A Study on the Constitutive Equation of Viscoelastic Fluids

(2nd Report, Converging and Diverging Flows in a V-Shaped Duct)

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The viscoelastic properties which would be important in the unsteady flow fields in the Lagrangian sense are studied for converging and diverging flows in a V-shaped duct.

When the viscoelastic fluid flows slowly in the duct, the velocity profile can be approximated by that of a purely viscous, non-Newtonian fluid, applying the Denn model as the constitutive equation; and the differential equation for the velocity profile can be solved by a perturbation method regarding the "power law index" as a perturbation parameter.

Using the velocity profile obtained, the stresses in the duct are analyzed. Results are in good agreement with the experimental results performed by Adams et al.

1. Introduction

Viscoelastic fluids have many rheological properties that cannot be observed in Newtonian fluids. Among these properties, the non-Newtonian viscosity, the normal stress effect and the memory effect are particularly important from an engineering viewpoint, so a constitutive equation must be able to explain them quantitatively\(^1\). However, in a certain special flow field, the effects of some of these properties on the flow may be negligible and in that case a rather simple constitutive equation can be used to solve the flow problems. Therefore, it may be significant to study what properties are important in such a special flow field.

From this point of view, in this paper, the viscoelastic properties required of a constitutive equation that is useful for the slow flow field are investigated by analyzing converging and diverging flows in a V-shaped duct.

Kalogirou\(^2\),\(^3\) analyzed a steady flow in a V-shaped channel and in a conical duct using the Oldroyd model as the constitutive equation. Adams et al.\(^4\) investigated the stress fields in a V-shaped duct experimentally and theoretically using the second-order theory of Coleman and Noll. But, there is a limit in the application of these models to the analysis of a flow behavior, because they cannot explain the non-Newtonian viscosity and the normal stress effect quantitatively except in the range of extremely low shear rates. Murai and Mori\(^5\) calculated numerically the velocity profile of the steady flow of a purely viscous non-Newtonian fluids in a V-shaped duct using the power law model.

Authors calculated analytically the velocity profile using a perturbation method and then analyzed the stresses in the channel. The results were compared with the experimental results by Adams et al.

2. Nomenclature

\(e^i\): contravariant components of rate-of-strain tensor
\(e_i\): covariant components of rate-of-strain tensor
\(g^i, g_i\): metric tensor
\(\mathbf{L}\): characteristic length of a channel
\(N_e\): Weissenberg number
\(n, s, \tau, \mu\): rheological constants in the Denn model
\(P_w\): pressure measured at the wall
\(p\): isotropic pressure
\(Q\): volumetric flow rate

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Newtonian viscosity and the normal stress difference.

The Denn model describes the memory effect because it contains a term of time derivative of stress. However, it is known that the memory effect cannot be explained quantitatively with such constitutive equations like the Denn model[9].

When the flow is so slow that the inertia is negligible, the second term of the left hand side in Eq. (1) may be small as compared with the first term. Then, expanding Eq. (1) in power series of \( \tau \), one gets the following equation as a first order approximation.

\[
\tau_{ij} = 2\mu \left( \frac{1}{2} \Pi \right)^{(n-1)/2} \varepsilon_{ij} - 2\tau \mu \left( \frac{1}{2} \Pi \right)^{(n-2)/2} \frac{\partial \varepsilon_{ij}}{\partial t}
\]

(7)

where the term containing \( D \Pi /Dt \) is neglected because \( D \Pi /Dt \) may be small when the inertia is negligible. Equation (7) may be useful for flows where the memory effect can be neglected.

2.2 Equations of motion and continuity

When the inertia and the gravity are negligible, the equation of motion can be given by

\[
0 = -g \delta_{ij} \rho \partial_{ij} + \tau_{ij}
\]

(8)

The equation of continuity is given by

\[
v_{r} = 0
\]

(9)

4. Velocity profile in a V-shaped channel

4.1 Derivation of a differential equation

The V-shaped duct is shown schematically in Fig. 1. It is assumed that half the vertex angle, \( \beta \), is sufficiently smaller than unity and the depth of the duct, \( w \), is so large compared with \( h \) that the flow is two-dimensional. If one assumes a radial flow, the velocity components are given by

\[
v_{r} = v(r, \theta), \quad v_{\theta} = v_{z} = 0
\]

(10)

