Flexural Vibrations of a Ring with Arbitrary Cross Section

By Mitsuru Endo

The paper deals with the flexural vibrations of a ring with arbitrary cross section and proposes an approximate theory to predict the natural frequencies by means of the Ritz's method. The frequency determinant is derived from the Hamiltonian principle by use of the stress-strain-displacement relations of three-dimensional theory of elasticity, and by expanding three components of the displacement in a finite double power series of the radial and axial coordinates with unknown coefficients.

Compared with the experimental results for twenty three kinds of rings which are classified into six sets with respect to the cross section, the theory was ascertained to be available for the practical use.

Furthermore, as an application to a general axi-symmetric elastic solid with free boundary surface, the frequencies of truncated conical shells and combined shells of hemispherical and circular cylindrical shells were calculated and compared with the already-known results, showing a good agreement between them.

1. Introduction

Vibrations of a ring have been treated mainly on the basis of the theory of a circular arc bar as a fundamental problem in the field of theory of elasticity\(^{1}(2)\). Nevertheless, as to a ring with arbitrary cross section, there are few practical methods to predict the natural frequencies. Recently, however, as the demand for noiseless machinery increases, the prediction of the frequencies of those rings which are used in ball bearings or rotating machines has actually become urgent from the practical point of view, in order to elucidate the characteristics of noises or vibrations\(^{3(9)}\).

Therefore, in the previous papers\(^{4(5)}\), the author proposed a few methods for a ring with rectangular cross section and for an arbitrary ring whose cross section is symmetrical with respect to the principal axis in the axial direction in addition to examining the applicability of existing theories. And from the same viewpoint, the present paper presents an effective method which can predict practically sufficient frequencies of general rings whose cross section is entirely arbitrary.

The energy method is supposed to be most suitable for arbitrary rings, and the present theory depends on the ordinary Ritz's method in which the components of displacement are approximated by a finite power series of the coordinates, and the frequency determinant is derived from the Hamiltonian principle. Generally speaking, it is difficult to estimate, in advance, the accuracy of approximation of an energy method, and besides in the actual numerical calculation the improvement of solutions is restrained by the capacity of a digital computer. So, in order to ascertain the practical validity of the method in addition to the theoretical propriety, experiments were carried out about twenty three kinds of rings classified into six sets with respect to the cross section, and the effects of the cross-sectional configuration on the accuracy of approximation were examined systematically.

Furthermore, it is expected that the present method is applicable to the vibrations of a general axi-symmetric elastic solid with free boundary surface. Hence, as an example, the frequencies of truncated conical shells with free edges and combined shells of hemispherical and circular cylindrical shells were calculated and compared with the already-known results, and its applicability was ascertained.

Nomenclature

Type A: vibrational mode in which the in-plane bending deformation is predominant
Type B: vibrational mode in which the out-of-plane twist-bending deformation is remarkable
\(\nu\): Poisson's ratio
\(E\): Young's modulus
\(G\): shear modulus
\(\gamma\): specific weight
\(g\): acceleration of gravity

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2. Theory

2.1 Method of analysis

A circular ring with arbitrary cross section as shown in Fig. 1 is considered. A point on the axis of the ring is taken as the origin O, and a point of the ring is defined by the coordinate system \((r, \theta, z)\), where \(r, \theta, z\) are the radial, angular and axial coordinates, respectively. And \(u, v, w\) are taken as the radial, tangential and axial displacements, respectively.

Flexural vibrations of a ring are qualitatively classified into two types, and in the paper one type with the in-plane bending deformation being predominant is designated as Type A and the other with the out-of-plane twist-bending deformation being remarkable is designated as Type B. On the other hand, mathematically, these types correspond to the first and the second axial modes and there exist an infinite number of the higher modes, but they are not treated in this paper.

The frequency determinant can be derived from the Hamiltonian principle by means of the Ritz’s method. If there is no external force acting on the ring, the Hamiltonian principle is given by

\[
\delta I = \delta \int_0^T (E_T - E_V) dt = 0 \quad (t : \text{time}) \quad \ldots \ldots \ldots \ldots (1)
\]

where \(\delta\) is the variational operator and \(E_T, E_V\) are the total kinetic and potential energies of the ring, respectively.

