Effect of Non-Uniform Inlet Velocity Profile on the Development of a Laminar Flow between Parallel Plates*

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This paper describes a theoretical analysis of a laminar developing flow in the entrance region between parallel plates, when the inlet flow has a constant velocity gradient normal to the channel axis. The analysis is performed within the framework of the boundary layer theory, and governing equations are solved by a finite-difference method. The following results are obtained from the analysis: (1) If the flow rate through the channel is constant, the additional pressure drop in the entrance region is considerably decreased as the normal velocity gradient at the entrance section is increased. (2) As is physically expected, when isothermal walls of the channel are considered, the mean heat transfer rate is much larger at the wall of higher inlet velocity side than at the smaller inlet velocity side. However, an average value of the heat transfer rate on the two walls as a whole is smaller by a few percent than that for the case of a uniform inlet velocity profile. (3) The entrance length shows a significant reduction with the increase of the transverse velocity gradient at the inlet section.

1. Introduction

The hydrodynamically developing flows in the entrance region of pipes and channels have typically been analyzed under the condition that the inlet velocity profile and subsequent downstream profiles are symmetric with respect to the channel centerline. In practice, most of the velocity profile at the channel inlet may be asymmetric, with the result that the developing velocity profiles are also asymmetric. The characteristics of development of such flows will be different from those of the flow in an entrance region with uniform inlet velocity profile. An example of this sort of situation will be found in the flow near mouths of pipes in a certain type of heat exchanger, where the velocity profile may generally have a velocity gradient in the direction normal to the flow owing to the non-uniform velocity profile that would exist in the upstream flow passage.

The purpose of the present paper is to theoretically describe the effects of asymmetric velocity profile at the channel inlet on its characteristics, such as pressure drop, viscous shear stress and heat transfer variations along the channel. For the convenience of analysis, the inlet velocity is assumed to include a constant transverse velocity gradient with respect to the channel centerline. A steady flow of a viscous incompressible fluid between parallel plates of constant temperature is considered. It is also assumed that the fluid properties such as density, viscosity, thermal conductivity etc. are independent of temperature or pressure.

2. Nomenclature

- $c_p$: specific heat at constant pressure
- $h_e$: heat transfer coefficient
- $k$: thermal conductivity
- $\rho$: density
- $p$: dimensionless pressure $=p'/(\rho W^2)$
- $p'$: pressure
- $v^*, v, w$: dimensionless velocity components $v^* = v'w$, $v = R_e v^*$, $w = w'/W$
- $v', w'$: velocity components in y' and z'-direction
- $y^*, z^*$: dimensionless coordinates $y = y'/H$, $z = z^*/R_e$
- $y'$, $z'$: Cartesian coordinates
- $C_a$: dimensionless additional pressure drop
- $H$: half-spacing between the upper and lower walls
- $K$: shear parameter $= \omega H/W$
- $L$: dimensionless inlet length

* Received 30th August, 1969.
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\[ N_{zt} : \text{local Nusselt number} = 2Hh_i/k \]
\[ \bar{N}_{zt} : \text{mean Nusselt number} = \left( \frac{1}{z} \right) \int_{0}^{z} N_{zt}dz \]
\[ P_r : \text{Prandtl number} = \mu c_p/k \]
\[ R_e : \text{Reynolds number} = WH/\nu \]
\[ T' : \text{temperature} \]
\[ T : \text{dimensionless temperature} = \left( T' - T_w \right)/\left( T_0 - T_w \right) \]
\[ T_0 : \text{temperature of fluid at the inlet} \]
\[ T_w : \text{temperature of the channel walls} \]
\[ T_m' : \text{bulk temperature} \]
\[ = \int_{-H}^{H} T' w' dy'/\int_{-H}^{H} w' dy' \]
\[ T_m : \text{dimensionless bulk temperature} \]
\[ = \left( T_m' - T_w \right)/(T_0 - T_w) \]
\[ W : \text{mean velocity} \]
\[ \mu : \text{viscosity} \]
\[ \nu : \text{kinematic viscosity} \]
\[ \rho : \text{density} \]
\[ \tau' : \text{shear stress} \]
\[ \tau : \text{dimensionless shear stress} = \left( \tau' / \rho W^2 \right) \]
\[ \omega : \text{velocity gradient at the entrance section} \]

Suffices + and − imply the values at the upper and lower walls, respectively.

3. Governing equations

Two-dimensional flows of a viscous incompressible fluid are described by the equations of continuity and two Navier-Stokes equations in two-coordinate direction. When heat transfer is considered, the thermal energy equation is further added as the fourth equation. In the case of a steady laminar flow, these equations can be written as

\[
\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{1.a}
\]

\[
v \frac{\partial w'}{\partial y} - w \frac{\partial v'}{\partial z} = -\frac{1}{\rho} \frac{\partial p'}{\partial z} + \nu \frac{\partial^2 w'}{\partial z^2} \tag{1.b}
\]

\[
v \frac{\partial v'}{\partial y} + w \frac{\partial w'}{\partial z} = -\frac{1}{\rho} \frac{\partial p'}{\partial y} + \nu \frac{\partial^2 w'}{\partial y^2} \tag{1.c}
\]

\[
\rho c_p \left( v' \frac{\partial T'}{\partial y} + w' \frac{\partial T'}{\partial z} \right) = k \frac{\partial^2 T'}{\partial y^2} + \mu \Phi' \tag{1.d}
\]

where \( \Phi' \) means the dissipation function and

\( \nu' = (\nu^2/\partial y^2) + (\partial^2/\partial z^2) \)

The configuration is shown in Fig.1, where \( z' \) corresponds to the channel centerline. The origin of \( z' \) is taken at the inlet section and \( y' \) is taken normal to the \( z' \)-axis. The spacing between the upper and lower walls of the channel is denoted by \( 2H \), and the velocity by \( W \). Then, the inlet velocity profile can be described by \( w' = W + \omega y' \)

\[ T_0, T_w \] which are assumed to be constant denote the fluid temperature at the inlet and the temperature of the channel walls, respectively. The laminar developing flow in the entrance region of a two-dimensional duct can generally be treated by means of the boundary layer equation derived by Prandtl. However, since a non-uniform inlet velocity profile is included in the present problem, it would be proper to first consider the applicability of Prandtl's boundary layer equation. To begin with, one introduces dimensionless variables and parameters with the following definitions:

\[
y = y'/H, \quad z = z'/H, \quad v = v'/W, \quad w = w'/W \]
\[
p = p'/(\rho W^2), \quad T = (T' - T_w)/(T_0 - T_w) \]

\[ K = \omega H/W, \quad R_e = WH/\nu \]