Using Eq. (9), one obtains

\[
v = -g \frac{\partial}{\partial r}
\]

(11)

Using Eqs. (10) and (11), one gets the physical components of the stresses in a cylindrical coordinate system as follows;

\[
\tau_{rr} = 2\mu \left( \frac{1}{2} \Pi \right)^{(n-1)/2} \left( -\frac{g}{r^3} \right) + 2\tau \mu \left( \frac{1}{2} \Pi \right)^{(n-2)/2} \frac{g_{(1,)}^{(1)}}{r^2}
\]

(12)

\[
\tau_{\theta r} = 2\mu \left( \frac{1}{2} \Pi \right)^{(n-1)/2} \left( \frac{g}{r^3} \right) + 8\tau \mu \left( \frac{1}{2} \Pi \right)^{(n-2)/2} \frac{g_{(1,)}^{(2)}}{r^2}
\]

(13)

\[
\tau_{r \theta} = \mu \left( \frac{1}{2} \Pi \right)^{(n-1)/2} \left( \frac{g_{(1,)}^{(1)}}{r^3} \right) + 4\tau \mu \left( \frac{1}{2} \Pi \right)^{(n-2)/2} \frac{g_{(2,)}^{(1)}}{r^4}
\]

(14)

\[
\tau_{zz} = \tau_{\theta \theta} = \tau_{rr} = 0
\]

(15)
where
\[ \Pi = \frac{2}{r^2} \left\{ 4g^2 + \left(g^{(1)} \right)^2 \right\} \] ............................. (16)
and \( g^{(n)} \) denotes the \( n \)th derivative of \( g \) with respect to \( \theta \). Using Eq. (15), one gets
\[ \frac{\partial p}{\partial r} = \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{r\theta} - \tau_{\theta r}}{r} \] ........................................ (17)
\[ \frac{\partial p}{\partial \theta} = \frac{\partial \tau_{r\theta}}{\partial \theta} + r \frac{\partial \tau_{\theta r}}{\partial r} + 2\tau_{\theta r} \] ........................................ (18)
from Eq. (8). If one uses the nondimensional variables,
\[ \nu = \frac{1}{Lw} \left( \frac{Q}{w} \right)^* \] ............................. (19)
\[ r = Rr^* \] ............................. (20)
\[ \Pi = \left( \frac{Q}{Lw} \right)^2 \] ............................. (21)
\[ \tau_{nn} = \rho \left( \frac{Q}{Lw} \right)^2 \tau_{n*}^* \] ............................. (22)
\[ g(\theta) = \left( \frac{Q}{Lw} \right)^2 g^*(\theta) \] ............................. (23)
where \( L \) denotes the characteristic length of the duct. Differentiating Eqs. (17) and (18) with respect to \( \theta \) and \( r \) respectively and eliminating \( p \), one gets
\[ \frac{\partial^2 (\tau_{r\theta} - \tau_{\theta r})}{\partial \theta^2} + 1 \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r^2} \frac{\partial \tau_{r\theta}}{\partial \theta} \] ............................. (24)
\[ r \frac{\partial \tau_{r\theta}}{\partial r} + 3 \frac{\partial \tau_{r\theta}}{\partial r} = 0 \] ............................. (25)
Since \( \partial p/\partial r \) is small as compared with \( 1/r(\partial \nu/\partial \theta) \) when \( \beta \leq 1 \), the term including \( \partial p/\partial r \) in \( \Pi \) may be negligible for the differentiability of \( \Pi \). Rewriting Eqs. (12) to (14) in a nondimensional form and substituting it into Eq. (21) gives
\[ 4(2n-1)(n-1)g^{(2)} + 4(1-3n+n^2)(g^{(1)})^2 + n(n-1)(g^{(2)})^2 + n(g^{(1)})^2 \] ............................. (26)
\[ + \frac{N_1}{[1 + \epsilon + \epsilon^2 \epsilon^{(n+1)}]} \left[ (8-3s^2 + 3s - 1)g^*(s-1)(g^{(1)}(s-1) + 2(s-1)) \right] \] ............................. (27)
\[ + 2(2s-1)g^*(s-1)(g^{(3)} - 2(s-1)) = 0 \] ............................. (28)
where \( N_1 \) is the Weissenberg number defined by
\[ N_1 = \frac{Q}{Lw} \] ............................. (29)
It is found that the flow along rays cannot exist except in the case of \( n = 1 \) and \( s = 2 \), since Eq. (22) contains \( r^* \) in the second term. However, if \( N_1/[1 + \epsilon + \epsilon^2 \epsilon^{(n+1)}] \ll 1 \), the second term may be neglected in comparison with the first term and the flow can be considered as radial. Actually, since for many polymer solutions it can be considered that \( \epsilon < 1 \) and \( s > 0 \), the value of \( N_1 \) is very small for slow flows. Therefore, it is reasonable to assume that \( N_1/[1 + \epsilon + \epsilon^2 \epsilon^{(n+1)}] \ll 1 \) except in the case where \( r^* \) is extremely small. Then, neglecting the second term in Eq. (22), one gets
\[ 4(2n-1)(n-1)g^{(2)} - 4(1-3n+n^2)(g^{(1)})^2 + n(n-1)(g^{(2)})^2 + n(g^{(1)})^2 \] ............................. (30)
\[ + g^{(3)} - 2(s-1)\right] \] ............................. (31)
\[ \int_{\theta_0}^{\theta} g^* d\theta = +1 \] (converging flow) ............................. (32)
\[ \int_{\theta_0}^{\theta} g^* d\theta = -1 \] (diverging flow) ............................. (33)
\[ g^*(\theta) = \frac{\cos 2\beta - \cos 2\theta}{\sin 2\beta - 2\beta \cos 2\beta} \] ............................. (34)
Equation (27) is identical to one for Newtonian fluids and the solution is given by
\[ g^* = \frac{\cos 2\beta - \cos 2\theta}{\sin 2\beta - 2\beta \cos 2\beta} \] ............................. (35)
Now the higher approximation will be obtained only for the converging flow. The solution for the diverging flow can be obtained from that for the converging flow by changing the sign. The differential equation for the first order approximation is given by
\[ 4g^{(1)} + g^{(3)} = \frac{8 \cos 2\beta \cdot \cos 2\theta}{\sin 2\beta - 2\beta \cos 2\beta \sin 2\theta} \] ............................. (36)
The boundary conditions for \( g_\theta^* (m=1, 2, 3, \ldots) \) are
\[
\begin{align*}
g_\theta^* &= 0 \quad \text{at} \quad \theta = \beta, \quad g_\theta^{*(1)} = 0 \quad \text{at} \quad \theta = 0 \\
\int_{-\beta}^{\beta} g_\theta^* d\theta &= 0
\end{align*}
\]
.....(31)

