\sigma_r - \frac{1}{2} \sigma_\theta \right)^2 \frac{1}{\sigma_i^2} \\ T &= \frac{1}{E} + \left(\frac{1}{E_T} - \frac{1}{E} \right) \left(\sigma_\theta - \frac{1}{2} \sigma_r \right)^2 \frac{1}{\sigma_i^2} \\ C &= -\nu \frac{1}{E} + \left(\frac{1}{E_T} - \frac{1}{E} \right) \\ &\quad \times \left(\sigma_\theta - \frac{1}{2} \sigma_r \right) \left(\sigma_r - \frac{1}{2} \sigma_\theta \right) \frac{1}{\sigma_i^2} \\ \sigma_i &= \sqrt{\sigma_r^2 + \sigma_\theta^2 - \sigma_r \sigma_\theta} \end{aligned} \right\} \quad (8)$$

The assumption of the plane stress state presupposes disregard of the transverse shear deformation.

3.1.2 For non-hardening materials

Using Eq. (4) and the von Mises yield criterion

$$\sigma_r^2 + \sigma_\theta^2 - \sigma_r \sigma_\theta = \sigma_p^2 \quad (9)$$

σ_p : uniaxial tensile yield stress

the following relation is obtained corresponding to Eq. (6):

$$\hat{\mathbf{e}} = \hat{\mathbf{A}} \hat{\mathbf{s}} \quad (10)^*$$

where

$$\left. \begin{aligned} \hat{\mathbf{e}} &= \begin{bmatrix} \Delta e_r \\ \Delta e_\theta \\ 0 \end{bmatrix} \\ \hat{\mathbf{A}} &= \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & \frac{2\sigma_r - \sigma_\theta}{3} \\ -\frac{\nu}{E} & \frac{1}{E} & \frac{2\sigma_\theta - \sigma_r}{3} \\ \frac{2\sigma_r - \sigma_\theta}{3} & \frac{2\sigma_\theta - \sigma_r}{3} & 0 \end{bmatrix} \\ \hat{\mathbf{s}} &= \begin{bmatrix} \Delta \sigma_r \\ \Delta \sigma_\theta \\ \Delta \lambda \end{bmatrix} \end{aligned} \right\} \quad (11)$$

Inversely, we obtain

$$\hat{\mathbf{s}} = \hat{\mathbf{A}}^{-1} \hat{\mathbf{e}} \quad (12)$$

where

$$\hat{\mathbf{A}}^{-1} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \quad (13)$$

$$\left. \begin{aligned} B_{13} &= B_{31} = \{3\nu(2\sigma_\theta - \sigma_r) + 3(2\sigma_r - \sigma_\theta)\} / z_0 \\ B_{23} &= B_{32} = \{3(2\sigma_\theta - \sigma_r) + 3\nu(2\sigma_r - \sigma_\theta)\} / z_0 \\ B_{33} &= -9(1 - \nu^2) / (E z_0) \\ z_0 &= (5 - 4\nu)(\sigma_r^2 + \sigma_\theta^2) - (8 - 10\nu)\sigma_r \sigma_\theta \end{aligned} \right\} \quad (14.a)$$

The quantities B_{11} , B_{12} , B_{21} and B_{22} are defined as follows:

* As for the general case, see Refs. (24) and (25).

$$(i) \quad \left. \begin{aligned} B_{11} &= E/(1-\nu^2), \quad B_{22} = E/(1-\nu^2) \\ B_{12} &= B_{21} = \nu E/(1-\nu^2) \end{aligned} \right\} \quad \dots\dots\dots(14.b)$$

in the elastic region

$$(ii) \quad \left. \begin{aligned} B_{11} &= (2\sigma_\theta - \sigma_r)^2 E/z_0 \\ B_{22} &= (2\sigma_r - \sigma_\theta)^2 E/z_0 \\ B_{12} &= B_{21} = -(2\sigma_r - \sigma_\theta)(2\sigma_\theta - \sigma_r)E/z_0 \end{aligned} \right\} \quad \dots\dots\dots(14.c)$$

in the plastic region

The flow of calculation is not affected essentially by the difference between two stress-strain laws.

3.2 Equilibrium equation

By assuming a small finite deflection, the equations of equilibrium of the plate subjected to a uniformly distributed pressure of intensity q can be obtained as follows⁽¹⁶⁾:

$$\left. \begin{aligned} N_r - N_\theta + r \frac{dN_r}{dr} &= 0 \\ M_r + \frac{dM_r}{dr} r - M_\theta - \hat{Q}r &= 0 \\ \hat{Q} &= -N_r \frac{dw}{dr} - \frac{1}{r} \int_0^r q r dr \end{aligned} \right\} \quad \dots\dots\dots(15)$$

or in the incremental form,

$$\left. \begin{aligned} \Delta N_r - \Delta N_\theta + r \frac{d}{dr}(\Delta N_r) &= 0 \\ \Delta M_r + r \frac{d}{dr}(\Delta M_r) - \Delta M_\theta - r \Delta \hat{Q} &= 0 \\ \Delta \hat{Q} &= -\Delta N_r \frac{dw}{dr} - N_r \frac{d}{dr}(\Delta w) - \frac{1}{r} \int_0^r \Delta q r dr \end{aligned} \right\} \quad \dots\dots\dots(16)$$

3.3 Constitutive equations of the plate

The strain-displacement relation on the middle surface under small strain and finite deflection is assumed in the following well-known form:

$$\epsilon_r = \frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2, \quad \epsilon_\theta = \frac{u}{r} \quad \dots\dots\dots(17)$$

for the membrane strains

$$\kappa_r = -\frac{d^2 w}{dr^2}, \quad \kappa_\theta = -\frac{1}{r} \frac{dw}{dr} \quad \dots\dots\dots(18)$$

for the changes of curvature

When their incremental relations are transformed into the matrix form, we obtain

$$\mathbf{K}_1 = \mathbf{C}_1 \mathbf{U} + \mathbf{C}_2 \mathbf{U}' \quad \dots\dots\dots(19)$$

Fig. 1 An element of circular plate

$$\mathbf{K}_2 = \mathbf{D}_1 \mathbf{U}' + \mathbf{D}_2 \mathbf{U}'', \quad ' = \frac{d}{dr} \quad \dots\dots\dots(20)$$

$$\left. \begin{aligned} \mathbf{K}_1 &= \begin{bmatrix} \Delta \epsilon_r \\ \Delta \epsilon_\theta \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} \Delta \kappa_r \\ \Delta \kappa_\theta \end{bmatrix} \\ \mathbf{U} &= \begin{bmatrix} \Delta u \\ \Delta w \end{bmatrix} \end{aligned} \right\} \quad \dots\dots\dots(21)$$

$$\left. \begin{aligned} \mathbf{C}_1 &= \begin{bmatrix} 0 & 0 \\ 1/r & 0 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 1 & dw/dr \\ 0 & 0 \end{bmatrix} \\ \mathbf{D}_1 &= \begin{bmatrix} 0 & 0 \\ 0 & -1/r \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \end{aligned} \right\} \quad \dots\dots\dots(22)$$