Now, we introduce the following two fundamental assumptions:

1. The stress-strain-displacement relations of three-dimensional theory of elasticity are directly applied to describe the total energies \(E_T, E_V\) by three components of the displacement \(u, v, w\).

2. Further, \(u, v\) and \(w\) are approximated by a finite double power series with respect to the radial and axial coordinates \(r, z\) with unknown coefficients.

By using the assumptions (1) and (2), an infinite-degree-of-freedom system of the vibrating ring is approximated by a finite-degree-of-freedom one with the unknown coefficients being variables, and from the simultaneous homogeneous equations with respect to those variables, the frequency determinant is derived.

2.2 Derivation of the frequency determinant

Taking \(R_0\) as a standard length of the ring, put

\[
\zeta = \frac{r}{R_0}, \quad \xi = \frac{z}{R_0}, \quad \dot{u} = \frac{u}{R_0}, \quad \dot{v} = \frac{v}{R_0},
\]

\[
\ddot{w} = \frac{w}{R_0}, \quad \ddot{\sigma}_r = \frac{\sigma_r}{G}, \quad \ddot{\tau}_{ij} = \frac{\tau_{ij}}{G} \quad \ldots \ldots \ldots \ldots (2)
\]

and \(E_T, E_V\) are expressed as follows:

\[
E_T = \frac{1}{2} \int \int \left\{ \frac{\partial \ddot{u}}{\partial \xi} \right\} \left\{ \frac{\partial \ddot{u}}{\partial \xi} \right\} d\xi d\zeta + \left\{ \frac{\partial \ddot{v}}{\partial \xi} \right\} \left\{ \frac{\partial \ddot{v}}{\partial \xi} \right\} d\xi d\zeta 
\]

\[
+ \left\{ \frac{\partial \ddot{w}}{\partial \xi} \right\} \left\{ \frac{\partial \ddot{w}}{\partial \xi} \right\} d\xi d\zeta \quad \ldots \ldots \ldots \ldots (3)
\]

\[
E_V = \frac{1}{2} G R_0^3 \int \int \left\{ \dot{\tau}_{rr} + \dot{\tau}_{r\theta} + \dot{\tau}_{\theta\theta} + \dot{\tau}_{zz} \right\} d\xi d\zeta 
\]

\[
+ \ddot{\tau}_{rr} + \ddot{\tau}_{r\theta} + \ddot{\tau}_{\theta\theta} + \ddot{\tau}_{zz} \right\} d\xi d\zeta \quad \ldots \ldots \ldots \ldots (4)
\]

And the stress-strain relations of three-dimensional theory of elasticity are given by

\[
\ddot{\sigma}_r = 2\left( \ddot{\varepsilon}_r + \frac{\nu}{1 - 2\nu} \ddot{\varepsilon}_\theta \right), \quad \ddot{\sigma}_\theta = 2\left( \ddot{\varepsilon}_\theta + \frac{\nu}{1 - 2\nu} \ddot{\varepsilon}_r \right), \quad \ddot{\sigma}_z = 2\left( \ddot{\varepsilon}_z + \frac{\nu}{1 - 2\nu} \ddot{\varepsilon}_\theta \right) \quad \ldots \ldots \ldots \ldots (5)
\]

where

\[
\varepsilon_r = \dot{u} + \dot{\varepsilon}_r + \ddot{\varepsilon}_r,
\]

and the strain-displacement relations are

\[
\varepsilon_r = \frac{\partial u}{\partial \xi}, \quad \varepsilon_\theta = \frac{1}{\zeta} \frac{\partial u}{\partial \theta} + \frac{u}{\zeta}, \quad \varepsilon_z = \frac{\partial u}{\partial z}
\]

\[
\gamma_{sr} = \frac{\partial \ddot{u}}{\partial \xi} + \frac{1}{\zeta} \frac{\partial \ddot{u}}{\partial \theta} + \frac{\ddot{u}}{\zeta} \quad \ldots \ldots \ldots \ldots (6)
\]