where \( K \) can be interpreted as the dimensionless velocity gradient at the inlet section and \( R_e \) is a Reynolds number. As far as an inviscid core exists in the central part of the developing region, the viscous flows near the upper and lower walls of the channel could be treated by the method described by van Dyke(1) and Devan(2) for the case of the laminar boundary layer over a body being placed in a uniform shear flow. They showed that the flow and temperature fields of the laminar boundary layer concerned can be written in the following form, when the thickness of the boundary layer is much less than the length scale \( W/\omega \):

\[ w(y, z'; R_e) = w_i(Y, z^*) + R_e^{-1/2} w_0(Y, z^*) + o(R_e^{-1/2}) \]

\[ v(y, z'; R_e) = R_e^{-1} v_i(Y, z^*) + o(R_e^{-1}) \]

\[ p(y, z'; R_e) = p_i(Y, z^*) + R_e^{-1/2} p_0(Y, z^*) + o(R_e^{-1/2}) \]

\[ T_i(Y, z^*) + R_e^{-1/2} T_0(Y, z^*) + o(R_e^{-1/2}) \]

where \( Y = R_e^{1/2} y \).

Substitution of these expansions into the dimensionless forms of Eqs. (1) gives the first approximation, when the coefficients of the same powers of \( R_e \) are compared:

\[
\frac{\partial w_i}{\partial z^*} + \frac{\partial v_i}{\partial Y} = 0 \tag{2.a}
\]

\[
w_i \frac{\partial w_i}{\partial z^*} + v_i \frac{\partial w_i}{\partial Y} = -\frac{\partial p_i}{\partial z^*} + \frac{\partial^2 w_i}{\partial Y^2} \tag{2.b}
\]

Fig. 1 Definition sketch and coordinate system
\begin{equation}
\frac{\partial T_1}{\partial Y} = \frac{1}{P_* \frac{\partial Y}{Y^2}} \quad (2.\text{c})
\end{equation}

\begin{equation}
w_1 \frac{\partial T_1}{\partial Y} + v_* \frac{\partial T_1}{\partial Y} = 0 \quad (2.\text{d})
\end{equation}

These are the Prandtl equations for the laminar boundary layer flow. In the same way, the second approximation is

\begin{equation}
\frac{\partial w_1}{\partial z^*} + \frac{\partial v_*}{\partial Y} = 0 \quad (3.\text{a})
\end{equation}

\begin{equation}
w_1 \frac{\partial w_1}{\partial z^*} + v_* \frac{\partial w_2}{\partial Y} + v_* \frac{\partial w_1}{\partial Y} = - \frac{\partial P_*}{\partial Y} \quad (3.\text{b})
\end{equation}

\begin{equation}
\frac{\partial P_*}{\partial Y} = 0 \quad (3.\text{c})
\end{equation}

\begin{equation}
\left( w_1 \frac{\partial T_1}{\partial z^*} + v_* \frac{\partial T_1}{\partial Y} \right) T_1 + \left( w_1 \frac{\partial T_1}{\partial z^*} + v_* \frac{\partial T_1}{\partial Y} \right) T_1
\end{equation}

\begin{equation}
= \frac{1}{P_* \frac{\partial Y}{Y^2}} \quad (3.\text{d})
\end{equation}

The terms of the viscous dissipation have been omitted from the thermal energy equations (2.\text{d}) and (3.\text{d}). The term \( \frac{\partial P_*}{\partial Y} \) which appears on the right-hand side of Eq. (3.\text{b}) will take the form of

\begin{equation}
\frac{\partial P_*}{\partial z^*} = -K \frac{d\delta_1}{dz^*} + \frac{dP_*}{dz^*} \quad (4)
\end{equation}

in which \( \delta_1 \) denotes

\begin{equation}
\delta_1 = \int_0^\infty \left[ W_1(z^*) - w_1(Y, z^*) \right] dY
\end{equation}

\( dP_*/dz^* \) implies a pressure gradient which would be originated from the flow due to displacement thickness of the boundary layer described by Eqs. (2.), while \( W_1(z^*) \) implies the velocity of an inviscid flow along the surface of the body. If it is intended to apply this scheme of solution to the laminar developing flow with the inlet velocity profile given by \( w = 1 + Ky \), solutions of Eqs. (2.) and (3.) would be obtained in different forms near the upper and lower walls of the channel, respectively, since the boundary conditions are different on each wall of the channel. The solutions for these two walls must be properly joined in order to solve the whole flow field in the entrance region. For example, when the variables \( y^* \) and \( y^- \) are introduced by defining as

\begin{equation}
y^* = 1 + y \quad y^- = 1 - y
\end{equation}

the boundary conditions at the upper and lower walls in the immediate channel neighbourhood of \( z^* = 0 \) become as follows so far as \( W_1(z^*) \) is judged to be almost constant in the region concerned. In terms of the new variables defined by \( Y^* = R_1^{1/2} y^* \) and \( Y^- = R_1^{1/2} y^- \) the boundary conditions to be applied to Eqs. (2.) and (3.) now become

\begin{equation}
Y^*, Y^- = 0 : w_1 = v_1^* = 0, T_1 = 0.
\end{equation}

\begin{equation}
w_1 = v_1^* = 0, T_2 = 0
\end{equation}

\begin{equation}
Y^* \to \infty : w_1 \to W_1^+, w_1 \to -K Y^*.
\end{equation}

\begin{equation}
T_1 = 0, T_2 = 0
\end{equation}

\begin{equation}
Y^- \to \infty : w_1 \to W_1^-, w_1 \to -K Y^-.
\end{equation}

\begin{equation}
T_1 = 0, T_2 = 0
\end{equation}

It is clear from these boundary conditions that Eqs. (2.) represent the flow and temperature fields in the laminar boundary layer along a flat plate placed in a uniform flow given by \( w = W_1^+ \) or \( w = W_1^- \), while Eqs. (3.) describe the effect of the velocity gradient in the \( y \)-direction existing in the main flow outside the boundary layer.

Application of the method just mentioned above to the entrance region flow considered in this paper would become a little complicated owing to the fact that the two solutions near the upper and lower walls must be matched appropriately in the manner to satisfy the continuity relation in the channel. Moreover, the expansion scheme leading to Eqs. (2.) and (3.) may not be valid in the region where the order of the boundary layer thickness is equal to or larger than that of the length scale \( W_1/o \). In this case, Eqs. (2.) and (3.) must be replaced by the boundary layer equations derived by Lu Ting(3) and Devan(4) which do not in general permit similarity solutions. However, at sections downstream of the position where the boundary layers along the upper and lower walls of the channel meet together, the flow will appropriately be described by Eqs. (2.).