Assume that the strains vary linearly through the plate thickness as

$$\mathbf{e} = \mathbf{K}_1 + z \mathbf{K}_2 \quad \dots\dots\dots(23)$$

and the following definitions are employed for the stress resultants and the stress couples:

$$\left. \begin{aligned} \Delta N_r &= \int_{-h/2}^{h/2} \Delta \sigma_r dz, \quad \Delta N_\theta = \int_{-h/2}^{h/2} \Delta \sigma_\theta dz \\ \Delta M_r &= \int_{-h/2}^{h/2} z \Delta \sigma_r dz, \quad \Delta M_\theta = \int_{-h/2}^{h/2} z \Delta \sigma_\theta dz \end{aligned} \right\} \quad \dots\dots\dots(24)$$

$$\mathbf{N} = \begin{bmatrix} \Delta N_r \\ \Delta N_\theta \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \Delta M_r \\ \Delta M_\theta \end{bmatrix} \quad \dots\dots\dots(25)$$

then, the constitutive equations of the plate are obtained as

$$\mathbf{N} = \left[\int \mathbf{A}^{-1} dz \right] \mathbf{K}_1 + \left[\int z \mathbf{A}^{-1} dz \right] \mathbf{K}_2 \quad \dots\dots\dots(26)$$

$$\mathbf{M} = \left[\int z \mathbf{A}^{-1} dz \right] \mathbf{K}_1 + \left[\int z^2 \mathbf{A}^{-1} dz \right] \mathbf{K}_2 \quad \dots\dots\dots(27)$$

since the stress-strain relation can be expressed by Eqs. (6) and (23) as follows:

$$\mathbf{s} = \mathbf{A}^{-1} \mathbf{K}_1 + z \mathbf{A}^{-1} \mathbf{K}_2 \quad \dots\dots\dots(28)$$

It should be noted that the stress resultants and the stress couples are coupled in Eqs. (26) and (27). This does not occur in the elastic bending problem of plates (Appendix I).

3.4 Basic differential equations

Similarly to the procedure proposed in ref. (17) by one of the authors in the case of finite axisymmetric bending of shells of revolution, the derived fundamental relations are reduced to a two-point boundary value problem. In what follows, the deriving method is presented.

Eliminating Δu from Eq. (19) and $\Delta \hat{Q}$ from the second and third equations in Eq. (16), the following intermediate relations, the first of which corresponds to the compatibility condition, are obtained:

$$\left[\begin{array}{cc} 1 & -1 \end{array} \right] \mathbf{K}_1 + \left[\begin{array}{cc} 0 & -r \end{array} \right] \mathbf{K}_2' + \left[\begin{array}{cc} 0 & -\frac{dw}{dr} \end{array} \right] \mathbf{U}' = 0 \quad \dots\dots\dots(29)$$

$$\left[\begin{array}{cc} 1 & 1 \end{array} \right] \mathbf{M} + \left[\begin{array}{cc} r & 0 \end{array} \right] \mathbf{M}' + \left[\begin{array}{cc} r \frac{dw}{dr} & 0 \end{array} \right] \mathbf{N} + \left[\begin{array}{cc} 0 & r N_r \end{array} \right] \mathbf{U}' + Q = 0 \quad \dots\dots\dots(30)$$

where

$$Q = \int_0^r \Delta q r dr$$

From Eqs. (26) and (27), K_1 and M are expressed as follows:

$$K_1 = \left[\int A^{-1} dz \right]^{-1} N - \left[\int A^{-1} dz \right]^{-1} \left[\int z A^{-1} dz \right] K_2 \dots\dots\dots (31)$$

$$M = \left[\int z A^{-1} dz \right] \left[\int A^{-1} dz \right]^{-1} \left\{ N - \left[\int z A^{-1} dz \right] K_2 \right\} + \left[\int z^2 A^{-1} dz \right] K_2 \dots\dots\dots (32)$$

Then, substituting them into Eqs. (29) and (30), the following differential equations are obtained respectively;

$$\begin{aligned} [1 \ -1] & \left\{ \left[\int A^{-1} dz \right]^{-1} N - \left[\int A^{-1} dz \right]^{-1} \left[\int z A^{-1} dz \right] K_2 \right\} + [0 \ -r] \\ & \times \left\{ \left[\int A^{-1} dz \right]^{-1} N - \left[\int A^{-1} dz \right]^{-1} \left[\int z A^{-1} dz \right] K_2 \right\}' + \left[0 \ -\frac{dw}{dr} \right] U' = 0 \dots\dots\dots (33) \end{aligned}$$

$$\begin{aligned} [1 \ -1] & \left\{ \left[\int z A^{-1} dz \right] \left[\int A^{-1} dz \right]^{-1} \left(N - \left[\int z A^{-1} dz \right] K_2 \right) + \left[\int z^2 A^{-1} dz \right] K_2 \right\} + [r \ 0] \\ & \times \left\{ \left[\int z A^{-1} dz \right] \left[\int A^{-1} dz \right]^{-1} \left(N - \left[\int z A^{-1} dz \right] K_2 \right) + \left[\int z^2 A^{-1} dz \right] K_2 \right\}' \\ & + \left[r \frac{dw}{dr} \ 0 \right] N + [0 \ r N_r] U' + Q = 0 \dots\dots\dots (34) \end{aligned}$$

which, in detail, are equivalent to

$$\begin{aligned} [1 \ -1] & \left\{ \begin{bmatrix} \frac{D_T}{D} & -\frac{D_c}{D} \\ -\frac{D_c}{D} & \frac{D_R}{D} \end{bmatrix} \begin{bmatrix} \Delta N_r \\ \Delta N_\theta \end{bmatrix} - \begin{bmatrix} \frac{D_T}{D} & -\frac{D_c}{D} \\ -\frac{D_c}{D} & \frac{D_R}{D} \end{bmatrix} \begin{bmatrix} F_R & F_c \\ F_c & F_T \end{bmatrix} \begin{bmatrix} \Delta \kappa_r \\ \Delta \kappa_\theta \end{bmatrix} \right\} \\ & + [0 \ -r] \left\{ \begin{bmatrix} \frac{D_T}{D} & -\frac{D_c}{D} \\ -\frac{D_c}{D} & \frac{D_R}{D} \end{bmatrix} \begin{bmatrix} \Delta N_r \\ \Delta N_\theta \end{bmatrix} - \begin{bmatrix} \frac{D_T}{D} & -\frac{D_c}{D} \\ -\frac{D_c}{D} & \frac{D_R}{D} \end{bmatrix} \begin{bmatrix} F_R & F_c \\ F_c & F_T \end{bmatrix} \begin{bmatrix} \Delta \kappa_r \\ \Delta \kappa_\theta \end{bmatrix} \right\}' \\ & + \left[0 \ -\frac{dw}{dr} \right] \begin{bmatrix} \Delta u \\ \Delta w \end{bmatrix}' = 0 \dots\dots\dots (35) \end{aligned}$$

