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Lifting Line Theory for a Wing with a Flat Plate Placed Near Its Tip*

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In order to obtain the blade characteristics of an axial fan with an orifice-type fan guide, the influence of the fan guide on the blade characteristics is investigated by establishing a theoretical model in which a flat plate parallel to the wing surface is placed near its tip. As the result, a Prandtl-type integral equation for this wing model is derived. In the present paper, the equation is solved when the wing and the plate are in the same plane. Furthermore, the wing planform and its aerodynamic forces satisfying the condition of minimum induced drag are also presented. Thus, the fundamental blade characteristics is clarified concretely.

1. Introduction

Usually, fans and blowers which have a cylindrical casing are used for industrial purposes. For domestic or agricultural use, we apply axial fans which have an orifice type or a bell-mouth type fan guide, or exposed non-pressure type fans which have neither a fan guide nor a cylindrical casing.

Many theoretical and experimental investigations have been made concerning the cylindrical-casing type hydrodynamical machines, and hence, it is not so difficult to design these fans and blowers. Contrary to these cylindrical casing type machines, only a few researches have been made about the exposed non-pressure type fans. Ishizawa⁽¹⁾ analyzed this type of fan. His theory is very helpful when we solve the practical design problems.

The characteristics of the fan with an orifice type or a bell-mouth type fan guide lie in the middle between those of the cylindrical-casing type fans and those of the exposed non-pressure type fans. In the non-pressure type fan, we can not get a static pressure rise, because it is completely exposed to the surrounding air. On the contrary, the static pressure rise can be obtained by a fan with an orifice type fan guide, but it is less than that obtained in the casing type fan.

About the fan with a fan guide, systematic investigations have not been made, because its hydrodynamical characteristics are very complicated. Especially, the influence of the fan guide upon the blade characteristics is not clarified at all.

In the present investigation, in order to clarify the fan guide effect on the blade characteristics, a theoretical wing model, i.e., a three dimensional wing in a uniform stream with a semi-infinite flat plate which is placed near the wing tip and parallel to the wing surface, is considered by developing the circumferential coordinate into a linear one (Fig. 2), and this model is analyzed theoretically by making some approximation. In this respect, for the cylindrical casing type fan, usually a wing model in which the vertical flat plate is placed near the wing tip is considered, and this model has been analyzed theoretically and experimentally by many researchers^{(2)~(4)}. In order to examine the difference among these three types of fans, the blade characteristics given in the present theory are compared with those given by Nishiyama and Takagi⁽⁵⁾ who investigated the blade characteristics of the cylindrical casing type machines, and also compared with the characteristics of a finite three dimensional wing which corresponds to the blade of an exposed non-pressure type fan.

2. The theoretical model and the coordinate system

In Fig. 1, a typical fan guide type axial fan is shown. When we develop the circumferential coordinate θ into a linear one, we get a theoretical wing model, that is, a cascade of finite wings near whose tip a flat plate is placed. Since we use the fan of this kind by fitting it on a wall, the flat plate can be considered semi-infinite. The fundamental purpose of the present investigation is to clarify the influence of the fan guide upon the blade characteristics. Hence, we shall pick up a single wing among the cascade, and shall study the single finite wing with a

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flat plate placed near its tip (Fig. 2). When we make the theoretical model in such a way, a fan-boss is replaced approximately by an infinite flat plate perpendicular to the fan guide surface. Further, actual relative air speed to the blades is not uniform; near the center of rotation, the relative air speed is small, and near the blade tip it is large. Hence, if we want to analyze the blade characteristics very precisely, we should study a wing which is placed in a shear flow. However, concerning this type of the wing model, the theoretical analysis is very difficult. In the present investigation, we analyze the characteristics of a wing which is placed in a uniform flow. The influence of non-uniformity of on-coming flow upon the wing characteristics can be determined qualitatively by the author's previous theoretical investigation⁽⁵⁾.

Since the aspect ratio of blades of such a fan is generally large, the blade (wing) characteristics can be analyzed reasonably by the lifting line theory. When a wing of finite span is placed in a flow, a sheet of the trailing vortex is shed from its trailing edge. The sheet of the trailing vortex inclines relative to the flat plate owing to the induced velocities of the trailing vortex itself and of the flat plate. Accordingly, a theoretical study about the wing whose trailing vortex inclines relative to the plate should be carried out. However, because it is only the trailing vortex very close to the wing that has great influence upon the wing characteristics, it is enough to study only a wing model in which the trailing vortex and the flat plate are parallel.

The flow field around a wing as shown in Fig. 2 is the same as that around the wing with two flat

Fig. 1 An axial flow fan with a fan guide

Fig. 2 The theoretical model of the axial flow fan with a fan guide

plates placed near its both tips, as shown in Fig. 3. In the present research, we shall make the theoretical analysis about the wing model illustrated in Fig. 3. Since the flat plate (the fan guide) and the wing (the blade) are generally not on the same plane, we shall treat a wing whose position relative to the flat plate is arbitrary.