The solution of Eq. (30) is
\[
\begin{align*}
g_\theta^* &= C_1 \cos 2\theta + \frac{\cos 2\beta}{\sin 2\beta - 2\beta \cos 2\beta} \\
&\times \left[ \log |\sin 2\theta| - \cos 2\theta \log |\tan \theta| \right] + C_2 \cdots (32)
\end{align*}
\]
where
\[
\begin{align*}
C_1 &= \frac{G(2\beta) \cdot \cos 2\beta}{(2\beta)^3} - \frac{\cos 2\beta \cdot \log |\tan \theta|}{\sin 2\beta - 2\beta \cos 2\beta} \\
C_2 &= -\frac{G(2\beta) \cdot \cos 2\beta}{(2\beta)^3} + \frac{\cos 2\beta \cdot \log |\sin 2\beta|}{\sin 2\beta - 2\beta \cos 2\beta} \\
G(2\beta) &= \frac{2s B_n(2\beta)^{2n+1}}{(2n+1)！}
\end{align*}
\]
and \( B_n \) is the Bernoulli's number. Consequently, the first-order solution of Eq. (24) is found from Eqs. (29) and (32) as
\[
\begin{align*}
g_\theta^* &= \cos 2\beta - \cos 2\theta + (1-n) \left[ C_1 \cos 2\theta \\
&+ \frac{\cos 2\beta \log |\sin 2\theta| - \cos 2\beta \log |\tan \theta|}{\sin 2\beta - 2\beta \cos 2\beta} \right] + C_2 \cdots (33)
\end{align*}
\]
The exact solution of Eq. (24) for \( n=0.5, \) or \( \varepsilon=0.5, \) is obtained by Murai and Mori, and given as
\[
g_\theta^* = \frac{2(\theta - \beta) + \sin 2\beta - \sin 2\theta}{1 + 2\beta - 2\sin 2\beta - 2\cos 2\beta} \cdots (34)
\]