Now, Eqs. (4) and (5) are substituted into Eq. (3) and then the following variable separation is introduced for a specific modal vibration with respect to \(n\):

\[
\ddot{u} = U(\zeta, \xi) \sin n\theta \cos pt,
\]

\[
\ddot{v} = V(\zeta, \xi) \cos n\theta \cos pt,
\]

\[
\ddot{w} = W(\zeta, \xi) \sin n\theta \cos pt \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6)
\]

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After substituting all the above relations into Eq. (1), the integration from 0 to 2π with respect to θ and at the interval of one period with respect to time t leads to the following expression of the Hamiltonian principle:

\[
\delta I_0 = \delta \int \left[ \frac{1}{2} \left( \zeta \left( \frac{\partial U}{\partial \zeta} \right)^2 + \frac{1}{\zeta} \left( U - nV \right)^2 + \zeta \left( \frac{\partial W}{\partial \zeta} \right)^2 \right) + \frac{2\nu}{1 - 2\nu} \frac{\partial U}{\partial \zeta} \left( U - nV \right) + \left( U - nV \right) \frac{\partial U}{\partial \zeta} \frac{\partial W}{\partial \zeta} + \frac{1}{2} \left( \zeta \left( \frac{\partial V}{\partial \zeta} + \frac{n}{\zeta} W \right)^2 + \zeta \left( \frac{\partial U}{\partial \zeta} + \frac{\partial V}{\partial \zeta} \right)^2 + \frac{1}{\zeta} \left( nU - V \right) + \zeta \frac{\partial V}{\partial \zeta} \right) \right] d\zeta d\xi = 0 \tag{7}
\]

Now, the components of deformation U, V, W in Eq. (7) are approximated by the following finite double power series of ζ and ξ, with unknown coefficients a_{ij}, b_{ij} and c_{ij}:

\[
U = \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} \zeta^{i-1} \xi^{j-1}, \quad V = \sum_{i=1}^{M} \sum_{j=1}^{N} b_{ij} \zeta^{i-1} \xi^{j-1}, \quad W = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} \zeta^{i-1} \xi^{j-1} \tag{8}
\]

And substituting Eq. (8) into Eq. (7), we obtain the homogeneous equations corresponding to the variations \( \delta a_{im}, \delta b_{lm}, \delta c_{im}, \) and \( \delta c_{im} \), as follows:

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}
\begin{bmatrix}
a_{i1} \\
b_{i1} \\
c_{i1}
\end{bmatrix}
= \begin{bmatrix}
M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & M_3
\end{bmatrix}
\begin{bmatrix}
a_{i1} \\
b_{i1} \\
c_{i1}
\end{bmatrix} \tag{9}
\]

where \( [a_{ij}] \), for instance, denotes the column vector with \( a_{ij} \) taken as the \( (N(i-1)+j) \)-th element, and the zero matrix \( 0 \), and \( K_{11} \sim K_{13}, M_1 \sim M_3 \) are the submatrices of order \( (MN) \times (MN) \). The elements of the I-th row and the J-th column of these matrices are expressed as follows:

\[
\begin{align*}
I &= N(i-1)+m \quad (i \& j = 1 \sim M, \quad j \& m = 1 \sim N) \\
J &= N(i-1)+j \\
\end{align*}
\]