$$\begin{aligned} [1 \ -1] & \left\{ \begin{bmatrix} F_R & F_c \\ F_c & F_T \end{bmatrix} \begin{bmatrix} \frac{D_T}{D} & -\frac{D_c}{D} \\ -\frac{D_c}{D} & \frac{D_R}{D} \end{bmatrix} \left(\begin{bmatrix} \Delta N_r \\ \Delta N_\theta \end{bmatrix} - \begin{bmatrix} F_R & F_c \\ F_c & F_T \end{bmatrix} \begin{bmatrix} \Delta \kappa_r \\ \Delta \kappa_\theta \end{bmatrix} \right) + \begin{bmatrix} K_R & K_c \\ K_c & K_T \end{bmatrix} \begin{bmatrix} \Delta \kappa_r \\ \Delta \kappa_\theta \end{bmatrix} \right\} \\ & + [r \ 0] \left\{ \begin{bmatrix} F_R & F_c \\ F_c & F_T \end{bmatrix} \begin{bmatrix} \frac{D_T}{D} & -\frac{D_c}{D} \\ -\frac{D_c}{D} & \frac{D_R}{D} \end{bmatrix} \left(\begin{bmatrix} \Delta N_r \\ \Delta N_\theta \end{bmatrix} - \begin{bmatrix} F_R & F_c \\ F_c & F_T \end{bmatrix} \begin{bmatrix} \Delta \kappa_r \\ \Delta \kappa_\theta \end{bmatrix} \right) + \begin{bmatrix} K_R & K_c \\ K_c & K_T \end{bmatrix} \begin{bmatrix} \Delta \kappa_r \\ \Delta \kappa_\theta \end{bmatrix} \right\}' \\ & + \left[r \frac{dw}{dr} \ 0 \right] \begin{bmatrix} \Delta N_r \\ \Delta N_\theta \end{bmatrix} + [0 \ r N_r] \begin{bmatrix} \Delta u \\ \Delta w \end{bmatrix}' + \int_0^r \Delta q r dr = 0 \dots\dots\dots (36) \end{aligned}$$

where, for work-hardening materials,

$$\left. \begin{aligned} D_R &= \int \frac{T}{RT-C^2} dz, & D_T &= \int \frac{R}{RT-C^2} dz, & D_c &= -\int \frac{C}{RT-C^2} dz \\ F_R &= \int z \frac{T}{RT-C^2} dz, & F_T &= \int z \frac{R}{RT-C^2} dz, & F_c &= -\int z \frac{C}{RT-C^2} dz \\ K_R &= \int z^2 \frac{T}{RT-C^2} dz, & K_T &= \int z^2 \frac{R}{RT-C^2} dz, & K_c &= -\int z^2 \frac{C}{RT-C^2} dz \end{aligned} \right\} \dots\dots\dots (37)$$

for non-hardening materials,

$$\left. \begin{aligned} D_R &= \int B_{11} dz, & D_T &= \int B_{22} dz, & D_c &= \int B_{12} dz, & F_R &= \int z B_{11} dz, & F_T &= \int z B_{22} dz \\ F_c &= \int z B_{12} dz, & K_R &= \int z^2 B_{11} dz, & K_T &= \int z^2 B_{22} dz, & K_c &= \int z^2 B_{12} dz \end{aligned} \right\} \dots\dots\dots (38)$$

and

$$D = D_R D_T - D_c^2 \quad (39)$$

Now, defining the stress function which satisfies the first of Eq. (16)⁽¹⁶⁾ and the incremental rotating angle as

$$\Delta N_r = \frac{h}{r} \Phi, \quad \Delta N_\theta = h \frac{d\Phi}{dr} \quad (40)$$

$$d(\Delta w)/dr = \varphi \quad (41)$$

Eqs. (35) and (36) constitute a set of simultaneous second order ordinary differential equations with respect to φ and Φ .

4. Numerical calculation

4.1 Finite difference scheme with iteration

The basic differential equations are summarized in non-dimensional form:

$$\begin{aligned} \frac{d^2 g}{dx^2} + \left\{ \frac{\bar{D}}{\bar{D}_R} \frac{d}{dx} \left(\frac{\bar{D}_R}{\bar{D}} \right) + \frac{\bar{D}_c}{x \bar{D}_R} + \frac{1}{x} \right\} \frac{dg}{dx} - \left\{ \frac{1}{x^2} \frac{\bar{D}_T}{\bar{D}_R} + \frac{\bar{D}}{x \bar{D}_R} \frac{d}{dx} \left(\frac{\bar{D}_c}{\bar{D}} \right) \right\} g + \left(\bar{F}_c - \frac{\bar{D}_c}{\bar{D}_R} \bar{F}_R \right) \frac{d^2 f}{dx^2} \\ + \left\{ \frac{1}{x} \left(\bar{F}_c - \frac{\bar{D}_T}{\bar{D}_R} \bar{F}_R - \bar{F}_T - \frac{\bar{D}_c}{\bar{D}_R} \bar{F}_R \right) + \frac{\bar{D}}{\bar{D}_R} \frac{d}{dx} \left(\frac{\bar{D}_R \bar{F}_c}{\bar{D}} - \frac{\bar{D}_c \bar{F}_R}{\bar{D}} \right) \right\} \frac{df}{dx} \\ + \left[\frac{1}{x} \frac{\bar{D}}{\bar{D}_R} \left\{ \frac{dW}{dx} + \frac{d}{dx} \left(\frac{\bar{D}_R \bar{F}_T}{\bar{D}} - \frac{\bar{D}_c \bar{F}_c}{\bar{D}} \right) \right\} - \frac{1}{x^2} \left(\frac{\bar{D}_T}{\bar{D}_R} \bar{F}_c - \frac{\bar{D}_c}{\bar{D}_R} \bar{F}_T \right) \right] f = 0 \quad (42) \\ \left\{ \frac{1}{\bar{K}_R} \left(\frac{\bar{D}_T}{\bar{D}} \bar{F}_R^2 - 2 \frac{\bar{D}_c}{\bar{D}} \bar{F}_c \bar{F}_R + \frac{\bar{D}_R}{\bar{D}} \bar{F}_c^2 \right) - 1 \right\} \frac{d^2 f}{dx^2} + \left[\frac{1}{\bar{K}_R x} \left\{ x \frac{d}{dx} \left(\frac{\bar{D}_T}{\bar{D}} \bar{F}_R^2 - 2 \frac{\bar{D}_c}{\bar{D}} \bar{F}_c \bar{F}_R + \frac{\bar{D}_R}{\bar{D}} \bar{F}_c^2 \right) \right. \right. \\ + \frac{\bar{D}_T}{\bar{D}} \bar{F}_R^2 - 2 \frac{\bar{D}_c}{\bar{D}} \bar{F}_c \bar{F}_R + \frac{\bar{D}_R}{\bar{D}} \bar{F}_c^2 \left. \right\} - \frac{1}{\bar{K}_R} \frac{d\bar{K}_R}{dx} - \frac{1}{x} \left. \right] \frac{df}{dx} + \frac{1}{\bar{K}_R x} \left\{ \frac{d}{dx} \left(\frac{\bar{D}_T}{\bar{D}} \bar{F}_R \bar{F}_c - \frac{\bar{D}_c}{\bar{D}} \bar{F}_c^2 \right. \right. \\ - \frac{\bar{D}_c}{\bar{D}} \bar{F}_R \bar{F}_T + \frac{\bar{D}_c}{\bar{D}} \bar{F}_c \bar{F}_T \left. \right\} - \frac{1}{x} \left(\frac{\bar{D}_T}{\bar{D}} \bar{F}_c^2 - 2 \frac{\bar{D}_c}{\bar{D}} \bar{F}_T \bar{F}_c + \frac{\bar{D}_R}{\bar{D}} \bar{F}_T^2 - \bar{K}_T \right) - \frac{d\bar{K}_c}{dx} + x \bar{N}_r \left. \right\} f \\ + \frac{1}{\bar{K}_R} \left(\frac{\bar{D}_R}{\bar{D}} \bar{F}_c - \frac{\bar{D}_c}{\bar{D}} \bar{F}_R \right) \frac{d^2 g}{dx^2} + \frac{1}{\bar{K}_R x} \left\{ \frac{\bar{D}_R}{\bar{D}} \bar{F}_c - \frac{\bar{D}_c}{\bar{D}} \bar{F}_R + \frac{\bar{D}_T}{\bar{D}} \bar{F}_R \right. \\ - \frac{\bar{D}_R}{\bar{D}} \bar{F}_T + x \frac{d}{dx} \left(\frac{\bar{D}_R}{\bar{D}} \bar{F}_c - \frac{\bar{D}_c}{\bar{D}} \bar{F}_R \right) \left. \right\} \frac{dg}{dx} + \frac{1}{\bar{K}_R x} \left\{ \frac{d}{dx} \left(\frac{\bar{D}_T}{\bar{D}} \bar{F}_R - \frac{\bar{D}_c}{\bar{D}} \bar{F}_c \right) \right. \\ \left. - \left(\frac{\bar{D}_T}{\bar{D}} \bar{F}_c - \frac{\bar{D}_c}{\bar{D}} \bar{F}_T \right) \frac{1}{x} + \frac{dW}{dx} \right\} g + \frac{x}{\bar{K}_R} \bar{Q} = 0 \quad (43) \end{aligned}$$