In Fig. 2, the exact solution and the first-order solution are shown setting \( \beta=15^\circ. \) The difference between them is relatively large. Therefore, when \( \varepsilon \) is as large as in this case, it is necessary to obtain a second-order solution. Though it is possible to obtain the second-order solution by continuing the above method, the solution seems to be so complex that it is useless, because the non-homogeneous term of the differential equation becomes very complex. As it is assumed that \( \beta \) is much smaller than unity, the solution can be expanded in terms of \( \beta \) and \( \theta. \) Then, the basic solution becomes
\[
g_\theta^* = \frac{3}{4\beta} \left[ 1 - (\frac{\theta}{\beta})^2 \right] \cdots (35)
\]
in place of Eq. (29), and the differential equation for \( g_\theta^* \) is given by
\[
4g_\theta^{*(1)} + g_\theta^{*(3)} = \frac{3(1-8\beta/5)}{2\beta^2 \theta} \cdots (36)
\]
Solving Eq. (36) under the boundary conditions, Eq. (31), and neglecting higher order terms, one gets
\[
g_\theta^* = \frac{3(1-8\beta/5)}{4\beta^2 \theta} \left[ \frac{1}{6} - (\frac{\theta}{\beta})^2 + (\frac{\theta}{\beta})^2 \log |\frac{\theta}{\beta}| \right] \cdots (37)
\]
The equation for \( g_\theta^* \) is given by
\[
4g_\theta^{*(1)} + g_\theta^{*(3)} = \frac{3}{2\beta^2} \left[ \frac{1}{9} \left( \frac{\theta}{\beta} - \frac{47}{5} \beta \right)^2 \frac{1}{\theta} \right. \right.
\]
\[
\left. + \left( 1 - \frac{8}{5} \beta^2 \right)^2 \log |\frac{\theta}{\beta}| \right] \cdots (38)
\]
and the solution is
\[
g_\theta^* = \frac{3}{8\beta} \left[ \left( \frac{1-2\beta^2/5}{6} - (\frac{\theta}{\beta})^2 \right) + (1-8\beta^2) \right] \times \left( \frac{\theta}{\beta} \right)^2 \log |\frac{\theta}{\beta}| \cdots (39)
\]
Consequently, the second-order solution is given by
\[
\begin{align*}
g_\theta^* &= \frac{3}{4\beta} \left[ 1 - (\frac{\theta}{\beta})^2 \right] + (1-n) \left[ \frac{3}{4\beta} \left( \frac{1-8\beta^2/5}{6} \right) \right. \right.
\]
\[
\left. + \left( \frac{1-2\beta^2/5}{6} - (\frac{\theta}{\beta})^2 \right) + (1-n) \right]^2 \times \left( \frac{\theta}{\beta} \right)^2 \log |\frac{\theta}{\beta}| \cdots (40)
\end{align*}
\]
In Fig. 2, the calculated curve of Eq. (40) for \( n=0.5 \) and \( \beta=15^\circ \) is shown. The figure shows that the result agrees well with the exact solution, Eq. (34), and it can be considered that Eq. (40) is useful when \( \beta \) is small. In Fig. 3 the second-order solutions for \( g_\theta^* \) are shown for different values of \( n \) by setting \( \beta=15^\circ. \)

5. Comparison with the experimental data by Adams et al.

Adams et al. obtained the stress data in a V-shaped duct, using the birefringent method and a highly birefringent material, polystyrene
in Aroclor, and compared the results with an analysis in terms of the second-order theory. In the paper, however, they claimed that the fluid exhibited the non-Newtonian viscosity and the normal stress difference in the simple shear flow was not proportional to the square of the shear rate. Therefore, the experimental data were compared with the results obtained in the preceding section. The second-order theory is given by

$$\tau_{ij} = 2\eta \epsilon_{ij} + 4\zeta \epsilon_{im} \epsilon_{mj} + 2\frac{\partial \epsilon_{ij}}{\partial t}$$

(41)

If $\zeta = 0$, Eq. (41) is equal to Eq. (7) when $n = 1$ and $s = 2$. In Table 1, rheological constants in Eq. (7) which were obtained by Adams et al. from the viscometric data and the data of the normal stress difference are shown together with rheological constants in Eq. (41), $\beta$, $Q$, and $\omega$.