These facts make it quite difficult to treat the entrance region flow considered here by means of the general procedure described above. Accordingly, application of Prandtl's boundary layer equations given by Eqs.(2.) is tried throughout the whole flow and temperature fields of the hydrodynamically developing region in the present case. The boundary conditions for Eqs. (2.) now become

\begin{equation}
z^* = 0 : w_1 = 1 + Ky, T_1 = 1
\end{equation}

\begin{equation}
z^* > 0, y^* = \pm 1 : w_1 = v_1^* = 0, T_1 = 0
\end{equation}

This approximation for the effects of the transverse velocity gradient in the inlet velocity profile may be called the method of a first-order treatment of the second-order effects. The validity of the approximation will be discussed in Chapter 6. When the new variables \( z \) and \( v \) are introduced by expressing as

\begin{equation}
z = z^*/R_1, \quad v = v^* R_1
\end{equation}

and \( Y \) is replaced by \( y \), Eqs. (2.) can be written as

\begin{equation}
w \frac{\partial w}{\partial z} - \frac{\partial w}{\partial y} \int_{-1}^1 \frac{\partial w}{\partial z} dy = - \frac{d \eta}{dz} + \frac{\partial^2 w}{\partial y^2} \quad (5)
\end{equation}
\[ w_0 \frac{\partial T}{\partial z} - \frac{\partial T}{\partial y} \int_1^y \frac{\partial w}{\partial y} \, dy = \frac{1}{p_r} \frac{\partial^2 T}{\partial y^2} \]  
(6)\\
where the suffix 1 is omitted, and the relation\\
\[ v = -\int_1^y \frac{\partial w}{\partial y} \, dy \]

derivable from the equation of continuity given by Eq. (2.1) is substituted. In terms of the new variables the boundary conditions now become\\
\[ z = 0 : w = w_1 + Ky, \ T = 1 \]  
(7.a)\\
\[ z > 0, \ y = \pm 1 : w = v = 0, \ T = 0 \]  
(7.b)\\

4. Finite-difference equations

In order to obtain an analytical solution of Eqs. (5) and (6) together with the boundary conditions (7.a) and (7.b), a perturbation method similar to that of Schlichting(4) or an appropriate linearization technique of inertia terms such as developed by Langhaar(5) and Sparrow et al.(6) will generally be employed. However, there is a little apprehension that these approaches would necessarily lead to some errors in the resulting solutions. Accordingly, the finite-difference method developed by Leigh(7) and Terrill(8) is applied in its extended form to the laminar entrance region flow considered in this paper with the object to minimize the expected errors.

The finite-difference forms of Eqs. (5) and (6) will now be obtained. Suppose that the solutions \( w = w_1, \ p = p_1 \) and \( T = T_1 \) at \( z = z_1 \) are known and the solutions \( w = w_2, \ p = p_2 \) and \( T = T_2 \) at \( z = z_2 \) are to be found. The derivations in the \( z \)-direction are replaced by finite-difference ratios, while the other quantities such as \( v, \ w, \ \partial w/\partial y \) etc. are replaced by averages of the two sections \( z_2 \) and \( z = z_2 \). By introducing the finite-difference approximations into Eqs. (5), one obtains\\
\[ \frac{w_1 + w_2}{2} \frac{w_2 - w_1}{1} = -\frac{p_2 - p_1}{2} \int_1^y \frac{w_2 - w_1}{l} \, dy \]  
(8)
where primes imply differentiation with respect to \( y \) and \( l = z_2 - z_1 \). Substituting the relations defined by\\
\[ \varphi = w_1 + w_2, \ w_2 = \varphi - w_1 \]  
(9)
and rearranging the terms in Eq. (8), one has\\
\[ \varphi (\varphi - 2w_1) - \varphi' \int_1^y (\varphi - 2w_1) \, dy = -2(p_2 - p_1) + l \varphi'' \]  
(10)
Since the flow quantity through each section along the channel must be constant, the following relation holds:\\\n\[ \int_1^y \varphi \, dy = 2 \int_1^y w_1 \, dy \]  
(11)
Once \( \varphi \) and \( p_2 \) are determined as solutions of Eqs. (10) and (11), one can easily obtain \( w_2 = \varphi - w_1 \).

Equations (10) and (11) are non-linear ordinary integro-differential equations for \( \varphi \) and \( p_2 \) which will be solved by an iterative process. If \( \varphi_n \) and \( p_{2,n} \) define the \( n \)th iterative approximations, \( \varphi_{n+1} \) and \( p_{2,n+1} \), can be computed from\\
\[ \varphi_n (\varphi_{n+1} - 2w_1) - \varphi_n \int_1^y (\varphi_{n+1} - 2w_1) \, dy = -2(p_{2,n+1} - p_1) + l \varphi_{n+1}'' \]  
(12.a)\\
\[ \int_1^y \varphi_{n+1} \, dy = -\frac{1}{2} \int_1^y w_1 \, dy \]  
(12.b)
A rectangular mesh of dimensions \( l, h \) is now formed by introducing differences in the \( y \)-direction. When the suffix \( k \) refers to the \( k \)th mesh-point in this direction, Eqs. (12.a) and (12.b) can be arranged as\\
\[ \int_1^y \varphi_{n+1} \, dy = (1/h^2) \int_1^y \varphi_{n+1} \, dy - \varphi_{n+1,1} + \varphi_{n+1,1+k} \]  
(13.a)\\
\[ -2p_{2,n+1} = -2p_1 - 2w_1 + \varphi_{n+1,1} + 2(\varphi_n)' \]  
(13.b)
where, \( h = 1 + kh \) and \( h = 2/n \). The following finite-difference approximations for \( (\varphi_{n+1}') \) and \( \int_1^y \varphi_{n+1} \, dy \) etc. are now made:\\\n\[ \varphi_{n+1}' = (1/h^2) \varphi_{n+1} \frac{1}{l} \int_1^y \varphi_{n+1} \, dy - \varphi_{n+1,1} + \varphi_{n+1,1+k} \]  
(14.a)\\
\[ \alpha_{n,k} = \varphi_{n+1} - \varphi_{n,k-1} \]  
(14.b)\\
\[ \int_1^y w_1 \, dy = h \varphi_{n+1,1} \]  
(14.c)
where the boundary conditions corresponding to Eq. (7.b)\n\[ w_{1,0} = w_{1,n} = 0, \ \varphi_{n,0} = \varphi_{n,n} = 0 \]  
(15.a)\\
\[ w_{1,0} = \varphi_{n+1,0} = 0 \]  
(15.b)\nhave been applied. When \( z_i = 0 \), some modifications must be made of Eqs. (15) which are given in the Appendix. Substitution of these approximations into Eqs. (13) and (12.b) and a little arrangement lead to\\
\[ -2p_{2,n+1} + \alpha_{n,0} \varphi_{n+1,1} + \alpha_{n,k} \varphi_{n+1,1+k} + \alpha_{n,k} \alpha_{n,1-k} + A_{n,k} \varphi_{n+1,1} + B_{n,k} \varphi_{n+1,1+k} \]  
(16.a)\\
\[ \varphi_{n+1} + \varphi_{n+1,1} + \varphi_{n+1,1+k} = 2\gamma_n \]  
(16.b)\nwhere\\
\[ \gamma_n = l/h^2 \]  
\[ A_{n,k} = \beta + \alpha_{n,k} \]
\[ B_{m,k} = -2\beta + \frac{1}{2} \alpha_{m,k} - \varphi_{m,k} \]
\[ C_{m,k} = -2\varphi_{m} + 2\varphi_{m+1} + 2\alpha_{m+1} \]