The coordinate system (x, y, z) is taken as shown in Fig. 3; the x -axis is parallel to the on-coming free flow, and y -axis is put on the flat plate. Let the wing span be $2s_0$ and the distance from the mid-span of the wing to the plate be b_0 . The symbol $c(y)$ and H are the wing chord and the interval between the wing surface and the flat plate, respectively. The other symbols are shown in Fig. 3.

3. The boundary conditions and the theoretical analysis

In the present research, the fluid is considered to be incompressible and inviscid. Let u , v and w denote the perturbation velocities in the x -, y - and z -directions, respectively. U_0 denotes the on-coming free flow velocity, which is constant.

In the lifting line theory, the following assumption is introduced: an each spanwise wing section is assumed to have the same characteristics as that of a two dimensional wing which is set at an angle which is equal to the sum of $w(o, y, H)/U_0$ and a geometrical angle of attack (actually w is negative). Therefore, on the wing surface, the precise boundary condition is not necessary.

On the contrary, on the surface of the flat plate, the boundary condition

$$z=0, |y| \geq b_0; w(x, y, 0)=0 \dots\dots\dots (1)$$

should be satisfied, because the fluid does not pass through the plate. This condition can be satisfied by the vorticities distributed continuously on the plate surface. In other words, the plates are regarded as the "lifting surface". Now, γ_p and δ_p denote the vorticities in the x - and y -direction respectively. Then, the velocity w_p induced in the z -direction by γ_p and δ_p is given in the form,

Fig. 3 The coordinate system and the theoretical wing model

$$w_p(x, y, z) = -\frac{1}{4\pi} \int_{S_p} \int \frac{(x-\xi)\gamma_p(\xi, \eta) + (y-\eta)\delta_p(\xi, \eta)}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{3/2}} d\xi d\eta \quad (2)$$

The integration region S_p is the whole region of the two plates. The vorticity γ_p is related to δ_p by the expression,

$$\frac{\partial \gamma_p}{\partial \eta} = \frac{\partial \delta_p}{\partial \xi} \quad (3)$$

When the wing is represented by the "lifting line", the velocities u' , v' and w' which are induced by the bound vortex and by the trailing vortex are,

$$u'(x, y, z) = \frac{z-H}{4\pi[x^2 + (z-H)^2]} \int_{-s_0}^{s_0} \frac{y-\eta}{R} \frac{d\Gamma}{d\eta} d\eta \quad (4)$$

$$v'(x, y, z) = \frac{z-H}{4\pi} \int_{-s_0}^{s_0} \frac{1}{(y-\eta)^2 + (z-H)^2} \left(1 + \frac{x}{R}\right) \frac{d\Gamma}{d\eta} d\eta \quad (5)$$

$$w'(x, y, z) = -\frac{1}{4\pi} \int_{-s_0}^{s_0} \frac{y-\eta}{(y-\eta)^2 + (z-H)^2} \frac{d\Gamma}{d\eta} d\eta - \frac{1}{4\pi} \frac{x}{x^2 + (z-H)^2} \int_{-s_0}^{s_0} \frac{y-\eta}{(y-\eta)^2 + (z-H)^2} \\ \times \left[R + \frac{(z-H)^2}{R} \right] \frac{d\Gamma}{d\eta} d\eta \quad (6)$$

where,

$$R = [x^2 + (y-\eta)^2 + (z-H)^2]^{1/2} \quad (7)$$

and $\Gamma(y)$ denotes the wing circulation.

Accordingly, the velocity w induced by both the wing and the plate is given by

$$w = w' + w_p \quad (8)$$

and hence, the boundary condition Eq. (1) yields

$$|y| \geq b_0; -\frac{1}{4\pi} \int_{-s_0}^{s_0} \frac{y-\eta}{(y-\eta)^2 + H^2} \frac{d\Gamma}{d\eta} d\eta - \frac{x}{4\pi(x^2 + H^2)} \int_{-s_0}^{s_0} \frac{y-\eta}{(y-\eta)^2 + H^2} \left(R_0 + \frac{H^2}{R_0}\right) \\ \times \frac{d\Gamma}{d\eta} d\eta - \frac{1}{4\pi} \int \int_{S_p} \frac{(x-\xi)\gamma_p(\xi, \eta) + (y-\eta)\delta_p(\xi, \eta)}{[(x-\xi)^2 + (y-\eta)^2]^{3/2}} d\xi d\eta = 0 \quad (9)$$

where

$$R_0 = [x^2 + (y-\eta)^2 + H^2]^{1/2} \quad (10)$$

This is an integral equation which determines the relation between $\Gamma(y)$ and γ_p [δ_p is related to γ_p by Eq. (3)]. If Eq. (9) were solved, the whole field around the wing with the flat plates placed near its tips could be found. However, it is very difficult to solve the equation about the "lifting surface" exactly. Then, the following approximation will be adopted. From Eqs. (4), (5) and (6), it can be concluded that, on the flat plate, the induced velocities u' , v' and w' increase gradually, and come to the values $u'(\infty, y, 0)$, $v'(\infty, y, 0)$ and $w'(\infty, y, 0)$ at infinitely downstream. Furthermore, the rate of their increase is very small. Accordingly, the flow pattern in the plane normal to the x -axis (Fig. 4) may be considered to be nearly two dimensional. Then, the vorticities δ_p , γ_p can be approximately written as follows;

$$\gamma_p(\xi, \eta) \cong \gamma_p(\eta; x), \quad \delta_p(\xi, \eta) \cong \delta_p(\eta; x) \quad (11)$$

This approximation means that, at an arbitrary position x , the vorticities $\delta_p(\xi, \eta)$, $\gamma_p(\xi, \eta)$ are substituted for the quasi-two-dimensional vorticities $\delta_p(\eta; x)$, $\gamma_p(\eta; x)$, but we retain the three dimensionality that the vorticities varies in the x -direction. The idea of this approximation is the same as that of the Jones' theory for a slender wing⁽⁶⁾.