The shear stress and the normal stress difference in terms of the Denn model can be calculated from Eqs. (7), (12), (13), (14), (16) and (20) and given by

$$\tau_{\theta \theta} = \frac{\mu}{r^{2n}} \left( \frac{Q}{w} \right)^n \left[ 4\left( \frac{\eta}{\psi} \right)^2 + \left( \frac{g^{(n)(1)}}{\psi} \right)^2 \right]^{(n-1)/2} \frac{g^{(n)(1)}}{\psi}$$

$$+ \frac{4\eta}{r^{2s}} \left( \frac{Q}{w} \right)^s \left[ 4\left( \frac{\eta}{\psi} \right)^2 + \left( \frac{g^{(1)(1)}}{\psi} \right)^2 \right]^{(s-1)/2} \frac{g^{(1)(1)}}{\psi}$$

(42)

The stress components in terms of the second-order theory are given by

$$\tau_{\theta \theta} = \frac{\eta}{r^{2n}} \left( \frac{Q}{w} \right)^n \frac{4\eta}{r^{2s}} \left( \frac{Q}{w} \right)^s \frac{g^{(n)(1)}}{\psi} \frac{g^{(1)(1)}}{\psi}$$

$$\tau_{\rho \rho} = \frac{\eta}{r^{2n}} \left( \frac{Q}{w} \right)^n \frac{4\eta}{r^{2s}} \left( \frac{Q}{w} \right)^s \frac{g^{(n)(1)}}{\psi} \frac{g^{(1)(1)}}{\psi}$$

(43)

In Figs. (4) to (7)*, the present analysis and the experimental data are plotted. The solid lines are obtained from Eqs. (42) and (43), using Eq.

![Image](image-url)

**Fig. 4**

![Image](image-url)

**Fig. 5**

<table>
<thead>
<tr>
<th>Table 1</th>
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<tr>
<td>Denn model</td>
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<tr>
<td>-----------------</td>
</tr>
<tr>
<td>$\mu$ dyn·sec$^2$ cm$^{-5}$</td>
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<td>270</td>
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* Obtained by private communication.
(40) for \( g^* \). When \( n \) is close to unity as in this case, the difference between the first and the second approximations of \( g^* \) is negligibly small. The figures show that the results of the present analysis in terms of the Denn model agree with the experimental results better than those in terms of the second-order theory. Though it may seem that the difference between the present analysis and the analysis in terms of the second-order theory is slight, it can be considered that the non-Newtonian viscosity and the normal stress effect affect the stress field in the duct, since the rheological constants, \( n \) and \( s \), of the fluid used in the experiment differ only slightly from unity and 2.0 respectively.

From the results, it is concluded that in a slow flow field, the non-Newtonian viscosity and the normal stress effect are more important than the memory effect and that the Denn model is a useful constitutive equation.

6. Pressure drop in the duct

The pressure measured at the side wall is given by

\[
P_\infty(r) = p(r, \beta) - \tau_{oo}(r, \beta)
\]

(46)

From Eq. (7), one finds

\[
\frac{\partial (p-\tau_{oo})}{\partial r} = \frac{\partial (\tau_{rr}-\tau_{oo})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{rr}-\tau_{oo}}{r}
\]

(47)

Substituting Eq. (47) into Eqs. (12) to (14), setting \( \beta = \beta \) and using Eq. (46), one obtains

\[
\frac{\partial P^*_{\infty}}{\partial r^*} = \frac{1}{R_*} F_1(n) N_i F_2(n, s)
\]

(48)

as the nondimensional pressure gradient at the side wall, where

\[
P^*_{\infty} = \frac{P_\infty}{\rho (Q/L)^2}
\]

(49)

\[
R_* = \frac{\rho L^2 (n-1)}{Q}
\]

(50)

\[
F_1(n) = n |g^{(2)}(\beta)|^{n-1} g^{(2)}(\beta)
\]

(51)

\[
F_2(n, s) = -2(3-2s) |g^{(1)}(\beta)|^s
\]