In view of the boundary conditions (15), the equations for \( k = 1 \) and \( n-1 \) respectively become
\[ -2p_{m+1} + B_{m+1} \varphi_{m+1} + B_{m+1,2} = C_{m,1} \] \hspace{1cm} (17-a)
\[ -2p_{m+1} + \alpha_{m+1} - \varphi_{m+1} + \alpha_{m+1,2} \varphi_{m+2} + \ldots + C_{m,n-1} \varphi_{m+n-2} = C_{m,2} \] \hspace{1cm} (17-b)

Equations (16-a) \sim (17-b) are a set of \( n \) simultaneous linear algebraic equations for the unknowns \( p_{2,m+1}, \varphi_{m+1,1}, \ldots, \varphi_{m+1,n+1} \). These simultaneous linear equations can be written in the matrix form;
\[ A_{n} \varphi_{n+1} = C_{n} \]
where, \( A_{n} \) is an \( n \times n \) matrix, while \( \varphi_{n+1} \) and \( C_{n} \) are column matrices given by
\[ \varphi_{n+1} = \begin{pmatrix} p_{2,m+1} \\ \varphi_{m+1,1} \\ \varphi_{m+1,2} \\ \cdots \\ \varphi_{m+1,n+1} \end{pmatrix} \]
\[ C_{n} = \begin{pmatrix} C_{m,1} \\ C_{m,2} \\ \cdots \\ C_{m,n-1} \\ 2\gamma_{n} \end{pmatrix} \]
The matrix \( A_{n} \) can be written easily from the left-hand sides of Eqs. (16-a) \sim (17-b). When the inverse matrix of \( A_{n} \) is denoted by \( A_{n}^{-1} \), the solution of the simultaneous equations is given by
\[ \varphi_{n+1} = A_{n}^{-1} C_{n} \]

This procedure is repeated until the following condition is satisfied:
\[ \max \{ |p_{2,m+1} - p_{3,m+1}|, |\varphi_{m+1,k} - \varphi_{m,k}| \} \leq \varepsilon \]
in which \( \max \{ \} \) means the maximum element contained in \( \{ \} \) and \( \varepsilon \) depends on the accuracy desired. This gives the values of \( p_{2} \) and \( \varphi \) at \( z = z_{2} \). The same process can then be applied to find the solution at \( z = z_{3} + l \) and so on. The value of \( \varphi \) thus obtained will be denoted by \( \varphi_{n} \) for later reference.

In the same manner, the finite-difference approximation can be applied to the thermal energy equation (6). The energy equation which corresponds to Eq. (8) becomes
\[ \frac{w_{1} + w_{2}}{2} T_{2} - T_{1} \left( \frac{T_{1}'' + T_{2}''}{2} \right) \frac{w_{2} - w_{1}}{l} dy = \frac{1}{P_{t}} \left( T_{1}'' \right) \]
The inclusion of the viscous dissipation term on the right-hand side of the above equation will not add any difficulty to the finite-difference procedure adopted here. The same process can be applied in order to evaluate the viscous dissipation. Let
\[ \varphi_{n} = w_{1} + w_{2}, \quad \theta = T_{1} + T_{2} \]
then one obtains
\[ \varphi_{n} (\partial_{n} - 2T_{1}) \theta' = \begin{pmatrix} \varphi_{n} - 2w_{1} \\ \varphi_{n} - 2w_{2} \end{pmatrix} dy = \frac{1}{P_{t}} \int_{-1}^{1} \partial_{n} \theta' \]
which can be written with respect to the \( k \)th mesh point from the lower wall of the channel as follows:
\[ \frac{l}{P_{t}} (\partial'_{n} + \partial'_{n}) \begin{pmatrix} \varphi_{n} - 2w_{1} \\ \varphi_{n} - 2w_{2} \end{pmatrix} dy = \varphi_{n} \theta_{n} \]

Substitution of the finite-difference approximations
\[ \begin{pmatrix} \theta'_{n} = (1/l) (\theta_{n+1} - 2\theta_{n} + \theta_{n-1}) \\ \theta'_{n} = l (\theta_{n+1} - \theta_{n}) \end{pmatrix} \]
into Eq. (18) yields
\[ D_{l} \theta_{n+1} + E_{l} \theta_{n} + F_{l} \theta_{n+2} = G_{l} \]
where
\[ D_{l} = \frac{\beta}{P_{t}} - \frac{1}{2} S_{n} + 7 \]
\[ E_{l} = -\frac{2\beta}{P_{t}} - 7 \]
\[ F_{l} = \frac{\beta}{P_{t}} + 1 + S_{n} - 7 \]
\[ G_{l} = -2T_{1} \theta_{n} \]

Since the boundary condition (7) requires that
\[ \varphi_{0} = \theta_{n} = 0 \]
the equations for \( k = 1 \) and \( n-1 \) are respectively
\[ E_{1} \theta_{1} + F_{1} \theta_{2} = G_{1} \]
\[ D_{n-1} \theta_{n-2} + E_{n-1} \theta_{n-1} = G_{n-1} \]
The set of \( n-1 \) simultaneous linear equations (20), (22-a) and (22-b) for the \( n-1 \) unknown quantities \( \theta_{1}, \theta_{2}, \ldots, \theta_{n-1} \) can be written in the matrix form:
\[ D \theta = G \]
in which \( D \) is an \( (n-1) \times (n-1) \) matrix that can easily be written from the left-hand sides of Eqs. (20), (22-a) and (22-b), and \( \theta \) and \( G \) are column matrices defined by
\[ \theta = (\theta_{1}, \theta_{2}, \ldots, \theta_{n-1}) \]
\[ G = (G_{1}, G_{2}, \ldots, G_{n-1}) \]
If the inverse of the matrix \( D \) is expressed by \( D^{-1} \), the solution vector can be obtained from
\[ \theta = D^{-1} G \]

It is obvious that the iteration procedure is not required, since the energy equation is linear with respect to \( T \).