By such an approximation, $w_p(x, y, z)$ becomes

$$w_p(x, y, z) = -\frac{1}{2\pi} \left\{ \int_{-\infty}^{-b_0} + \int_{b_0}^{\infty} \right\} \frac{(y-\eta)\delta_p(\eta; x)}{(y-\eta)^2 + z^2} d\eta \quad (12)$$

Then, Eq. (9) yields

Fig. 4 Flow pattern in the plane parallel to the yz -plane

Fig. 5 A horse shoe vortex

$$|y| \geq b_0; -\frac{1}{4\pi} \int_{-s_0}^{s_0} \frac{y-\eta}{(y-\eta)^2 + H^2} \frac{d\Gamma}{d\eta} d\eta - \frac{x}{4\pi(x^2 + H^2)} \int_{-s_0}^{s_0} \frac{y-\eta}{(y-\eta)^2 + H^2} \left(R_0 + \frac{H^2}{R_0}\right) \frac{d\Gamma}{d\eta} d\eta - \frac{1}{2\pi} \left\{ \int_{-\infty}^{-b_0} + \int_{b_0}^{\infty} \right\} \frac{\delta_p(\eta; x)}{y-\eta} d\eta = 0 \quad (13)$$

In the lifting line theory, only the "down wash" $w(0, y, H)$ at the lifting line is necessary, and is obtainable, if $\delta_p(\eta; 0)$ is acquired from Eq. (13).

In the next place, $\delta_p(\eta; 0)$ will be solved supposing that only a single horse shoe vortex exists, and the down wash $w(0, y, H)$ induced by the continuously distributed trailing vortex will be acquired by integrating the velocity which is induced by a single horse shoe vortex.

Since, in the present investigation, only the wing of symmetric planform is taken into consideration, it is enough to consider only a symmetric horse shoe vortex which flows away at $y=y_0$ and $-y_0$ as shown in Fig. 5. For the same reason, the vorticity $\delta_p(\eta; 0)$ can be written by the expression

$$\delta_p(\eta; 0) = -\delta(-\eta; 0) \quad (14)$$

Therefore, the integral equation for a single horse shoe vortex of strength γ_w yields

$$-\frac{\gamma_w}{2\pi} \frac{y_0(y^2 - y_0^2 - H^2)}{(y^2 - y_0^2 - H^2)^2 + 4y^2 H^2} - \frac{1}{\pi} \int_{b_0}^{\infty} \frac{\eta \delta_p(\eta; 0)}{y^2 - \eta^2} d\eta = 0 \quad (|y| \geq b_0) \quad (15)$$

Here, we introduce the following dimensionless variables and constants:

$$r_* = (y/b_0)^2 - 1 \quad (16)$$

$$s_* = (\eta/b_0)^2 - 1 \quad (17)$$

$$\xi_* = y_0/b_0 \quad (18)$$

$$\sigma_0 = H/b_0 \quad (19)$$

$$\phi(s_*) = b_0 \delta_p(\eta; 0) \gamma_w \quad (20)$$

Then, Eq. (15) becomes

$$\int_0^{\infty} \frac{\phi(s_*)}{r_* - s_*} ds_* + \frac{\xi_*(r_* + 1 - \xi_*^2 - \sigma_0^2)}{(r_* + 1 - \xi_*^2 + \sigma_0^2)^2 + 4\xi_*^2 \sigma_0^2} = 0 \quad (21)$$

This integral equation should be solved under the condition that $\phi(r_*)$ tends to zero at infinity. When a suitable variables change is made in Eq. (21), the same integral equation as given in the two dimensional wing theory is obtainable. The solution $\phi(s_*)$ is acquired by the Söhngen's inversion formula⁽⁷⁾, that is,

$$\phi(r_*) = \frac{\xi_*}{\pi^2 \sqrt{r_*}} \int_0^{\infty} \frac{\sqrt{s_*}}{r_* - s_*} \frac{s_* + 1 - \xi_*^2 - \sigma_0^2}{(s_* + 1 - \xi_*^2 + \sigma_0^2)^2 + 4\xi_*^2 \sigma_0^2} ds_* = -\frac{\xi_*}{\pi \sqrt{2(R_* + \sqrt{S_*})}} \frac{1}{\sqrt{r_*}} \times \frac{(R_* + \sqrt{S_*} + 2\sigma_0^2)r_* + R_* + \sqrt{S_*} - 2\sigma_0^2}{r_*^2 + 2R_*r_* + S_*} \quad (22)$$