(52)

and \( R_* \) is the generalized Reynolds number. The first term of right hand side in Eq. (48) is a term resulting from viscosity and the second term is a term resulting from elasticity. In Fig. 8, \( F_1(n) \) for converging flow vs. \( n \) is shown by setting \( \beta = 15^\circ \). The relation between \( F_1(n) \) for diverging flow and \( n \) can be obtained only by adding the minus sign to \( F_1(n) \) for converging flow. In Fig. 9, \( F_2(n, s) \) vs. \( n \) is shown for various values of \( s \) by setting \( \beta = 15^\circ \). The figure shows that \( F_2(n, s) \) varies from minus to plus values according to an increase of the value of \( s \). Figure 9 and Eq. (48) show that in the case of the converging flow, the elastic effect contributes to a decrease in the pressure drop if \( F_2(n, s) > 0 \)
and to its increase if \( F_2(n, s) < 0 \), and that in the case of the diverging flow the elastic effect contributes negatively to the converging flow. It is interesting that the elastic effect does not affect the pressure drop when \( s = 1.5 \).

In order to examine the effect of the elastic effect on the pressure drop quantitatively, the experimental result by Adams et al. was used; if one chooses the unit length as the characteristic length of the duct, one gets

\[
N_1 = 0.0232
\]

The ratio of the second term and the first term of right-hand side in Eq. (48) at the point of \( r^* = 1 \) is

\[
\frac{\text{elastic term}}{\text{viscous term}} = \frac{N_1 F_2(n, s)}{F_1(n)} \approx 0.11
\]

and it is found that the pressure drop by the elastic effect is about 10 per cent of the total.

## 7. Conclusions

(1) It is shown that the velocity profile of viscoelastic fluids in the V-shaped duct can be approximated by that of purely viscous non-Newtonian fluids when the flow is so slow that the inertia force can be neglected.

(2) And then the differential equation for the velocity that is derived by Murai and Mori was solved by a perturbation method.

(3) Using the results, the stresses in the duct were analyzed. The results are in good agreement with the experimental results by Adams et al.

(4) It is concluded that for a slow flow in which the inertia force can be neglected the memory effect is negligible, but it is necessary to take account of the non-Newtonian viscosity and the normal stress effect. When the flow rate increases, however, the flow differs from that along rays, and in the extreme case unsteady phenomena will occur. In such a case, the memory effect can not be neglected. One of the authors is at present working on this flow behavior.

## References


## Discussion

### I. Ashidate (University of Tokyo):

(1) Is Eq. (7) obtained by expanding \( \varepsilon^{ij} \) and \( \tau^{ij} \) in Taylor series with respect to \( \varepsilon \) and substituting them into Eq. (1), by assuming that they are functions of \( \tau \)? Is the second term of the left hand side in Eq. (1) neglected? Or, is Taylor expansion of this term taken into account?

(2) When \( n=1 \) and \( s=2 \), Eq. (22) becomes

\[
4(g^{(2)})^2 + g^{(1)}g^{(3)} + \frac{N_1}{r^*} [g^{(1)}g^{(1)}]^2 + 2g^{(1)}g^{(3)} = 0
\]

and the solution may be given by the form as \( g^* = g^*(\theta, r^*) \). Please show the reason why the flow is radial.

### T. Tanaka (Nagoya University):

Analytical results of the shear stresses and the normal stress differences are compared with the experimental results by Adams et al. in Figs. 4 to 7. Most of these figures show that the theory and experiment agree very well. However, in the case of the normal stress difference on the midway line, \( \theta/\beta = 1/2 \), for the diverging flow, both disagree very much. With respect to this point, the following questions arise:

(3) Good agreement is shown even for the
diverging flow in the cases of the center line and of the side wall in the whole range from just before the entrance to the exit; but only in the case of the midway line, a large deviation is found especially near the entrance and after the middle portion the deviation disappears.

(4) Though the experimental result shows that the sign of $(\tau_{rr} - \tau_{\theta\theta})$ changes near the entrance, both analytical results by the Denn model and the second-order theory do not change the sign at least in the range of $r > 0.6$ cm.

(5) Both analytical results show that $\tau_{rr} - \tau_{\theta\theta} = 0$ at $r = 0.6$ cm.