and

\[
\begin{align*}
(K_{11})_{ij} &= \left[ \frac{1}{2} \left( (i-1)(i-1) - \nu(i-2)(i-2) \right) + \frac{n^2}{2} \left( j+m-2 \right) \right] + \frac{1}{2} \left( j-1 \right) \left( m-1 \right) \left( i+1 \right) \left( j+m-4 \right) \\
(K_{12})_{ij} &= \left[ \frac{1}{2} \nu(i-2) - \frac{1}{2} \nu \left( j-1 \right) \left( m-1 \right) \right] \left( i+1 \right) \left( j+m-3 \right) \\
(K_{13})_{ij} &= \left[ \frac{1}{2} \nu(i-2) - \frac{1}{2} \nu \left( j-1 \right) \left( m-1 \right) \right] \left( i+1 \right) \left( j+m-2 \right) \\
(K_{22})_{ij} &= \left[ \frac{1}{2} \left( j-1 \right) + \frac{1}{2} \nu \left( j-1 \right) \left( m-1 \right) \right] \left( i+1 \right) \left( j+m-3 \right) \\
(K_{23})_{ij} &= \left[ \frac{1}{2} \nu(i-2)(i-2) + \frac{n^2}{2} \left( j+m-2 \right) \right] + \frac{1}{2} \nu \left( j-1 \right) \left( m-1 \right) \left( i+1 \right) \left( j+m-4 \right) \\
(K_{33})_{ij} &= \left[ \frac{1}{2} \left( j-1 \right) + \frac{1}{2} \nu \left( j-1 \right) \left( m-1 \right) \right] \left( i+1 \right) \left( j+m-3 \right) \\
(M_{11})_{ij} &= (M_{21})_{ij} = (M_{31})_{ij} = \left( i+1 \right) \left( j+m-4 \right) \\
(M_{22})_{ij} &= \left( i+1 \right) \left( j+m-3 \right) \\
(M_{33})_{ij} &= \left( i+1 \right) \left( j+m-2 \right) \\
\end{align*}
\tag{10}
\]

The submatrices \( K_{11}, K_{21}, K_{31} \) and \( K_{22} \) are equal to the transposed matrices of \( K_{12}, K_{13}, \) and \( K_{32} \), respectively, but in the actual calculation we can utilize the symmetric property of the total matrix.

From the condition that Eq. (9) may have no trivial solution, we obtain the following frequency determinant for a general ring:

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}
\begin{bmatrix}
M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & M_3
\end{bmatrix}
\begin{bmatrix}
a_{i1} \\
b_{i1} \\
c_{i1}
\end{bmatrix}
= \begin{bmatrix}
M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & M_3
\end{bmatrix}
\begin{bmatrix}
a_{i1} \\
b_{i1} \\
c_{i1}
\end{bmatrix} \tag{11}
\]

where the order of this determinant is \((3MN) \times (3MN)\). Depending on either Eq. (11) or Eq. (9), we can calculate eigenvalues \( \Delta \), and then we know the real frequencies \( f \) of the ring.

Considering demands for the frequencies of higher modes or the determination of eigen vectors, we ought to adopt an appropriate method of numerical calculation. In the numerical calculation, it is necessary to carry out double integration with fairly high order weights over the cross section of a ring. So, it is most desirable to use the numerical integration for the universality of the program. In order to improve the accuracy of solutions, we have only to take larger \( M, N \) in the input data.

In the actual calculation, a HITAC 5020E digital computer was used.

\section*{2-3 Frequency determinants for a special case}

As a special case of section 2-2, a ring having a symmetrical cross section with respect to the central
plane of the ring is considered. Such a ring is actually used, for instance, as an inner or outer race of ball bearings.

If the intersect of the central plane and the axis of the ring is taken as the origin O, the relations

\[(2j-1) = \int \xi^2 \xi_{j} dz_{1} dz_{2} = 0 \quad \cdots \quad (12)\]

should hold because of the symmetry of the cross section. Accordingly, each element of the submatrices \(K_{ij}, M_{i}\) in Eq. (10) becomes zero corresponding to either even or odd number of \((j+m)\). Hence, exchanging the rows and columns appropriately and rearranging the row vector of Eq. (9), we obtain the following simultaneous equations:

\[
\begin{bmatrix}
K^A & 0 & d^A \\
0 & K^B & d^B
\end{bmatrix} = A \begin{bmatrix}
M^A & 0 \\
0 & M^B
\end{bmatrix} \begin{bmatrix}
d^A \\
d^B
\end{bmatrix}
\cdots \cdots (13)
\]

where

\[
\begin{bmatrix}
a_{i1}a_{i-1} \\
b_{i1}a_{i-1} \\
c_{i1}a_{i-1}
\end{bmatrix}, \begin{bmatrix}
a_{i1}a_{i-1} \\
b_{i1}a_{i-1} \\
c_{i1}a_{i-1}
\end{bmatrix} (h: \text{integer})
\]