where

$$\left. \begin{aligned} (\bar{K}_R, \bar{K}_T, \bar{K}_c) &= (1/Eh^3)(K_R, K_T, K_c) \\ (\bar{F}_R, \bar{F}_T, \bar{F}_c) &= (1/Eh^2)(F_R, F_T, F_c) \\ (\bar{D}_R, \bar{D}_T, \bar{D}_c) &= (1/Eh)(D_R, D_T, D_c) \\ \bar{D} &= D/(E^2 h^2) \end{aligned} \right\} \quad (44)$$

$$\left. \begin{aligned} W &= w/h, \quad \bar{N}_r = N_r a^2 / (Eh^3) \\ f &= \varphi a / h, \quad g = \Phi a / (Eh^2) \\ x &= r/a, \quad \bar{Q} = \Delta q a^4 / (2Eh^4) \end{aligned} \right\} \quad (45)$$

Since the non-dimensional quantities $\bar{D}_R, \dots, \bar{F}_R, \dots, \bar{K}_R, \dots, \bar{N}_r$ and W are those referred to the required equilibrium state, that is, unknown quantities, the derived equations are essentially nonlinear. To avoid the difficulties which arise from this situation, linearization and iteration procedures are adopted; the solution obtained in the last stage is substituted as approximations of the quantities $\bar{D}_R, \bar{D}_T, \dots$ etc., and the process is repeated until the solution converges⁽¹⁷⁾ (Appendix 2). At each stage of the iteration, the solution is obtained numerically by the finite difference method⁽¹⁷⁾. Twenty-point mesh in the radial direction and eight-point mesh in the Z -direction are selected

within the discretization error admissible in the practical computation; according to the test calculation, the differences between the solutions due to the radial subdivisions of 20, 50 and 100 do not exceed 1%. Simpson's rule is used to perform the integration with respect to z . The pressure increment $\bar{Q}$ is set up as 10% of the limiting value obtained by the elastic pure bending theory.

The iterative procedure is stopped by the criterion

$$\frac{||\Sigma f_i^{(l)}| - |\Sigma f_i^{(l-1)}||}{|\Sigma f_i^{(l-1)}|} \leq \eta \quad (46)$$

l : iteration number

When the solutions f and g are obtained, the stresses, the strains, the stress resultants, the stress couples and the deflection can be calculated by the following relations:

$$\Delta \bar{N}_r = g/x, \quad \Delta \bar{N}_\theta = dg/dx \quad (47)$$

$$\Delta \bar{K}_r = -df/dx, \quad \Delta \bar{K}_\theta = -f/x \quad (48)$$

$$\begin{aligned} \begin{bmatrix} \Delta \bar{\epsilon}_r \\ \Delta \bar{\epsilon}_\theta \end{bmatrix} &= \begin{bmatrix} \bar{D}_R & \bar{D}_c \\ \bar{D}_c & \bar{D}_T \end{bmatrix}^{-1} \begin{bmatrix} \Delta \bar{N}_r \\ \Delta \bar{N}_\theta \end{bmatrix} \\ &= \begin{bmatrix} \bar{D}_R & \bar{D}_c \\ \bar{D}_c & \bar{D}_T \end{bmatrix}^{-1} \begin{bmatrix} \bar{F}_R & \bar{F}_c \\ \bar{F}_c & \bar{F}_T \end{bmatrix} \begin{bmatrix} \Delta \bar{K}_r \\ \Delta \bar{K}_\theta \end{bmatrix} \quad (49) \end{aligned}$$

$$\begin{bmatrix} \Delta \bar{M}_r \\ \Delta \bar{M}_\theta \end{bmatrix} = \begin{bmatrix} \bar{F}_R & \bar{F}_c \\ \bar{F}_c & \bar{F}_T \end{bmatrix} \begin{bmatrix} \bar{D}_R & \bar{D}_c \\ \bar{D}_c & \bar{D}_T \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \Delta \bar{N}_r \\ \Delta \bar{N}_\theta \end{bmatrix} - \begin{bmatrix} \bar{F}_R & \bar{F}_c \\ \bar{F}_c & \bar{F}_T \end{bmatrix} \begin{bmatrix} \Delta \bar{\kappa}_r \\ \Delta \bar{\kappa}_\theta \end{bmatrix} \right\} + \begin{bmatrix} \bar{K}_R & \bar{K}_c \\ \bar{K}_c & \bar{K}_T \end{bmatrix} \begin{bmatrix} \Delta \bar{\kappa}_r \\ \Delta \bar{\kappa}_\theta \end{bmatrix} \quad (50)$$

$$\Delta \bar{e}_r = \Delta \bar{e}_r + Z \Delta \bar{\kappa}_r, \quad \Delta \bar{e}_\theta = \Delta \bar{e}_\theta + Z \Delta \bar{\kappa}_\theta \quad (51)$$

$$\begin{bmatrix} \Delta \bar{\sigma}_r \\ \Delta \bar{\sigma}_\theta \end{bmatrix} = \begin{bmatrix} \bar{R} & \bar{C} \\ \bar{C} & \bar{T} \end{bmatrix}^{-1} \begin{bmatrix} \Delta \bar{e}_r \\ \Delta \bar{e}_\theta \end{bmatrix} \quad (52)$$

$$\Delta W = - \int f dx \quad (53)$$

where

$$(\bar{N}_r, \bar{N}_\theta) = (a^2/Eh^3)(N_r, N_\theta)$$

$$(\bar{M}_r, \bar{M}_\theta) = (a^2/Eh^4)(M_r, M_\theta)$$

$$(\bar{\kappa}_r, \bar{\kappa}_\theta) = (a^2/h)(\kappa_r, \kappa_\theta)$$

$$(\bar{\sigma}_r, \bar{\sigma}_\theta, \bar{\sigma}_i) = (a^2/Eh^2)(\sigma_r, \sigma_\theta, \sigma_i)$$