**Authors' closures**

(1) Equation (7) is not derived by the Taylor series expansion but by a perturbation expansion: That is, by assuming that $\tau^{ij}$ is given by an expansion of power series in $\tau$, $\tau^{ij} = \tau_0^{ij} + \tau_1^{ij} + \tau_2^{ij} + \cdots$ 

and substituting Eq. (B) into Eq. (1), one obtains

$$[\tau_0^{ij} + \tau_1^{ij} + \cdots] + \tau \left[ \frac{1}{2} \Pi \right]^{(s-1)/2} \frac{\partial}{\partial t}$$

$$\times [\tau_0^{ij} + \tau_1^{ij} + \cdots] = 2 \mu \left[ \frac{1}{2} \Pi \right]^{(s-1)/2} e^{ij}$$

Arranging Eq. (C) as power series in $\tau$ and putting the coefficient of $\tau^n (n = 0, 1, 2, \ldots)$ equal to zero, one gets

$$\tau_0^{ij} = 2 \mu \left[ \frac{1}{2} \Pi \right]^{(s-1)/2} e^{ij}$$

$$\tau_1^{ij} = -2 \mu \left[ \frac{1}{2} \Pi \right]^{(s-2)/2} \frac{\partial e^{ij}}{\partial t}$$

$$+ \frac{1}{2} \left[ \frac{1}{2} \Pi \right]^{(s-1)/2} e^{ij}$$

where $D/Dt$ denotes the substantial derivative. In the case of a slow flow in which the inertia force can be neglected, it may be considered that $D \Pi / Dt = 0$, and Eq. (E) becomes

$$\tau_1^{ij} = -2 \mu \left[ \frac{1}{2} \Pi \right]^{(s-1)/2} \frac{\partial e^{ij}}{\partial t}$$

Consequently, the first order approximation for $\tau^{ij}$ is given by

$$\tau^{ij} = 2 \mu \left[ \frac{1}{2} \Pi \right]^{(s-1)/2} e^{ij} - 2 \tau \mu \left[ \frac{1}{2} \Pi \right]^{(s-2)/2} \frac{\partial e^{ij}}{\partial t}$$

from Eqs. (B), (D) and (F).

(2) Equation (A) can be rewritten as

$$4(g^{(1)} + g^{(2)} - 2g^{(3)})[1 + \frac{N_r}{r^{s-1}} \cdot 2g^{*}] = 0$$

Therefore,

$$4(g^{(1)} + g^{(2)} - 2g^{(3)}) = 0$$

and the flow is radial, since $g^*$ is a function of $\theta$ only.

(3) The reason may be that the value of $(\tau_{rr} - \tau_{\theta\theta})$ must decrease abruptly when the upstream parallel flow (Coullet flow) changes to a radial flow. The value of $(\tau_{xx} - \tau_{yy})$ on the midway line in the Coullet flow in terms of Denn model is given by

$$\frac{(\tau_{xx} - \tau_{yy})_{x=1/4}}{3} = 3.229 \text{ dyn/cm}^2$$

This value is nearly equal to that of

$$\frac{(\tau_{rr} - \tau_{\theta\theta})_{r=2/1}}{3}$$

at the entrance (Append. Fig. 1). On the other hand, in the radial flow, it is found from Fig. 6 that $\tau_{rr} - \tau_{\theta\theta} = -1.000 \text{ dyn/cm}^2$ at $r = 0.8$ cm. Actually, a discontinuous change of the normal stress difference is impossible and a complex flow may occur near the entrance. But, as the fluid flows downstream, this entrance effect may decrease, so that the analytical and the experimental results come to agree well.

The reason why the analytical results agree well with the experimental data in both cases of the shear stress on the midway line and the normal stress difference on the centerline or at the wall may be given as follows: The calculated value of the shear stress on the midway line in the Coullet flow is $\tau_{xy} = -4.185 \text{ dyn/cm}^2$ and the corresponding value in the radial flow does not change very much from this value. On the center line, since the direction of motion of the fluid particle does not change and the velocity at the wall is zero, the entrance effect does not affect so much the velocity or the velocity gradient at these points.

(4) The reason may be explained that the flow near the entrance is affected by the upstream Coullet flow. Since the calculated value of $(\tau_{rr} - \tau_{\theta\theta})$ at $r = 0.6$ cm in terms of the Denn model is positive (43.0 dyn/cm$^2$), the change of the sign occurs.

(5) Actually, both analytical results show that

$$(\tau_{rr} - \tau_{\theta\theta}) = 43.0 \text{ dyn/cm}^2 \quad \text{(Denn model)}$$

$$(\tau_{rr} - \tau_{\theta\theta}) = -64.3 \text{ dyn/cm}^2 \quad \text{(Second-order theory)}$$