As Eq. (13) can be separated into two sets of simultaneous equations, we finally obtain the following two frequency determinants:

\[
[K^A] - A[M^A] = 0 \quad \cdots \cdots (14a)
\]

\[
[K^B] - A[M^B] = 0 \quad \cdots \cdots (14b)
\]

Referring to Eq. (13), we find that only the even powers of \(\xi\) are considered for \(u, v\) and only the odd powers for \(w\) in Eq. (14a), and the situation is reversed as to Eq. (14b). Hence, Eq. (14a) and Eq. (14b) turn out to be the frequency determinants for Type A and Type B, respectively.

By using the same notation as in Eq. (9), each element of the determinants of Eqs. (14a), (14b) are expressed as follows:

Put

\[
I = N(i-1)+h \quad (i & l = 1 \sim M, \& k = 1 \sim N)
\]

and

\[
(K_{11}^A)_{IJ} = \left[\frac{1+i+3}{2h+2k-4} + 2h(h-1)(k-1)\right] + \left[\frac{i}{2}\right]
\]

\[
(K_{12}^A)_{IJ} = \left[\frac{1}{2}n(i-1) - \frac{1}{2}n(2h-1)(k-1)\right] + \left[\frac{i+1}{2h+2k-4}\right]
\]

\[
(K_{22}^A)_{IJ} = \left[\frac{1}{2}n(i-1) - \frac{1}{2}n(2h-1)(k-1)\right] + \left[\frac{i}{2h+2k-4}\right]
\]

\[
(M_1^A)_{IJ} = (M_2^A)_{IJ} = \left[\frac{i+l-1}{2h+2k-4}\right]
\]

As in section 2-2, the other submatrices \(K_{11}^{A\beta}, K_{12}^{A\beta}, K_{22}^{A\beta}\) are equal to the transposed matrices of \(K_{11}^{A\beta}, K_{12}^{A\beta}, K_{22}^{A\beta}\) respectively.

Separating Eq. (11) into Eq. (14a) and Eq. (14b), we can decrease the amount of numerical calculation.
remarkably, for the number of rows and columns in Eqs. (14·a), (14·b) can be taken half of that in Eq. (11) for the purpose of obtaining a solution with the same accuracy of approximation.

2.4 The case of solid cross section

An axi-symmetric body whose cross section is solid as illustrated in Fig. 2 is considered. As shown in Eq. (5), the strain-displacement relations include the minus 1st power of ζ. So, if Eq. (8) is directly used as an approximate function of the displacement, some strain components must be infinite at ζ=0, excepting the case that the coefficient of zero power of ζ vanishes. On the other hand, we can realize from a logical consideration that all the points on the axis of the vibrating ring must stand still all the time for n≥2. Accordingly, in the case of solid cross section we have only to exclude the terms of zero power of ζ and use the following relations in place of Eq. (8):

\[
U = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{s=1}^{S} a_{ij}^{s} \xi_{i}^{s} \xi_{j}^{s-1}
\]

\[
V = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{s=1}^{S} b_{ij}^{s} \xi_{i}^{s} \xi_{j}^{s-1}
\]

\[
W = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{s=1}^{S} c_{ij}^{s} \xi_{i}^{s} \xi_{j}^{s-1}
\]

By the above generalization of the theory, it becomes possible to obtain the natural frequencies of solid spheres, circular cylinders, circular plates and so on.

3. Experiment

3.1 Method of measurement

The acoustic measurement of natural frequencies is as follows: First, the acoustic signal generated by hitting a hanging ring is picked up by means of a microphone, and this input is introduced into an oscilloscope through an amplifier and a filter. Then by using a standard oscillator of digital-type, natural frequencies are determined from Lissajous' figures.