where

$$R_* = 1 - \xi_*^2 + \sigma_0^2 \quad (23)$$

$$S_* = R_*^2 + 4\xi_*^2 \sigma_0^2 \quad (24)$$

Making use of $\phi(r_*)$ thus obtained, we have the down wash at the lifting line,

$$w(0, y, H) = -\frac{\gamma_w}{2\pi b_0} \frac{\xi_*}{y_*^2 - \xi_*^2} + \frac{\gamma_w}{4\pi b_0} \int_0^{\infty} \frac{r_* + Q_*}{r_*^2 + 2P_*r_* + Q_*} \phi(r_*) dr_* = -\frac{\gamma_w}{2\pi b_0} \times \left[\frac{\xi_*}{y_*^2 - \xi_*^2} + \frac{\xi_*(R_* + \sqrt{S_*} + 2\sigma_0^2)}{\sqrt{2(R_* + \sqrt{S_*})}} \left\{ B \left(\frac{1}{\sqrt{2(P_* + \sqrt{Q_*})}} - \frac{1}{\sqrt{2(R_* + \sqrt{S_*})}} \right) + \frac{E}{\sqrt{Q_*}} \frac{1}{\sqrt{2(P_* + \sqrt{Q_*})}} - \frac{G}{\sqrt{S_*}} \frac{1}{\sqrt{2(R_* + \sqrt{S_*})}} \right\} \right] \quad (25)$$

where,

$$y_* = y/b_0 \quad (26)$$

$$P_* = 1 - y_*^2 + \sigma_0^2 \quad (27)$$

$$Q_* = P_*^2 + 4\sigma_0^2 y_*^2 \quad (28)$$

$$B = (1/\Delta) [2(R_* - P_*)(P_*K_* - Q_*) - (S_* - Q_*)(K_* - P_*)] \quad (29)$$

$$E = (1/\Delta) [2(R_* - P_*)(2P_*^2K_* - Q_*P_* - Q_*K_*) + (S_* - Q_*)(Q_* - P_*K_*)] \quad (30)$$

$$G = (1/\Delta) [2(R_* - P_*)(2R_*P_*K_* - S_*P_* - S_*K_*) + (S_* - Q_*)(S_* - P_*K_*)] \quad (31)$$

$$\Delta = 4(R_* - P_*)(S_*P_* - R_*Q_*) - (S_* - Q_*)^2 \quad (32)$$

$$K_* = \{(R_* + \sqrt{S_*} - 2\sigma_0^2)/(R_* + \sqrt{S_*} + 2\sigma_0^2)\} \sqrt{S_*} \quad (33)$$

As the above is the down wash which is induced by a single horse shoe vortex, the down wash induced by continuously distributed trailing vortex can be derived by integrating Eq. (25) along the span.

$$w(0, y, H) = -\frac{1}{2\pi b_0} \int_0^{s_0/b_0} \frac{\xi_*}{y_*^2 - \xi_*^2} K_0(y_*^2, \xi_*^2; \sigma_0) \frac{d\Gamma}{d\xi_*} d\xi_*$$

$$= -\frac{1}{4\pi b_0} \int_{-s_0/b_0}^{s_0/b_0} \frac{K_0(y_*^2, \xi_*^2; \sigma_0)}{y_* - \xi_*} \frac{d\Gamma}{d\xi_*} d\xi_* \quad (34)$$

Where, s_0 denotes the semi-span, and

$$K_0(y_*^2, \xi_*^2; \sigma_0) = 1 + \frac{(R_* - P_*)(R_* + \sqrt{S_*} + 2\sigma_0^2)}{\sqrt{2}(R_* + \sqrt{S_*})} \left[B \left(\frac{1}{\sqrt{2}(P_* + \sqrt{Q_*})} - \frac{1}{\sqrt{2}(R_* + \sqrt{S_*})} \right) \right.$$

$$\left. + \frac{E}{\sqrt{Q_*}} \frac{1}{\sqrt{2}(P_* + \sqrt{Q_*})} - \frac{G}{\sqrt{S_*}} \frac{1}{\sqrt{2}(R_* + \sqrt{S_*})} \right] \quad (35)$$

Now, as shown in Fig. 6, the effective angle of attack α_e is given by

$$\alpha_e = \alpha + \frac{w(0, y, H)}{U_0} \quad (36)$$

Substituting $w(0, y, H)$ given in Eq. (34) into Eq. (36), we have

$$\alpha_e = \alpha - \frac{1}{4\pi b_0 U_0} \int_{-s_0/b_0}^{s_0/b_0} \frac{K_0(y_*^2, \xi_*^2; \sigma_0)}{y_* - \xi_*} \frac{d\Gamma}{d\xi_*} d\xi_* \quad (37)$$

In the lifting line theory, the characteristics of a wing at each section are supposed to be equal to those of a two dimensional wing working at an angle α_e , and hence, the local circulation $\Gamma(y)$ can be written as

$$\Gamma(y) = \frac{1}{2} U_0 c(y) \kappa \alpha_e \quad (38)$$

where $c(y)$ and κ denote the local chord length and the lift curve slope $\partial C_L / \partial \alpha_e$ of a two-dimensional wing, respectively. Substituting α_e given in Eq. (37) into Eq. (38), we have an integral equation whose unknown function is $\Gamma(y)$.

$$\Gamma(y_*) = \frac{1}{2} U_0 c(y) \kappa \left[\alpha(y_*) - \frac{1}{4\pi b_0 U_0} \int_{-s_0/b_0}^{s_0/b_0} \frac{K_0(y_*^2, \xi_*^2; \sigma_0)}{y_* - \xi_*} \frac{d\Gamma}{d\xi_*} d\xi_* \right] \quad (39)$$

Naturally, when we solve this equation for every value of σ_0 , we can obtain the aerodynamic forces acting on the wing whose position relative to the plate is arbitrary. However, it is very difficult to solve this equation analytically.