$$(\bar{R}, \bar{T}, \bar{C}) = E(R, T, C)$$

$$Z = z/h$$

4.2 Integration of increments

If an iteration number and a stage number are denoted by l and k respectively, total quantities such as total stresses and total strains are calculated by⁽¹⁸⁾

$$\psi^{(l+1)}(k+1) = \psi(k) + \frac{1}{2} \{ \Delta \psi^{(l)}(k+1) + \Delta \psi(k) \} \quad (54)$$

where $\psi(k)$ and $\Delta \psi(k)$ are a total quantity and an increment which are determined in the k -th incremental stage. It is noted that this is applicable only if the pressure increment is constant. In calculating $\psi^{(l)}(k+1)$, we must employ the following relation:

$$\psi^{(l)}(k+1) = \psi(k-1) + 2\Delta \psi(k) \quad (55)$$

and especially, when $k=1$,

$$\psi^{(l)}(k+1) = \Delta \psi(k) \quad (56)$$

4.3 Work-hardening law

As the simplest example, the case of isotropic work-hardening is studied. The double formula proposed by Holmquist and Nadai⁽¹⁹⁾ is employed in the present paper:

$$\bar{e}_i = \bar{\sigma}_i + K(\bar{\sigma}_i - \bar{\sigma}_p)^n \quad (57)$$

Hence, a non-dimensional tangent modulus can be expressed by

$$\bar{E}_T = \frac{\Delta \bar{\sigma}_i}{\Delta \bar{e}_i} = \frac{1}{1 + nK(\bar{\sigma}_i - \bar{\sigma}_p)^{n-1}} \quad (58)$$

where the parameters n , K and $\bar{\sigma}_p$ are determined by the uniaxial stress-strain curve under the restraint that $K=0$ in the elastic range. In numerical analysis, the disadvantage that the stress-strain curve is separated into two parts and the complication arising from three parameters is canceled out by the feature that it gives a more accurate description to the mechanical property of materials than any other simple formula like the Ramberg-Osgood law⁽²⁰⁾.

The results calculated by Ohashi et al.⁽¹¹⁾

will be cited in Section 5.1; they are based on the following formula:

$$\bar{\sigma}_i = a_1 \{ 1 - \exp(-a_2 \bar{e}_i) \} \quad (59)$$

that leads to

$$\bar{E}_T = 1 - a_2 \bar{h}^2 \bar{\sigma}_i \quad (60)$$

The parameters are determined in ref. (11) as

$$a_1 = 10.43, \quad a_2 = 1.021$$

4.4 Boundary conditions

The following boundary conditions of the edge ($x=1$) are studied:

i) Clamped edge

$$f=0, \quad \bar{u}=0 \quad (61)$$

(no slope and no radial displacement)

ii) Pinned edge free to rotate

$$\bar{M}_r=0, \quad \bar{u}=0 \quad (62)$$

(no bending moment and no radial displacement)

iii) Simply supported edge

$$\bar{M}_r=0, \quad g=0 \quad (63)$$

(no bending moment and no radial membrane stress)

Since there is no singularity at the center of the plate, the following condition must be imposed:

$$f=0, \quad g=0 \quad (64)$$

4.5 Unloading

(a) Local unloading during the loading process

It is possible that a structure is locally subjected to unloading while it is globally in a loading condition as the case may be. Then, after recognizing that the local unloading occurs when the second term in the right hand side of Eq. (54) becomes negative, the term corresponding to the plastic strain is ignored.

(b) Usual unloading

When the pressure increment is altered to be negative, Eq. (56) is used at the starting point of the unloading. The other procedures are the same as those given in the local unloading.

5. Results and discussions

The calculated results for aluminum specimens which are tested in the following experiment are shown for the three kinds of boundary condition given in Eqs. (61)~(63).

Fig. 2 Specimen and clamping apparatus

5.1 Comparisons with experimental results

A clamped aluminum circular plate with a radius of 150 mm and a thickness of 2 mm is tested under air pressure (Fig. 2). The parameters in Eq. (57) of this material are determined by applying the least square method to

Fig. 3 Load-surface strain diagram

Fig. 4 Load-maximum deflection diagram

Fig. 5 Load-maximum deflection diagram

Fig. 6 Load-surface strain diagram ($x=0$)

the results of uniaxial tensile test as follows:

$$n=2.6, \quad K=0.98, \quad \bar{\sigma}_p=6.75$$

$\bar{\sigma}_p$ is defined as a stress, at which a permanent set of 0.03% may be observed.

Figures 3 and 4 show the radial strain on the upper surface ($Z=0.5$) of the plate and the maximum deflection, respectively. The difference between theoretical and experimental results occurring in a higher pressure region suggests that it is difficult to satisfy the condition of clamped edge completely.

Next, to compare with the experimental results obtained by Ohashi et al.⁽¹¹⁾ a clamped plate with a radius of 125 mm and a thickness of 10 mm is calculated by employing the stress-

Table 1 Parameters of the stress-strain curve

a/h	n	K	$\bar{\sigma}_p$
75	2.6	0.98	6.75
37.5	2.54	2.87	1.4
18.75	2.085	7.4	0.3516

Fig. 7 Deflection distribution

Fig. 8 Distribution of stress resultant

strain curve defined in Eq. (59). According to the concept of effective diameter⁽¹¹⁾, it is assumed that $a/h=14.25$. The results are shown in Figs. 5~6. The present solution agrees well with the experiment. From the comparison of the present solution ($q=0.1013$) with Ohashis' solu-

tion ($q=0.10$), it can be shown that the stress couple at the center of plate is overestimated up to 60%, if the membrane force is not taken into account (see also Fig. 7).

5.2 Clamped edge

The calculated results for a clamped plate of $a/h=37.5$ are shown, and the results for $a/h=18.75$ and 75 are also supplemented. The parameters of stress-strain curve are chosen as listed in Table 1.

Figures 8~10 show the stress resultants, the stress couples and the strain distributions at

Fig. 9 Distribution of stress couple

Fig. 10 Distribution of surface strain ($z=0.5$)

A	$qa^4/Eh^4=1.07 \times 4$	B	$qa^4/Eh^4=1.07 \times 8$
C	$qa^4/Eh^4=1.07 \times 12$	D	$qa^4/Eh^4=1.07 \times 16$
E	$qa^4/Eh^4=1.07 \times 20$	F	$qa^4/Eh^4=5.1 \times 16$
G	$qa^4/Eh^4=0.27 \times 6$		

Fig. 11 Load-deflection characteristics

(a) Load-stress resultant diagram ($x=0$)

(b) Load-stress resultant diagram ($a/h=37.5$)

Fig. 12

$Z=0.5$, respectively, where the classifications A, B, ... show that the results correspond to each

(a) Load-stress couple diagram ($x=0$)

(b) Load-stress couple diagram ($a/h=37.5$)
Fig. 13

(a) Load-surface strain ($x=0$)

(b) Load-surface strain ($a/h=37.5$)
Fig. 14

loading stage defined in Fig. 11, and the ordinate of Fig. 8 must be multiplied by ten for the solution of F ($a/h=75$).