3.2 Rings used in the experiment

In the experiment, twenty three kinds of rings made of carbon steel were used. Those kinds are classified into six sets with respect to the cross section. The configurations and the nondimensional aspect ratios are shown in Fig. 3, where the standard length R0 is 50 mm. In all the sets, the maximum thickness-to-mean-diameter ratios are nearly equal to 0.1.
Three kinds of Set $T$ shown in Fig. 3(a) are examples of general rings and particularly the cross section of Kinds TI and TO does not have any symmetry. And the cross-sectional contour of those rings is taken as a continuous function with respect to the coordinates $\zeta$ and $\xi$. Sets OI, IC, OC, IS and OS shown in Fig. 3(b)~(f) are examples of special rings whose cross section is symmetrical with respect to the central plane of the ring, and the cross-sectional contour is taken as a discontinuous function of $\zeta$ and $\xi$, and their configurations are changed systematically to examine the accuracy of the theory. As the name of four kinds in each set varies from A to D, the ratio of the width of the protruding cross-sectional part to the maximum length of the ring becomes small. In all the sets, the ratio of the maximum thickness to the minimum thickness is 2:1 and the ratio of the length to $2R_b$ (100 mm) varies from 0.1 to 1.0.

4. Considerations

4.1 Comparisons of theory and experiment

In order to obtain a real frequency from $\omega$, the estimated material constants shown in Table 1 are used.

Equation (11) was used to calculate the frequencies of three kinds of rings in Set T, and the first and the second eigenvalues corresponded to the frequency of Type A and that of Type B, respectively.

Table 1 Estimated material constants of rings

<table>
<thead>
<tr>
<th>Material</th>
<th>$\nu$</th>
<th>$E$ kg/mm²</th>
<th>$\gamma$ kg/mm²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbon steel</td>
<td>0.3</td>
<td>$2.1 \times 10^4$</td>
<td>$7.86 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

As an example, Fig. 4 shows the theoretical and experimental results for Kind TI. The notation $M(3) \times N(5)$ in the figure means that $i$ is taken from 1 to 3 (i.e. from $\xi^0$ to $\xi^3$) and $j$ is taken from 1 to 5 (i.e. from $\zeta^0$ to $\zeta^4$) in Eq. (8). This makes the total matrix of 45 rows and 45 columns. The accuracy of the results for Kinds TOI and TO was the same degree as that for Kind TI. Hence, we can reasonably conclude that, as far as those rings are concerned, it is possible to predict nearly accurate frequencies for both Type A and Type B.

Next, as to Sets OI IC OC, IS and OS, Eq. (14·a) and Eq. (14·b) were used for Type A and Type

![Fig. 4 Comparison of theory and experiment](image1)

![Fig. 5 Comparison of theory and experiment](image2)
B, respectively. The results for Sets IC, OC, and IS are shown in Fig. 5 ~ Fig. 7, where the figures (a) and (b) show the results for Type A and Type B, respectively. The notation $M(3) \times N(4~6)$ is almost the same as that in Fig. 4, but in this case the power series expansion with respect to the axial coordinate is taken up to $\xi^{2N-1}$.

Now, observing each figure (a), we can find that the theoretical results show a good agreement with those of the experiment, and so it turns out possible to obtain nearly accurate frequencies as to Type A by means of the present method. Meanwhile, it is observed in each figure (b) that the theoretical results are higher, in general, than those of the experiment and the deviation between them is larger than that for Type A as far as $M$ and $N$ shown in the figures are concerned. However, even the maximum error, which is observed in the second mode for Kind OCD, proves to be less than 7% and it appears to have sufficient accuracy for practical use. For Sets OI and OS, almost the same results were obtained.

Now, observing those results for Type B, we can make the following considerations concerning the effects of cross-sectional configuration on the accuracy of approximation:

The error for Set OC is larger than that for Set IC, in general, and the error for Set OS is larger than that for Set IS. The error for Sets OI, IC and OC

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Fig. 6 Comparison of theory and experiment

Fig. 7 Comparison of theory and experiment
becomes larger as the name of four kinds varies from A to D, whereas the error for Sets IS and OS is nearly uniform or becomes larger as D→A. This generally means that the error due to approximate power series expansion of displacements is larger when the cross-sectional configuration has a discontinuity on the outer line than on the inner line and when near the central part than near both the edges. The above-mentioned tendency ought to be noted when natural frequencies of an arbitrary ring are calculated.