In the present chapter, the total lift force, the induced drag, etc. will be presented, and further, the condition for the minimum induced drag will also be shown. The typical solution of the equation (39)

Fig. 6 The wing section and its angle of attack

will be given in the next chapter.

First of all, the spanwise lift distribution $l(y)$ is given in the form

$$l(y) = \rho U_0 \Gamma(y) \quad (40)$$

and by integrating $l(y)$ along the span, we get the total lift force.

$$L = \rho U_0 \int_{-s_0}^{s_0} \Gamma(y) dy \quad (41)$$

As shown in Fig. 6, the direction of the local lift $l(y)$ inclines relative to the z -axis at an angle $\Delta \alpha_i$. Then, the induced drag D_i is given as

$$D_i = \rho U_0 \int_{-s_0}^{s_0} \Gamma(y) \Delta \alpha_i(y) dy \quad (42)$$

where, the induced angle of attack is

$$\Delta \alpha_i = -\frac{w(0, y, H)}{U_0} \quad (43)$$

As commonly known, the condition for the wing to realize the minimum induced drag is

$$\Delta \alpha_i(y) = \Delta \alpha_0 \quad (\text{constant}) \quad (44)$$

that is, the minimum induced drag is given by the condition that the induced angle of attack $\Delta \alpha_i(y)$ is constant along the wing span. Accordingly, $\Gamma(y)$ which satisfies this condition is derived from Eqs. (34) and (43).

$$\int_{-s_0/b_0}^{s_0/b_0} \frac{K_0(y_*^2, \xi_*^2; \sigma_0)}{y_* - \xi_*} \frac{d\tilde{\Gamma}}{d\xi_*} d\xi_* = 4\pi b_0 U_0 \Delta \alpha_0 \quad (45)$$

where the symbol " $\sim$ " is affixed to $\Gamma(y)$ of the wing with minimum induced drag. The induced angle of attack $\Delta \alpha_0$, which is an unknown constant in Eq. (45), can be determined such that $\tilde{\Gamma}(y)$ may satisfy the integral equation (39).

Now, when the integral equations (39) and (45) are solved, the aerodynamic forces acting on the wing will all be obtainable. Then, in the next chapter, Eqs. (39) and (45) will be solved in the case of $\sigma_0 = 0$, that is, in the case that the wing (the blade) and the flat plates (the fan-guide) are in the same plane, and we shall examine the wing characteristics.

4. The wing characteristics in the case that the wing and the plates are in the same plane

4.1 The solution of the integral equation and the aerodynamic forces on the wing

When σ_0 tends to zero in Eq. (35), $K_0(y_*^2, \xi_*^2; \sigma_0)$ becomes

$$K_0(y_*^2, \xi_*^2; 0) = \sqrt{\frac{1-\xi_*^2}{1-y_*^2}} = \sqrt{\frac{b_0^2-y_0^2}{b_0^2-y^2}} \quad (46)$$

and the integral equation (39) yields

$$\Gamma(y) + \frac{\kappa c(y)}{8\pi\sqrt{b_0^2-y^2}} \int_{-s_0}^{s_0} \frac{\sqrt{b_0^2-y_0^2}}{y-y_0} \frac{d\Gamma}{dy_0} dy_0 = \frac{1}{2} \kappa U_0 c(y) \alpha(y) \quad (47)$$

Here, the following new dimensionless values and variables ϕ and θ are introduced.

$$\mu_0 = \frac{\kappa c_0}{8s_0}, \quad h = \frac{s_0}{b_0}, \quad \alpha_*(y) = \frac{\alpha(y)}{\alpha_0}, \quad c_*(y) = \frac{c(y)}{c_0}, \quad c_0 = c(0), \quad \alpha_0 = \alpha(0) \quad (48)$$

$$y = s_0 \cos \phi, \quad y_0 = s_0 \cos \theta \quad (49)$$

Using this new variable θ , we represent the circulation $\Gamma(y_0)$ as the following series.

$$\Gamma(y_0) = \Gamma(\theta) = \frac{1}{2} \kappa U_0 c_0 \alpha_0 \frac{1}{\sqrt{1-h^2 \cos^2 \theta}} \sum_{m=0}^{\infty} A_{2m+1} \sin(2m+1)\theta \quad (50)$$