The maximum deflections of the plate (the deflections at the center) are shown in Fig. 11. For a thinner circular plate, the elastic solution based on the von Karman theory⁽²¹⁾ is valid for a wide range of loading parameters. It might be possible that the limiting load is defined at the point with a specified amount of the discrepancy between the two theories.

(1) $qa^4/2Fh^4=5.1 \times 4$ (3) $qa^4/2Eh^4=5.1 \times 12$
(2) $qa^4/2Eh^4=5.1 \times 8$ (4) $qa^4/2Eh^4=5.1 \times 16$
Fig. 15 Progressive yielding ($a/h=75$)

(1) $qa^4/2Eh^4=1.07 \times 2$ (3) $qa^4/2Eh^4=1.07 \times 10$
(2) $qa^4/2Eh^4=1.07 \times 6$ (4) $qa^4/2Eh^4=1.07 \times 14$
Fig. 16 Progressive yielding ($a/h=37.5$)

(1) $qa^4/2Eh^4=0.27 \times 2$ (4) $qa^4/2Eh^4=0.27 \times 5$
(2) $qa^4/2Eh^4=0.27 \times 3$ (5) $qa^4/2Eh^4=0.27 \times 6$
(3) $qa^4/2Eh^4=0.27 \times 4$
Fig. 17 Progressive yielding ($a/h=18.75$)

(1) $qa^4/2Eh^4=1.07 \times 8$ (4) $qa^4/2Eh^4=1.07 \times 2$
(2) $qa^4/2Eh^4=1.07 \times 6$ (5) $qa^4/2Eh^4=0$
(3) $qa^4/2Eh^4=1.07 \times 4$
Fig. 18 Progressive yielding ($a/h=37.5$, unloading)

Figures 12~14 give the stress resultants, the stress couples at the center of the plate and the strains on the upper surface for several values of the pressure. In Figs. 12(a), 13(a) and 14(a), the curves obtained for $a/h=75, 37.5$, of which the results for $a/h=37.5$ are enlarged separately in Figs. 12(b), 13(b) and 14(b), are compared with the unloading behavior of clamped and simply supported plates to give the residual deflection, the residual strain and the residual stress after the first cycle of a loading-unloading process, which are discussed later.

In Figs. 15~18, the progressive yielding is studied. What is shown in Fig. 18 for an unloading case is very interesting. The uniform pressure is applied on the lower surface ($Z=-0.5$). When the membrane force exists, the plastic ranges propagate asymmetrically with respect to the middle surface as shown in these figures by the contour curves on which $\bar{\sigma}_i = \bar{\sigma}_p$. In Fig. 16, there can be observed a local unloading region in the course of (3) and (4), while Fig. 18 shows a local loading region during unloading (3) to (5) in the lower part in the vicinity of the plate center.

Figure 19 shows the stress paths at several

Fig. 19 Surface stress path ($a/h=18.75, Z=0.5$)

Fig. 20 Load-maximum deflection diagram (simply supported edge)

points on the upper surface during the loading of the plate $a/h=18.75$. In the vicinity of the edge, the stress trajectory runs apart from the proportional loading path.

5.3 Simply supported edge

A part of the results obtained about a plate of $a/h=37.5$ the stress-strain relation of which is approximated by Eqs. (57) and (58) is discussed. Figure 20 gives the load-deflection curve and Fig. 21 the strain distribution on the upper surface of the plate to clarify that e_r and $e_θ$ have maximum and minimum in the vicinity of

(1) $qa^4/2Eh^4=1.07 \times 2$ (4) $qa^4/2Eh^4=1.07 \times 8$
(2) $qa^4/2Eh^4=1.07 \times 4$ (5) $qa^4/2Eh^4=1.07 \times 10$
(3) $qa^4/2Eh^4=1.07 \times 6$

Fig. 21 Distribution of surface strain ($Z=0.5$, simply supported edge)

(1) $qa^4/2Eh^4=1.07 \times 2$ (4) $qa^4/2Eh^4=1.07 \times 8$
(2) $qa^4/2Eh^4=1.07 \times 4$ (5) $qa^4/2Eh^4=1.07 \times 10$
(3) $qa^4/2Eh^4=1.07 \times 6$

Fig. 22 Progressive yielding (simply supported edge)

Fig. 23 Load-maximum deflection (pinned edge free to rotate)

$\alpha=0.8$, respectively, which is not the case of a clamped plate. The progressive yielding is discussed in Fig. 22. It can be also shown that the stress resultant and the stress couple are positive throughout the radius except that the circumferential stress resultant is negative only in the neighborhood of the edge.

5.4 Pinned edge free to rotate

The same plate as used in the simply supported edge is analyzed. The load-deflection curve and the strain distribution are shown in Figs. 23 and 24, respectively. The progressive yielding is discussed in Fig. 25.

5.5 Supplements to the numerical solutions

The convergence of the iterative scheme is an important point in such a problem as considered in the present paper. In order to obtain a rapid convergence rate, the so-called interpolated parameter is introduced; if it is denoted by λ , then the solution $f^{(l)}$ of the l -th stage of iteration is calculated by

$$f_i^{(l)} = \lambda \bar{f}_i^{(l)} + (1-\lambda)f_i^{(l-1)}, \quad 0 < \lambda \leq 1$$

where $\bar{f}_i^{(l)}$ is a solution directly obtained by Eqs. (42) and (43). This procedure is adopted in all cases including the non-hardening material. The concrete procedure is outlined in the following; put $\lambda=1$ at the first stage of the calculation and examine whether a convergence occurs within thirty iterations, and if the expected result is not obtained the parameter is modified as $(\lambda-0.1)$ and the iteration is continued until λ diminishes to 0.5, before the computation is stopped. This procedure, however, is unnecessary in most cases. The larger values of a/h promise a relatively rapid convergence, while for the smaller values, especially in the case of a clamped edge, it is difficult to obtain a rapid convergence. In a simply supported plate, the convergence is always good.

Finally, it should be stated that the authors are indebted to Marcal and his co-workers for guiding them to the required end through their investigations⁽²²⁾⁽²³⁾ which are formulated under the assumption including a similar constitutive law but a different solving procedure.

6. Conclusions

The following conclusions are made as a result of this study:

(1) From the comparisons among the present solutions, the pure bending solutions and the experimental results, it is found that the pure bending solutions fail at the initial part of the increasing load.

(2) The difference in progressive yielding due to the variation of plate thickness indicates that the contour pattern of a thin plate differs significantly from what is presumed by the bending solution. This is based on the effect of the membrane force.

(3) The load-deflection characteristic of a thin plate has an inclination to have good agreement with the elastic finite bending solution (the von Karman solution) over a wide range of pressures. The load at which the solution deviates abruptly from the von Karman solution corresponds approximately to the load at which the yielding zone due to the $\bar{\sigma}_p$ -criterion spreads over the whole section of the plate.

(4) The convergence of the iterative procedure is likely to be good for the adequately chosen increment of the loading parameter, but it is noted that an instability of the solution was experienced for large values of h/a of the clamped plate.