5. Numerical calculations

Numerical calculations are performed for five values of \( K = 0, 0.3, 0.6, 0.8, 1.0 \). The energy equation is treated with the assumption that the temperatures of the channel walls and fluids at the inlet section are constant, respectively. The Prandtl number is taken as 0.72.

In the immediate neighbourhood of the entrance section, the mesh size of \( n = 72 \), \( l = 10^{-4} \) was employed, and the values of \( n \) and \( l \) were varied in the downstream direction as are shown in Table 1. The value of \( e \) for convergence was chosen to be \( 10^{-4} \). The number of iterations 8 in the first step
was changed to 2 or 3 in the range of $z \geq 10^{-2}$, when the first approximate solution at $z = z_1$ was taken to be the same as at $z = z_2$. The calculations were carried out on the HITAC 5020E computer installed at the Computer Center of University of Tokyo, which took about forty minutes to proceed from $z = 0$ to $z = 0.4$. Library Program No. 25, F1/TC/INV of the Computer Center was utilized in order to obtain the inverse matrices $A^{-1}$ and $D^{-1}$.

6. Results and discussions

Figure 2 shows the profiles of the axial velocity component and the temperature at several sections for $K = 0.8$. The asymptotic approach of $w$ to the fully-developed parabolic velocity profile $w = 1.5 (1 - y^2)$ is clearly seen. It should be noted that a significant asymmetry of the velocity profile in the vicinity of the entrance section is gradually reduced as the flow develops and a completely symmetrical profile is realized in the fully-developed flow regime. The inviscid core in the central part of the channel is found to be accelerated in the downstream direction with a constant inclination given by $\partial w / \partial y = K$, which can be interpreted as the result of Thomson’s theorem on the constancy of circulation. The variations of $w$ in the axial direction for $y = 0$, $\pm 0.5$, $\pm 0.7$ and $\pm 0.9$ are shown in Fig. 3 (a) and (b) for two values of $K = 0$ and 0.8. The result for $K = 0$ corresponds to a flow in the entrance region with uniform inlet velocity profile, and its overall development agrees well with previous calculations\(^{11}\). Although a number of numerical analyses have been reported since the work of Bodoia & Osterle\(^{11}\) for the laminar developing flow in the entrance region between parallel plates, the inertia terms of the equation for $z = z_2$ are in most cases linearized by the velocity at $z = z_1$ to solve these linearized equations. This process will correspond roughly to the first iteration in the scheme presented in

<table>
<thead>
<tr>
<th>$z$</th>
<th>$l$</th>
<th>$n$</th>
</tr>
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<tbody>
<tr>
<td>$(0.2) \times 10^{-4}$</td>
<td>$1 \times 10^{-4}$</td>
<td>72</td>
</tr>
<tr>
<td>$(0.6) \times 10^{-4}$</td>
<td>$2 \times 10^{-4}$</td>
<td>72</td>
</tr>
<tr>
<td>$(0.6) \times 10^{-4}$</td>
<td>$4 \times 10^{-4}$</td>
<td>50</td>
</tr>
<tr>
<td>$(1.1 \sim 5.4) \times 10^{-2}$</td>
<td>$8 \times 10^{-4}$</td>
<td>50</td>
</tr>
<tr>
<td>$(5.1 \sim 11.1) \times 10^{-2}$</td>
<td>$12 \times 10^{-4}$</td>
<td>32</td>
</tr>
<tr>
<td>$(11.1 \sim 1) \times 10^{-1}$</td>
<td>$24 \times 10^{-4}$</td>
<td>25</td>
</tr>
</tbody>
</table>

Fig. 3 Comparison of axial development of velocity for $K = 0.8$ with that for $K = 0$.

Fig. 2 Distributions of axial velocity component and temperature for $K = 0.8$ and $P_r = 0.72$. 

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the present paper. So far as the number of iterations in the vicinity of the entrance section and the value of $e$ employed in the present computation are concerned, it could be judged that the present result is more accurate than the previous calculations.

The flow in the entrance region approaches asymptotically the fully-developed regime, so that the theoretical fully-developed flow can be attained at a section infinitely downstream. However, for practical purposes, it may be reasonable to define the entrance length as an axial distance where the centerline velocity attains 98% or 99% of its fully-developed value. Table 2 contains the dimensionless entrance lengths $L_{50}$ and $L_{90}$ thus defined, together with the value obtained by other investigators. It should be mentioned that these entrance lengths decrease as the value of $K$ increases. For instance, the value of $L_{90}$ for $K = 0.6$ is smaller by 10% than that for $K = 0$, which is the case of a uniform inlet velocity profile.

Figure 4 shows the variation of temperature in the axial direction for the cases of $K = 0$ and 0.8. An asymmetry of the temperature profile with respect to the channel centerline is well demonstrated by this figure. Although the asymmetry is significant near the entrance section owing to the large asymmetry of the velocity profile, it becomes negligibly small in the region of $z > 0.1$.

The variation of pressure in the axial direction is given in Fig. 5. The pressure drop decreases as $K$ increases. If the flow were fully developed immediately from the entrance section, the pressure distribution would be

$$p(0) - p(z) = 3z$$

The pressure distribution curves shown in Fig. 5 are seen to approach a straight line of gradient of 3 for all values of $K$. It may be of practical interest to introduce a dimensionless additional pressure drop $C_\mu$ in the entrance region defined by

$$C_\mu = \lim_{z \to 0} [p(0) - p(z) - 3z]$$

The values of $C_\mu$ numerically computed at $z = 0.4$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$L_{50}$</th>
<th>$L_{90}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.139</td>
<td>0.178</td>
</tr>
<tr>
<td>0.3</td>
<td>0.136</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.125</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.114</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.102</td>
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* Collins-Schwalbert

<table>
<thead>
<tr>
<th>Method of solutions</th>
<th>$L_{50}$ ($K = 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present result</td>
<td>0.178</td>
</tr>
<tr>
<td>Bodea-Osterlee</td>
<td>0.175</td>
</tr>
<tr>
<td>Hwang-Fan</td>
<td>0.169</td>
</tr>
<tr>
<td>Roid-Cess</td>
<td>0.182</td>
</tr>
<tr>
<td>Schlichting</td>
<td>0.160</td>
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</table>