Since the circulation $\Gamma(y)$ is symmetrical, the above series is naturally given by the sum of odd terms only. Substituting Eq. (50) into Eq. (47), we obtain the algebraic equality

$$\sum_{m=0}^{\infty} A_{2m+1} G_{2m+1}(\phi) = c_*(\phi) \alpha_*(\phi) \sqrt{1-h^2 \cos^2 \phi} \quad (51)$$

where,

$$G_{2m+1}(\phi) = \sin(2m+1)\phi + \mu_0 c_*(\phi) \left[(2m+1) \frac{\sin(2m+1)\phi}{\sin \phi} - \frac{h \{a^{2m+1} - h \cos \phi \cos(2m+1)\phi\}}{1-h^2 \cos^2 \phi} \right] \quad (52)$$

$$a = \frac{h}{1+\sqrt{1-h^2}} \quad (53)$$

From Eq. (51), we should obtain the unknown constants A_{2m+1} 's such that Eq. (51) may be valid for an arbitrary value of ϕ . However, it is very difficult to obtain these unknowns analytically. Only numerical calculation is available. The results of numerical calculation will be shown later.

The aerodynamic forces are presented in terms of A_n .

$$L = \frac{1}{2} \kappa \rho U_0^2 c_0 s_0 \alpha_0 \sum_{m=0}^{\infty} A_{2m+1} (Q_m - Q_{m+1}) \quad (54)$$

where,

$$Q_m = \frac{\pi}{2} (1+\beta) \beta^m \sum_{k=0}^{\infty} \frac{(2k-1)!! (2m+2k-1)!!}{(2k)!! (2m+2k)!!} \beta^{2k} \quad (55)$$

$$\beta = \frac{h^2}{(2-h^2)^2 + 2\sqrt{1-h^2}}$$

$$D_i = \frac{\pi}{4} \kappa \rho U_0^2 c_0 s_0 \mu_0 \alpha_0^2 \frac{1+a^2}{1-a^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1} A_{2n+1} [(2m+1)a^{2|m-n|} + h(-1)^m a^{2n+1}] \quad (56)$$

4.2 Minimum induced drag

As for the wing realizing minimum induced drag, the circulation $\tilde{\Gamma}(y_0)$ will be also presented by a series,

$$\tilde{\Gamma}(y_0) = \tilde{\Gamma}(\theta) = \frac{1}{2} \kappa U_0 c_0 \alpha_0 \frac{1}{\sqrt{1-h^2 \cos^2 \theta}} \times \sum_{m=0}^{\infty} \tilde{A}_{2m+1} \sin(2m+1)\theta \quad (57)$$

When $\tilde{\Gamma}(y_0)$ given in Eq. (57) is substituted into Eq. (45), we get an algebraic equality, which contains the unknown constant $\angle \alpha_0$, the induced angle of attack. However, from this algebraic equality, we can remove the unknown constant $\angle \alpha_0$, if $\tilde{A}_{2m+1}$'s are put as follows;

$$\tilde{A}_{2m+1} = \frac{\angle \alpha_0}{\mu_0 \alpha_0} \tilde{A}_{2m+1}^* \quad (58)$$

The algebraic equality for $\tilde{A}_{2m+1}^*$ is

$$\sum_{m=0}^{\infty} \tilde{A}_{2m+1}^* H_{2m+1}(\phi) = \sqrt{1-h^2 \cos^2 \phi} \quad (59)$$

where

$$H_{2m+1}(\phi) = (2m+1) \frac{\sin(2m+1)\phi}{\sin \phi} - \frac{h \{a^{2m+1} - h \cos \phi \cos(2m+1)\phi\}}{1-h^2 \cos^2 \phi} \quad (60)$$

From Eq. (59), we can obtain $\tilde{A}_{2m+1}^*$ regardless of μ_0 and $\angle \alpha_0$. Meanwhile, $\angle \alpha_0$ is determined under the requirement that $\tilde{A}_{2m+1}$'s should also satisfy Eq. (51). Substituting Eq. (58) into Eq. (51), we get

$$\frac{\angle \alpha_0}{\alpha_0} = \frac{\mu_0 c_*(\phi)}{\tau(\phi) + \mu_0 c_*(\phi)} \alpha_*(\phi) \quad (61)$$

where,

$$\tau(\phi) = \frac{1}{\sqrt{1-h^2 \cos^2 \phi}} \sum_{m=0}^{\infty} \tilde{A}_{2m+1}^* \sin(2m+1)\phi \quad (62)$$

Because $\angle \alpha_0$ given in Eq. (61) is constant, the right hand side of Eq. (61) should be constant for every value of ϕ . When we put $\phi = \pi/2$ ($y=0$, mid-span) in Eq. (61), $\angle \alpha_0$ is

$$\frac{\angle \alpha_0}{\alpha_0} = \frac{\mu_0}{\tau_0 + \mu_0} \quad (63)$$

where,

$$\tau_0 \equiv \tau \left(\frac{\pi}{2} \right) = \sum_{m=0}^{\infty} (-1)^m \tilde{A}_{2m+1}^* \dots\dots\dots (64)$$

As the unknown constant $\Delta\alpha_0$ is thus determined, $\tilde{A}_{2m+1}$ is

$$\tilde{A}_{2m+1} = \frac{1}{\tau_0 + \mu_0} \tilde{A}_{2m+1}^* \dots\dots\dots (65)$$