(5) The loading path plotted in the stress

(1) $qa^4/2Eh^4=1.07 \times 2$ (4) $qa^4/2Eh^4=1.07 \times 8$
(2) $qa^4/2Eh^4=1.07 \times 4$ (5) $qa^4/2Eh^4=1.07 \times 10$
(3) $qa^4/2Eh^4=1.07 \times 6$

Fig. 24 Distribution of surface strain ($Z=0.5$, pinned edge free to rotate)

(1) $qa^4/2Eh^4=1.07 \times 2$ (4) $qa^4/2Eh^4=1.07 \times 8$
(2) $qa^4/2Eh^4=1.07 \times 4$ (5) $qa^4/2Eh^4=1.07 \times 10$
(3) $qa^4/2Eh^4=1.07 \times 6$

Fig. 25 Progressive yielding (pinned edge free to rotate)

(a) Comparison between the theories with and without the couple terms of the stress resultants and the stress couples ($a/h=75$)

(b) Comparison between the theories with and without the couple terms of the stress resultants and the stress couples ($a/h=37.5$)
Fig. 26

Fig. 27 Maximum deflection in the computations with and without the iterative procedure

Appendix 1

The values of the maximum deflection of a clamped plate obtained by the theories with and without the coupling effects of the stress resultants and the stress couples are compared in Fig. 26 for $a/h=75$ and 37.5 (see section 3.3).

Appendix 2

The values of the maximum deflection of a clamped plate resulting from the computations with and without iterative procedure are compared in Fig. 27.

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space shows that the proportional loading does not occur if the calculation is performed up to a large loading parameter, which suggests that the deformation theory of plasticity is hard to apply to such type of problems as investigated in the present paper.

(6) The present analysis corresponds to the tangent modulus method of plasticity. The tangent modulus method without the iteration procedure which has been often employed in complicated problems gives sometimes an inaccurate solution as the case may be (Appendix 2).

Finally, the calculation of one cycle of the loading-unloading process in the present paper can be extended to examining a shake-down characteristics of structures and the present approach can be also extended to an analysis of shells of revolution under axisymmetrical loading condition associated with finite strain, shear deformation, anisotropic hardening plasticity and so on.

The authors are deeply indebted to the discussers who gave their helpful advices on all matters relating to the work described in this paper and pointed out the deficiencies of the paper.

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 (24) Yamada, Y., *Jour. Institute of Industrial Sci., Univ. Tokyo* (in Japanese), Vol. 19, No. 3 (1967), p. 75.
 (25) Yamada, Y., *Jour. Institute of Industrial Sci., Univ. Tokyo* (in Japanese), Vol. 20, No. 5 (1968), p. 199.

Discussion

T. SEKIYA: (University of Osaka Prefecture)

(1) There are some notations whose definitions are not given explicitly. Are they defined as follows?

$$(\bar{\sigma}_r, \bar{\sigma}_\theta, \bar{\sigma}_i) = (a^2/Eh^2)(\sigma_r, \sigma_\theta, \sigma_i)$$

$\dot{\psi}$ (in section 4.2): symbol of total quantities, e.g. total stresses

$$(\bar{R}, \bar{C}, \bar{T}) = E(R, C, T)$$

(2) Does the notation Σ mean the summation of the quantities defined at each mesh point? If so, it could not be the criterion of convergence for the value at each mesh point but one for mean value. How do the authors think about the criterion $\Sigma\{f_i^{(l)} - f_i^{(l-1)}\}^2 / \Sigma\{f_i^{(l)} - f_i^{(0)}\}^2 \leq \eta$ from this point of view?

(3) In Figs. 3 and 4, there is good agreement between the calculated and experimental results, but a discrepancy in the surface strains is observed near the edge point. Does this discrepancy only come from insufficient clamping of the specimen?

(4) In section 4.2, does the stage of $k=1$ correspond to the natural state?

M. TANAKA (Osaka University):

(5) In the present paper, the stress-strain relations of non-hardening material are also derived. Did the authors perform the numerical computation for such a material? Is it possible to apply the linearization technique as described in the paper to such an idealized material as the elastic-perfectly-plastic solid?

(6) The discussor thinks that the results obtained might be affected by the magnitude of pressure increment. Did the authors examine the accuracy of the numerical computation using such a factor?

(7) The discussor would like to ask about the computation time. i) How does it depend upon the mesh number? ii) Total time to finish a cycle of the loading-unloading process.

(8) Does the answer (5) mean that the convergence is not good even if the interpolated parameter is employed? By the way, how the

parameter is employed in the iterative procedure and what is the value of the parameter?

S. TAIRA (Kyoto University)

(9) There is no question so far as the theory is concerned. However, the following two questions arise on the experiment which examines the validity of the theory; i) influence of the anisotropy of the material tested (initial anisotropy and progressive anisotropy) and ii) boundary condition of the circular plate. In order to avoid the influence of material anisotropy, it is desirable to employ an isotropic material. To this end, isn't it necessary to examine the material anisotropy prior to the experiment?

For the problem ii), it might be required to decide a position where the plate is effectively clamped (an effective diameter). Is it impossible to carry out the preliminary experiment of the elastic bending using the wire strain gage?

Y. OHASHI and N. KAMIYA (Nagoya University)

(10) The authors assume that the flow theory is correct but the deformation theory is not in the elastic-plastic analysis. However, according to the recent investigations^(*), the classical flow theory is incorrect since it can not explain the general elastic-plastic deformation rigorously. That is, the flow theory as well as the deformation theory can not be correct in the elastic-plastic deformation except in the case of the proportional loading.

In Fig. 19, the stress trajectories at several points of plate surface are shown to conclude that the deformation theory is not adequate since proportional loading can not exist, for example, as shown at $x=0.8$. It appears, however, that the equivalent stress at this point is so small that the deformation of element is elastic or is in the state immediately after the plate begins to yield. If the element is elastic,

(*) Doshchinsky, G.A. and Maksak, V.I., *Inz. Zhur. Mekh. Tvyord. Tela*, Vol. 3 (1966), p. 153; Vol. 5 (1966), p. 118.

(*) Ilyushin, A.A., *Plasticity*, *Idz. AN SSSR*, (1963).

the deformation theory is undoubtedly correct even if the stress trajectory is not proportional.

On the other hand, even from the view-point that the flow theory is to be correct, the solution based on the deformation theory can be used approximately if the various discrimination criterions proposed by Budiansky^{(*)3}, Zhukov^{(*)4} and Ilyushin^{(*)2} are satisfied in cases except in the proportional loading. Hence, since the results obtained by the discussers^{(*)5} satisfy Budiansky's condition, the discussed problem is immaterial from this point of view.

The solution which is compared by the authors does not include the effect of membrane force so that, naturally, it will break down beyond a limiting deformation. The discussers believe that the discrepancy between the flow theory and the deformation theory can not be attributed to the difference of the theories and their inconsistencies, since the applicability of the deformation theory to the present problem is ensured as discussed above.

(11) Previously, one of the discussers solved the finite bending problem of a circular plate of mild steel using the analytical technique except the Runge-Kutta method applied at the final stage of integration with respect to radius^{(*)6}. In ref. ^{(*)5}, the exponential stress-strain law was used to make possible an analytical integration with respect to plate thickness. The Runge-Kutta method is applicable to calculation of behavior of a clamped circular plate which makes abrupt and nonlinear changes near its clamped edge because of the capability to select the mesh increment arbitrarily [for example, ref. (7)].