<table>
<thead>
<tr>
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<th>$C_\mu$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
<td>0.3</td>
<td>0.301</td>
</tr>
<tr>
<td>0.6</td>
<td>0.206</td>
</tr>
<tr>
<td>0.8</td>
<td>0.107</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.019</td>
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</table>

<table>
<thead>
<tr>
<th>Method of solutions</th>
<th>$C_\mu$ ($K = 0$)</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>Schiller</td>
<td>0.314</td>
</tr>
<tr>
<td>Schlichting</td>
<td>0.301</td>
</tr>
<tr>
<td>Collins-Schwalter</td>
<td>0.338</td>
</tr>
<tr>
<td>Ishizawa</td>
<td>0.339</td>
</tr>
<tr>
<td>Bodea-Osterlee</td>
<td>0.338</td>
</tr>
<tr>
<td>Hwang-Fan</td>
<td>0.313</td>
</tr>
<tr>
<td>Roid-Cess</td>
<td>0.315</td>
</tr>
<tr>
<td>Sparrow et al</td>
<td>0.325</td>
</tr>
</tbody>
</table>
are tabulated in Table 3 (a). \( C_w \) decreases appreciably as \( K \) increases, and the value of \( C_w \) for \( K=0.8 \) is only 32\% of the one for \( K=0 \). In Table 3 (b), values of the additional pressure drop obtained by previous investigators are compared with the result of the present analysis. The value of \( C_w \) given by Bodoia & Osterle\(^{11}\) is 0.338 and is larger by 1.5\% than the present result. The value obtained by Ishizawa\(^{16}\) by means of the momentum integral equation is found to be closest to the present calculations. It is noteworthy that the additional pressure drop defined by \( C_w \) becomes negative when \( K=1.0 \), and \( C_w=0 \) when \( K=0.98 \) as shown in Fig. 6.

The asymmetry of the temperature profile with respect to the channel centerline leads to different heat transfer rates at the upper and lower walls of the channel. The bulk temperature is introduced by the usual definition

\[
T_m = \frac{\int_{-H}^{H} T'w' dy'}{\int_{-H}^{H} w' dy'}
\]

and the local heat transfer coefficient \( h_l \) and the local Nusselt number \( N_{ux} \) are defined by

\[
h_l = q'/(T_m - T_w)
\]

\[
N_{ux} = 2Hh_l/k
\]

where \( q' \) is the heat flux at the channel walls. When the dimensionless bulk temperature is written as

\[
T_m = (T_m - T_w)/(T_0 - T_w)
\]

the local Nusselt numbers at the upper and lower walls of the channel take the following forms:

\[
N_{ux}^+ = -\frac{2}{T_m} \left( \frac{\partial T}{\partial y} \right)_{y=1}
\]

\[
N_{ux}^- = \frac{2}{T_m} \left( \frac{\partial T}{\partial y} \right)_{y=-1}
\]

which are shown in Figs. 7 (a) and (b) for these values of \( K=0, 0.6 \) and 1.0. As was mentioned already, the Prandtl number is taken as 0.72 which is an acceptable value for air. The variation of the mean Nusselt number defined by

\[
\bar{N}_{ux} = \frac{1}{z} \int_{0}^{z} N_{ux} dz
\]

is shown in Fig. 8 for three cases of \( K=0, 0.6 \) and 1.0. The mean Nusselt number is a little larger on the upper wall than on the lower one.
A more detailed comparison between $N_{u_r^+}$ and $N_{u_r^-}$ is made in Table 4 at three sections: $z=10^{-2}, 1.02 \times 10^{-1}$ and $10^{-1}$. As will be seen in this Table, average values of $N_{u_r^+}$ and $N_{u_r^-}$ decrease as $K$ increases, the difference $(N_{u_r})_{z=0}-N_{u_r^+}$ at the section $z=10^{-3}$. Although this amount of reduction is significant in the immediate vicinity of the entrance section, it can be neglected in the region of $z>0.05$.

The variations of the bulk temperature are shown in Fig. 9 for two extreme cases of $K=0$ and 1.0. Since these two curves are seen to be almost identical, the inlet velocity profile will have negligible effects on the bulk temperature.

As was already suggested in Chapter 3, the flows along the upper and lower walls of the channel will approach asymptotically the laminar boundary layer flows along a flat plate located in a uniform shear flow, when $z \to 0$. Van Dyke[10] and Murray[15], among others, solved Eqs. (3) for this case to clarify the effects of the transverse velocity gradient of the main flow upon the shear stress and heat transfer at the surface of the plate. In terms of the present notation, their results can be written in the following forms:

$$\tau^+ = 0.332 (1+K)^{3/2} z^{1/2} - 3.126 K$$ \hfill{(23.a)}

$$\tau^- = 0.332 (1-K)^{3/2} z^{1/2} + 3.126 K$$ \hfill{(23.b)}

$$N_{u_r^+} = 0.592 (1+K)^{1/2} z^{-1/2} - 1.822 \frac{K}{1+K}$$ \hfill{(24.a)}

$$N_{u_r^-} = 0.964 z^{-1/3}$$ \hfill{(24.b)}

Equations (23-2) and (24-2) cannot be applied when $K=1$ which means the inlet velocity profile given by $w=x=1+y$. For this case, Lighthill's expression[17] for the Nusselt number

$$N_{u_r^-} = 0.964 z^{-1/3}$$ \hfill{(24.c)}

can be utilized in place of Eq. (24-2). Since the numerical results shown in Figs. 10 and 7 (a) approach the asymptotic curves given by Eqs. (23) and (24), it could be judged that the entrance region flow of a channel with a linearly varying inlet velocity profile was appropriately treated by the assumption introduced in Chapter 3. This fact suggests that the same method of analysis may also be applied to analyze the entrance region flows including more general velocity profiles at the inlet section.

7. Conclusions

In this paper, the development of a laminar flow in the entrance region between parallel plates with a constant transverse velocity gradient at the inlet section is theoretically considered to obtain numerical solutions based on the laminar boundary layer theory. The results obtained can be summarized as follows:

1) If the flow rate through the channel is constant, the additional pressure drop in the entrance region is significantly decreased as the transverse velocity gradient at the inlet section is increased. For example, the additional pressure drop for $K=0.6$ is about 62% of the value for $K=0$ which means a uniform velocity profile at the inlet.

2) When isothermal walls are considered, the mean rate of heat transfer is much larger at
the wall of higher velocity side than that at the smaller velocity side, and the difference increases
with $K$. However, an average value of the heat
transfer rates of the two walls is smaller by only
a few percent than that for the case of $K=0$.