4.2 Natural frequencies of general axisymmetric elastic solids

The natural frequencies of the truncated conical shells shown in Fig.8(a) are measured by W.C.L. Hu, et al.(7) and Fig. 8(b) shows the comparison of the results and the present theoretical results for the first axial mode of flexural vibrations, where frequency factor $f$ is taken as the ordinate and $n$ is taken as the abscissa. It is observed that there is a little difference between them as to the shell of No. 1, but we can reasonably conclude that the present theory can predict the practically accurate frequencies of those shells.

As another example, the flexural vibrations of the combined shells of hemispherical and circular cylindrical shells shown in Fig.9(a) were considered, and Fig.9(b) shows the present theoretical results and those of the theory of inextensional vibrations of a shell(6), where $f/K$ is taken as the ordinate because the results of the inextensional theory are in proportion to $K$, and the nondimensional length $\rho_o$ of the cylindrical part is taken as the abscissa. For the case of thin shells (i.e. $\kappa=0.01$), the two results are almost the same. As the thickness of the shell wall increases, the frequency curves of the present theory become relatively lower, but by virtue of an intuitive consideration this tendency appears to be right.

As shown in the above examples, the present theory is applicable to various axisymmetric elastic solids with free-boundary surface.

4.3 Availability of the theory

The fundamental assumptions of the present method are only (1) and (2) shown in 2.1, and so it
may be expected that it is possible to obtain a solution which converges to an analytical solution based on the three-dimensional theory of elasticity. However, the convergence of solutions ought to be examined in the actual numerical calculation because a computer has its own limit in capacity. As an example, Table 2 shows the numerical results for Kind TI, where \( M \) and \( N \) are changed variously. Considering the cross-sectional configuration of rings, \( N \) was taken larger than \( M \), in advance, as shown in Table 2. However, it turns out ineffectual to adopt an \( N \) larger than 5, and instead \( M \) ought to be taken larger in order to improve the solutions. In the present paper, the combination of \( M \) and \( N \) is regarded as optimal when the number of \( (3MN) \) becomes as small as possible and besides practically sufficient solutions are obtained by using those \( M \) and \( N \), and so in Fig. 4 the results for \( M=3 \) and \( N=5 \) are shown. On the other hand, \( M \) was taken as 3 invariably in all the examples shown in Figs. 5~7, although the optimal combination of \( M \) and \( N \) depends on the individual cross-sectional configuration of rings. The convergence of the solutions for those sets was worse than that for Set T, and it appears to be possible to decrease the error in approximation further if \( M \) is taken larger instead of \( N \). The accuracy for \( M=3 \) in the present method is almost the same as that of the previous energy method which depends on the theory of a circular cylindrical shell considering the influence of rotary inertia and transverse shear deformation.

Particularly in contrast with the finite element method, the defects of the present method are, that (i) it is impossible to take general boundary conditions into account and, (ii) it is clumsy to describe the partial and sharp change of displacement curves. But the merits are, that (i) it is possible to obtain practically sufficient solutions by operating relatively small scale matrices and, (ii) the procedure of programming, treating input data and improving the accuracy of solutions is relatively easy. Accordingly, we may reasonably conclude that the method is pretty available for the prediction of the natural frequencies of an arbitrary ring including a general axi-symmetric elastic solid with free-boundary surface.

5. Conclusions

In order to predict the natural frequencies of an arbitrary ring, an approximate theory by means of the Ritz's method was proposed and it was examined by the experimental and already-known results. The conclusions are summarized as follows:

(1) By this method, it is possible to predict practically sufficient frequencies of various rings whose cross section is entirely arbitrary.

(2) When the cross-sectional contour of a ring is continuous with respect to the coordinates \( r, z \), the error of solutions is small in comparison with that for the discontinuous cross section. And the error is larger when the contour has a discontinuity on the outer line and near the central part than on the inner line and near both the edges.

(3) It is possible to obtain the natural frequencies of general axi-symmetric elastic solids with free boundary surface if the cross-sectional configuration is not so complicated.

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References