In the authors' finite difference solutions, the mesh subdivisions of eight in the direction of plate thickness and twenty in the direction of radius are chosen, and the latter is proved to be sufficient through the test calculations for a finer mesh. However, there is a doubt about it in the vicinity of clamped edge. For example, the progressive yielding of Figs. 17 and 22 show a remarkable difference from the above-mentioned results^{(*)6}, and moreover the curve

(2) in Fig. 18 must be perpendicular to the Z-axis at $x=0$. Is it possible to explain the situations as based on the change of plate thickness but not on the computational error?

Authors' closure

(1) Yes, they are. The first and the third equations were supplemented at the last part of section 4-1.

(2) Since Σ denotes the summation of the quantities obtained at all the mesh points, Eq. (46) can not be generally the criterion of convergence. But, in the present calculation where f has the same sign for all the mesh points, it can be used as a special case. For the general purposes, the relation pointed out by the discussers should be employed.

(3) There can be other than the boundary condition; for example, the assumption of an axisymmetric plane stress and the disregard of the transverse shear deformation. But, possibly the influence of the boundary condition may be most important.

(4) Yes, it corresponds to the natural state.

(5) As pointed out in the paper, it is possible to do so. After the authors received the question, they carried out a calculation of the elastic-perfectly plastic material in the case of a clamped edge, which is shown in Append.-Fig. 1. The solution using the deformation theory obtained by Ohashi and Murakami [ref. (8), p. 514] is compared to show that there is good agreement between the two theories within the calculated loading parameter.

But, naturally, the solution converges slowly as soon as the plate begins to yield. This may be due to the technique that the pressure increment is chosen as a constant value. To avoid this difficulty, the authors suppose an

Append.-Fig. 1 Load-maximum deflection diagram ($a/h=12.5$, clamped edge)

(*)3 Budiansky, B., *Trans. ASME, Ser.E*, Vol.26, No. 2 (1959), p. 259.

(*)4 Zhukov, A.N., *Problems in Theory of Plasticity, Izd. AN USSR*, (1961), p. 30.

(*)5 Ohashi, Y. and Kamiya, N. *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 33, No. 253 (1967-9), p. 1379.

(*)6 Ohashi, Y. and Murakami S., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 31, No. 224 (1965-4), p. 501, 514.

(*)7 Tamate, O. and Abe, H., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 30, No. 215 (1964-7), p. 828.

alternative method such as specifying the strain increment should be applied.

(6) No, the authors did not, but they presume that the pressure increment affects the convergence of the iterative method significantly but not so much the final results.

(7) i) Approximately, it increases proportionally with the mesh number. ii) The computing time for the case of the clamped edge (Section 5.2, $a/h=37.5$) using HITAC-5020 of Tokyo University is as follows:

* 14.87 minutes in the case of loading only (from $k=1$ to 200).

* 10.95 minutes in the case of unloading after loading (from $k=1$ to 100).

(8) Please refer to Section 5.5, which was revised in the present form after the question was received.

(9) i) Because of the difficulty to obtain an isotropic material, the following procedure is carried out: in order to examine the anisotropy, two specimens whose axes are perpendicular to each other, one of which coincides with rolling direction, are prepared and the stress-strain curve calculated as the mean value of the results obtained by two uniaxial tensile tests is employed. The discrepancy between the stresses of two curves is at most 5-8% at the strain of 0.3%.

The theoretical evaluation of the progressive anisotropy is a quite difficult problem. When the suitable plastic law is established in future, it will be and should be capable of being combined with the present formulation.

ii) It is possible to carry out such an experiment. The concept of effective diameter is introduced when the comparison is made with the results obtained by Ohashi and Kamiya.

(10) The discussers conclude, in the authors' opinion, that the conventional flow theory cannot be true for a real metal since the yield locus depends on the distribution of microscopic internal stresses and ultimately both two theories can not be true. As far as an ideal elastic-plastic body is concerned, however, the flow theory should be true as a mathematical model and the deformation theory corresponding to it should be approximately valid for the small curvature of the strain trajectory. Hence, one of the authors' purposes lays in denying the fact that one of the reasons why the deformation theory is preferred by some investigators is the complexity of computation procedure. As shown in Append.-Fig. 1, the difference between the two theories is not so remarkable within relatively

small effective strain as far as the load-deflection curve is concerned.

To explain the situation shown in Fig. 19, it is rewritten as Append.-Fig. 2. It shows that the curve of $x=0.8$ enters the yielding zone beyond the dotted curve and at the final stage the effective strain reaches as large as 2.45 times the initial yield strain.

Finally, the main purpose to refer to the results without the effect of membrane force is to compare the present solutions with the experimental results derived by the discussers.

(11) One of the features of the present formulation is in the capability of combining arbitrary plastic laws and arbitrary boundary conditions in the elastic-plastic analysis as well as the simplified computing procedure using the high speed digital computer. Then, the authors should like to emphasize that the basic situation based on is naturally different from the discussers'.

In the text, the authors state that the mesh number of twenty is sufficiently accurate for practical purposes. But this is based only on the results at the load where the yielding zone begins to appear. According to the latest test calculation at a large loading parameter, a rather remarkable error occurs as the case may be (Append.-Table 1). Consequently, unexpected errors may occur occasionally in the resultant couple which is calculated by differentiating the solution f . However, when the deflection is compared among the experimental result, the elastic finite bending solution and the solution using the deformation theory, the mesh number of twenty is found to be satisfactory for the practical purposes. Then, it is clear that the effect of plate thickness is much

Append.-Fig. 2 Stress path ($Z=0.5$)

Append.-Table 1 Solutions using various mesh numbers

(i) $qa^4/2Eh^4=1.07 \times 10$

mesh number	10		20		50	
x	f	g	f	g	f	g
0	0	0	0	0	0	0
0.2	-0.021 4	0.004 81	-0.021 5	0.004 85	-0.021 5	0.004 84
0.4	-0.038 7	0.008 05	-0.038 8	0.008 08	-0.038 8	0.008 06
0.6	-0.046 1	0.009 95	-0.046 2	0.009 98	-0.046 2	0.009 93
0.8	-0.036 2	0.010 8	-0.036 2	0.010 9	-0.036 3	0.010 8
1.0	0	0.011 5	0	0.011 5	0	0.011 4

(ii) $qa^4/2Eh^4=1.07 \times 40$

mesh number	10		20		50	
x	f	g	f	g	f	g
0	0	0	0	0	0	0
0.2	-0.011 3	0.005 57	-0.010 6	0.005 78	-0.009 05	0.006 18
0.4	-0.024 0	0.010 6	-0.023 5	0.011 2	-0.021 6	0.012 5
0.6	-0.033 9	0.013 9	-0.034 7	0.014 6	-0.033 2	0.016 0
0.8	-0.038 0	0.014 8	-0.040 0	0.015 1	-0.039 4	0.015 8
1.0	0	0.016 2	0	0.016 6	0	0.018 1

greater than that of the finite difference error. The authors wish to discuss the accuracy of the finite difference solution in a future work.

Finally, according to the discussers' opinion, the curve (2) in Fig. 18 is corrected as shown in the text.