(3) The additional pressure drop is found to
decrease by a few tens of percent in the presence of
a normal velocity gradient at the entrance section,
while the average heat transfer rate decreases by
only a few percent. This fact suggests that the
flow resistance in the entrance region can be
greatly decreased by suitably controlling the inlet
velocity profile at a small deduction of the heat
transfer.

(4) The entrance length shows a significant
reduction with the increase of the normal velocity
gradient at the inlet section. For example, the
entrance length $L_{98}$ for $K=0.6$ is about 90% of
the value for $K=0$.

Appendix

Since the velocity and temperature profiles at
the inlet section are $u=1+Ky$ and $T=1$ in dimen-
sionless variables, the boundary conditions for $\varphi_{m}$,
$\varphi_{m+1}$ and $\theta$ become

$$
\begin{align*}
\varphi_{m,0} &= 2w_{1,0} = 2(1-K) \\
\varphi_{m,n} &= 2w_{1,n} = 2(1+K) \\
\varphi_{m+1,0} &= w_{1,0} \\
\varphi_{m+1,n} &= w_{1,n} \\
\theta_{0} &= \theta_{n} = 2
\end{align*}
$$

when $z_1=0$. Accordingly, Eqs. (14.b), (14.c),
(16.a), (16.b), (17.a), (17.b), (19), (22.a) and
(22-b) should be modified as follows:

$$
\int_{0}^{1} \varphi_{m+1,0} dy = H \left( \frac{1}{2} \varphi_{m+1,0} + \varphi_{m+1,1} + \varphi_{m+1,2} + \cdots \right)
+ \varphi_{m+1,k+1} + \frac{1}{2} \varphi_{m+1,k}
$$

$$
\Gamma_{n} = \frac{1}{2} w_{1,0} + w_{1,1} + w_{1,2} + \cdots + w_{1,k-1} + \frac{1}{2} w_{1,k}
$$

$$
-2p_{2,m+1} + \alpha_{m,n} \varphi_{m+1,1} + \cdots + \alpha_{m,n} \varphi_{m+1,k-1} + \beta \varphi_{m+1,k+1}
= C_{m,n} - \frac{1}{2} \alpha_{m,n} \varphi_{m+1,0}
$$

$$
\varphi_{m+1,1} + \varphi_{m+1,2} + \cdots + \varphi_{m+1,n-1}
= 2\gamma_{n} - \frac{1}{2} (\varphi_{m+1,0} + \varphi_{m+1,n})
$$

$$
-2p_{2,m+1} + B_{m,n} \varphi_{m+1,1} + \beta \varphi_{m+1,2}
= C_{m,n} - \frac{1}{2} \alpha_{m,n} \varphi_{m+1,0} - \beta \varphi_{m+1,0}
$$

$$
-2p_{2,m+1} + \alpha_{m,n-1} \varphi_{m+1,1} + \cdots + \alpha_{m,n-1}
\times \varphi_{m+1,n-2} + A_{m,n} - \varphi_{m+1,n-1} + B_{m,n-1} \varphi_{m+1,n-1}
= C_{m,n-1} - \frac{1}{2} \alpha_{m,n-1} \varphi_{m+1,0} - \beta \varphi_{m+1,0}
$$

References


Discussion

M. Murakami (Nagoya University):

(1) Transition from laminar to turbulent flows may occur at lower Reynolds number in the
entrance region as the value of $K$ increases. What kind of relation is expected between the critical
Reynolds number and the value of $K$?
(2) The additional pressure drop is shown to become negative when $K>0.98$. What is the
physical explanation of this phenomenon?
S. Ishizawa (Hitachi Ltd.):

(3) The additional pressure drop in the en-

trance region has usually been defined as the difference in static pressure between the inlet section
and the fully-developed flow regime, as was also
employed in this paper. However, from a phys-
ical point of view, one of the most interesting
things will be how the energy loss increases in the
downstream direction. Since a non-uniform inlet
velocity is considered in this paper, the total
pressure depends on the individual velocity profile
even if the same flow rate is realized. Then an
additional energy loss defined as the difference in

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total pressure between the inlet section and the fully-developed flow regime should be considered in
addition to $C$. Since the dimensionless velocity profile at the inlet section is $w=1+Ky$, its dimen-
sionless mean dynamic pressure is written as
\[
p_d(0) = \frac{1}{4} \int_1^{\infty} w^2 dy = \frac{1}{2} K^2 \frac{3}{6}.
\]
On the other hand, the fully-developed velocity profile is a parabola given by $w = \frac{3}{2}(1-y^2)$ ir-
respective of the inlet velocity profile. Then, the dimensionless mean dynamic pressure becomes
\[
p_d(\infty) = \frac{1}{4} \int_1^{\infty} w^2 dy = \frac{3}{5}.
\]
Accordingly, the additional energy loss is larger by $p_d(0) - p_d(\infty) = \frac{K^2}{6} - \frac{1}{10}$ than the additional pres-
sure drop shown in Fig. 6. In view of the nature of the entrance region flow treated in this paper, it
would be desirable to show the variation of energy loss together with Fig. 6 for convenience of
physical interpretation of the results.

(4) The equations of line 28 from top in page 325 of the text have originally been derived for the laminar boundary layer along a flat plate placed in a uniform shear flow. In this situation, what would be the physical meanings of $W$ and
$H$ which are included in the definitions of $R_c = WH$, $y = \frac{y'}{H}$, $z^* = \frac{z'}{H}$ etc.? The third terms on the right-
hand side of these equations should be $O(R_c^{-1})$ and $O(R_c^{-3/2})$, respectively, although the authors
wrote these terms as $O(R_c^{-1/2})$ and $O(R_c^{-1})$.

(5) At line 8 from top of page 332 of the text, the authors concluded that the assumption employed in Chapter 3 is good through the entire flow field of the entrance region in view of the fact that the numerical solution thus obtained tends to the solutions of Eqs. (2) and (3), when the inlet section $z = 0$ is approached. However, as is seen from the solutions of van Dyke and Murray, it should be noted that the second terms on the right-hand sides of the equations mentioned in (4) vanish when $z = 0$ and only the first terms remain, which correspond to the Blasius' solution of the laminar boundary layer along a flat plate in a uniform flow. What is obtained in this paper is essentially a solution of Eqs. (2) with the boundary conditions given by Eqs. (7) instead of a uniform main flow. Since the boundary layer thickness becomes infinitely thin when $z = 0$, it is clear that the effect of vorticity in the main flow vanishes and the solution will approach the Blasius' solution just mentioned. Even if the assumption that Eqs. (2) prevail throughout the entrance region were not correct, the same asymptotic behavior of the numerical solution would be observed. Accordingly, the fact that the present solution tends to the solution of Eqs. (2) and (3) as $z \to 0$ does not necessarily guarantee the validity of the assumption employed in this paper. (Whether the assumption is essentially proper or not will not be discussed here.)