The relation between the optimum wing planform $c_*(\phi)$ and its aerodynamic twist $\alpha_*(\phi)$ is given as follows, when we equate Eq. (61) with Eq. (63).

$$c_*(\phi) = \frac{\tau(\phi)}{(\tau_0 + \mu_0)\alpha_*(\phi) - \mu_0} \dots\dots\dots (66)$$

Finally, the aerodynamic forces are

(i) The total lift force

$$\tilde{L} = \frac{1}{2} \kappa \rho U_0^2 c_0 s_0 \alpha_0 \frac{1}{\tau_0 + \mu_0} \sum_{m=0}^{\infty} \tilde{A}_{2m+1}^* (Q_m - Q_{m+1}) \dots\dots\dots (67)$$

(ii) The induced drag

$$\begin{aligned} \tilde{D}_i &= \frac{1}{2} \kappa \rho U_0^2 c_0 s_0 \alpha_0^2 \left(\frac{1}{\tau_0 + \mu_0} \right)^2 \\ &\times \sum_{m=0}^{\infty} \tilde{A}_{2m+1}^* (Q_m - Q_{m+1}) \dots\dots\dots (68) \end{aligned}$$

where Q_m 's are given in Eq. (55).

4.3 The numerical examples and the wing characteristics

In the present section, the characteristics of the rectangular wings will be presented and discussed. At the same time, the planforms and the distributions of the circulation of the optimum wing will also be presented. As for the numerical procedure of calculating A_{2m+1} in Eqs. (51) and (59), several methods have been proposed. In the present investigation, the following method is adopted: the infinite series given in Eqs. (51) and (59) are approximately substituted for a finite series, say a series of N terms, supposing that A_n of more than N th order are negligibly small. Then, we acquire A_{2m+1} under the condi-

tion that Eqs. (51) and (59) thus obtained are valid at N points along the wing span (N distinct values of ϕ). The coefficients A_{2m+1} are obtainable with fairly good accuracy when we put $N \cong 8 \sim 10$.

In Fig. 7, the circulations of the rectangular wing whose aspect ratio is 6 are shown. The fan blade corresponds to the half side of the present model wing. Hence, the circulations illustrated in Fig. 7 coincide with that of the blade whose aspect ratio R_B is 3. The parameter is the dimensionless value s_0/b_0 or ϵ_0/s_0 , which represents the clearance from the wing tip to the edge of the plate, i.e. the tip clearance of a fan blade. The circulation Γ for $s_0/b_0 = 0$ corresponds to the case for $b_0 = \infty$, that is, the case that the fan has no fan guide. From these results, we find as follows: the influence of the fan guide upon the blade characteristics is rather small at $s_0/b_0 = 0 \sim 0.8$, whereas it becomes rapidly large according as s_0/b_0 becomes closer to unity.

Usually, the parameter s_0/b_0 is set larger than 0.9 when we design a fan with such an orifice type fan guide. For such practical purposes, the circulation of the wing (the blade) whose tip clearance parameter s_0/b_0 is larger than 0.9 is illustrated in Fig. 8. When the plate (the fan-guide) is very close to the wing tip, it is concluded that even a small change of the tip clearance gives great influence on the wing characteristics.

In the next place, we shall debate on the wing with minimum induced drag. In Fig. 9, we illustrate the circulation of the optimum wing for $s_0/c_0 = 3.0$. In this figure, the value of the parameter s_0/b_0 or ϵ_0/s_0 is limited to that of the ordinary fans. The circulation $\tilde{\Gamma}$ indicated in a dotted line corresponds to the case that the flat plate (the fan guide) is absent, and the distribution is elliptic. The optimum planforms are given in Eq. (66), and those of the untwisted

Fig. 7 The spanwise circulation distributions of a rectangular wing ($AR=6.0$)

Fig. 8 The spanwise circulation distributions of a rectangular wing ($AR=6.0$)

wing [$\alpha_*(y)=1$] are shown in Fig. 10. Both circulations and planforms of the optimum wing change their shape in a certain degree, when s_0/b_0 varies a little. Therefore, when we design a fan with such a fan guide, we should pay full attention to the determination of the planform and of the tip clearance. The above-mentioned are several numerical examples of the present theory for the wing which is placed in the same plane with the flat plates ($\sigma_0=0$).

As stated before, the axial flow hydrodynamical machines can be classified into three types: a fan housed in the casing of cylindrical type [type (a)], a fan with an orifice type fan guide [type (b)], and an exposed non-pressure type fan [type (c)]. In Fig. 11, these fans and their theoretical models are shown. The aerodynamic characteristics of the model wing of type (a) have been studied by several researchers. Meanwhile, making use of the ordinary finite wing theory, we can easily get the characteristics of the model wing of type (c). As for a wing model of type (b), theoretical studies have not been made up to date, but they are achieved successfully in the present investigation. Then, we shall examine the difference in the characteristics of the model wings of each type. In Fig. 12, the model wing circulations are illustrated. In this figure, the circulation of the wing model (a) is quoted from Nishiyama and Takagi⁽⁸⁾. From

Fig. 9 The circulations of the optimum wing ($s_0/c_0=3.0$)

Fig. 10 The planforms of the optimum wing without the aerodynamic twist: $\alpha_*(y)=1$

Fig. 12, we come to the conclusion that the characteristics of the fan (b) rank between that of the fan (a) and that of the fan (c), and are rather akin to the characteristics of the fan (a). As the result, we find that the orifice type fan guide minimizes leakage through the blade tip clearance, and we can get a comparatively high pressure from the fan of the present type.