(6) It is hoped that the values obtained in the numerical calculation are added in Appendix.

Table 1.

Authors' closure

(1) Establishment of the relation between the critical Reynolds number of transition and the value of $K$ is not yet undertaken and remains as a future study. However, it does not seem to the authors that a larger value of $K$ is always accom-
panied by a smaller value of the critical Reynolds number. Stability of the entrance region flow with uniform inlet velocity in a circular pipe was discussed by Tatsumi (\cite{a1}), while that between parallel plates was considered by Chen & Sparrow (\cite{a2}). Their methods of analysis could be extended to the case of non-uniform inlet velocity profile (\cite{a3}).

Along with this problem, the stability of a laminar boundary layer along a flat plate placed in a uniform shear flow is also important. However, these are the subjects of the future study.

(2) As was pointed out by S. Ishizawa, the additional energy loss defined as the difference in total pressure is always positive in the range of $K$ treated in this paper. A more detailed description on this point will be given in the authors' closure (3).

(3) Since the main object of this paper was to find the effect of non-uniformity of the inlet velocity profile on the pressure drop in the entrance region, only the additional pressure drop was presented in the text. However, as was pointed out by the discussers, an aspect of the characteristics

\cite{a3} After the discussion with M. Murakami was finished, the work of Chen & Sparrow (Phys. Fluids, Vol. 13, No. 3 (1970), p. 827.) came to the authors' attention. They investigated the linear stability of hydrodynamically developing laminar channel flows characterized by asymmetric velocity profiles induced by linearly varying inlet velocity profiles and found instability occurs at a higher Reynolds number as the value of $K$ becomes larger.
of the entrance region flow will be well understood by the additional energy loss. When the additional energy loss is denoted by \( E_* \), it can be written as
\[
E_* = C_* + (K^2/6) - (1/10)
\]
which is shown in Appendix-Fig. 1 as a function of \( K \). \( E_* \) is found to decrease as \( K \) increases in the same manner as \( C_* \). In this connection, it is worth mentioning that a linear relationship exists between \( E_* \) and \( K^2 \). Since the mean total pressure at the inlet section is larger by \( K^2/6 \) than in the case of uniform inlet velocity profile, one can define this quantity as the excess energy \( \Delta E \). Appendix-Fig. 2 shows the difference in the additional pressure drop \( \Delta C = (C)_K - C_* \) and that in the additional energy loss \( \Delta E = (E)_K - E_* \) as functions of the excess energy \( \Delta E = K^2/6 \). This figure clearly shows that both \( \Delta C \) and \( \Delta E \) are proportional to \( \Delta E \). The same nature of relationship is expected to hold for other shapes of the inlet velocity profile.

(4) The equations which are referred to by the discussers include, as a special case, the laminar boundary layer flow along a flat plate placed in a uniform shear flow \( w' = W + Oy' \). In this case, \( W \) means the velocity at \( y' = 0 \) of the undisturbed shear flow. In the same manner as the Blasius solution, it has been known that the laminar boundary layer of the uniform shear flow is described by the two dimensionless variables defined by \( \eta = \sqrt{y'/v} \) and \( K R_e^{-1/2} \). When these variables are expressed by the dimensional variables, one obtains
\[
\eta = y'(W/\nu)^{1/2} \\
K R_e^{-1/2} = (\omega/W)(\nu'W)^{1/2}
\]
in which \( H \) is not contained at all. Accordingly, \( H \) should be understood as an artificial length scale conventionally introduced to define the dimensionless variables without any particular physical meaning.

\( o(R_e^{-1/2}) \) and \( o(R_e^{-1}) \) denote small quantities of higher order than \( R_e^{-1/2} \) and \( R_e^{-1} \), respectively. A more detailed explanation of the symbols \( o \) and \( O \) will be found elsewhere (94). It is possible that the third terms of the expansions which are referred to by the discussers happen to be of the order \( O(R_e^{-1} \ln R_e) \) and \( O(R_e^{-3/2} \ln R_e) \) (95). They cannot necessarily be written as \( O(R_e^{-1}) \) and \( O(R_e^{-3/2}) \). Therefore, the notations \( o(R_e^{-1/2}) \) and \( o(R_e^{-1}) \) are employed in the expansions concerned.

(5) When \( z = 0 \), the Blasius terms certainly become dominant and the second terms which represent the contribution of the shear flow tend to vanish. As is shown in Appendix-Table 1, \( N_{\infty} \) for \( K=1.0 \) is almost identical to the Blasius term in the range \( z = 10^{-3} \sim 10^{-4} \), while that for \( z = 10^{-3} \) is smaller by about 3% than the Blasius term. The present numerical solutions are found to approach the corrected values for the second terms as \( z \to 0 \). Although this fact lends a support to the method of analysis employed in this paper, one cannot conclude that the method is correct throughout the entrance region merely from this fact, as was pointed out by the discussers. However, in connection with this point, the authors have the following opinions. Since Eqs. (3-b) and (3-d), analogous to Eqs. (2-b) and (2-d), are parabolic, the solutions at any section are determined from the solutions at the section immediately upstream. If the entrance region flow is to be treated by means of the boundary layer approximation as was assumed in this paper, correct solutions at one arbitrary section will necessarily lead to correct solutions at all downstream sections.

Appendix-Table 1

<table>
<thead>
<tr>
<th>( Z )</th>
<th>( N_{\infty} ) for ( K=1.0 ) as ( z \to 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-1}</td>
<td>264.8</td>
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<tr>
<td>10^{-2}</td>
<td>83.72</td>
</tr>
<tr>
<td>10^{-3}</td>
<td>26.48</td>
</tr>
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</table>
tions. Therefore, the authors suppose that the result of the present paper is valid throughout the entrance region. Moreover, it should be added that an attempt was made successfully by Lewis\(^{(6)}\) to treat the effects essentially included in \(w_z, v_z^*\) by means of \(w_1, v_1^*\) with proper alternation of the boundary conditions, just as the one in the present analysis.

\((6)\) The numerical calculations were started from \(z=0\) with \(l=10^{-4}\). Some fluctuations in the

values of \(N_{uz}\) were unavoidable in the neighbourhood of \(z=10^{-4}\). The values of \(N_{\text{in}}=25.8\) were obtained at \(z=10^{-3}\) when \(K=1.0\).