5. Conclusions

In order to obtain the blade characteristics of an axial flow fan with an orifice type fan guide, the influence of the fan guide upon the blade characteristics is investigated by analyzing the theoretical wing model in which near the wing tip a semi-infinite flat plate is placed parallel to the wing surface.

As the result, the Prandtl-type integral equation for the wing circulation has been acquired, when the position of the wing relative to the plate is arbitrary. When the wing and the plate are on the same plane, the integral equation is solved actually. Then, we obtain the aerodynamic forces acting on a wing with an arbitrary planform. As numerical examples, the characteristics of rectangular wings are shown in the Chapter 4.

Fig. 11 The axial flow fans of different types and their theoretical model

Fig. 12 The circulation of an each type fan blade

Furthermore, the equation for the wing which realizes the minimum induced drag is obtained and solved. Consequently, the relation between the optimum wing planform and the aerodynamic twist is clarified, and the aerodynamic forces acting on this wing are given.

As the result of the present investigation, the blade characteristics of an axial flow fan with an orifice type fan guide can be made clear fundamentally.

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Discussion

M. NAMBA (Kyushu University):

(1) When we develop the blades of an axial fan with an orifice type fan guide into a linear cascade model, the flat plate (the fan guide) meets the free flow velocity at a finite angle, which is equal to a supplement of the cascade stagger angle. Accordingly, the inclination of the flat plate relative to the free flow velocity is not negligibly small.

(2) The questioner does not make any objection to the idea that a single wing is used as the first approximate model instead of the cascade. In the author's explanation, it seems that the inclination angle can be neglected when we adopt a single wing model. However, the fact that the wing meets the flat plate at a finite angle should not be overlooked even if a single wing model is adopted. We want a more detailed explanation about this point.

T. NISHIYAMA (Tohoku University):

(3) We want a more detailed and general explanation of how the theoretical model is made. The wing model should be made without losing the generality of the fan guide shape and of the position relative to the fan blades. For instance, as shown in Append-Fig. 1, denote the fan guide shape by (θ_1, θ_2, t) and its relative position to the fan blades by (δ_0, H) , then

$\theta_1=0, t=\infty; \delta_0=0, h\neq 0$: the cylindrical casing
 $\theta_1=0, t=0, \theta_2\neq 0; \delta_0\neq 0, h\neq 0$: the flat plate

Append.-Fig. 1.

and,

$$\theta_1 \neq 0, \theta_2 \neq 0, t = \text{finite}; \delta_0 \neq 0, h \neq 0$$

is considered to be of the general form which includes the bell-mouth type fan guide. How faithfully does the flat plate represent the real fan guide?

(4) In order to solve the integral equation (9), it seems to be inevitable to make some approximation. However, it is doubtful that two dimensional approximation, which is expressed by Eq. (11), is reasonable especially in the region very close to the lifting line. On the other hand, in the region far from the lifting line, this approximation is regarded as reasonable, because in that region the flow field varies very slightly in the x -direction.

Therefore, according as the tip clearance becomes smaller, the present analysis also seems to deteriorate in its accuracy.

Y. SUGIYAMA (Nagoya University):

(5) In the region $-s_0 < y < s_0$, the velocity induced in the z -direction by the trailing vortex seems to change slowly in the x -direction, but in the same region, the velocity induced by the bound vortex turns from up wash into down wash at $x=0$. Therefore, when ε_0 and H are small, the total induced velocity seems to vary slightly even on the flat plate ($-s_0 > y$ and $s_0 < y$) because of the bound vortex.

When we notice the cross flow, up wash is realized in the upstream of the lifting line, and down wash in the downstream. Therefore, it seems unreasonable that the present author considers the cross flow to be nearly two dimensional applying the idea of the Jones' wing theory.

Author's Closure

(1) The present author thinks that, when we want to study the cascade model precisely, the angle

Append.-Fig. 2 The typical axial flow fan with a fan guide

which the flat plate meets the free flow velocity should not be neglected. However, the main purpose of the present investigation is to obtain the fundamental influence of the fan guide upon the blade characteristics. Therefore, first of all, the theoretical model is simplified: in the next place, the cascade model should be studied.

(2) The trailing vortex may flow away from the wing inclining at a finite angle relative to the fan guide plate. Therefore, as pointed out, we should study the wing whose trailing vortex meets the flat plate at a finite angle. However, in order to simplify the theoretical procedure, we neglect the inclination angle of the trailing vortex. As mentioned in the present paper, the wing model in which the inclination angle is finite will be studied on another occasion.

(3) There are fans having several type fan guides. The axial fans with such a fan guide as shown in Append.-Fig. 2 are used for various purposes. The present author makes the fans with such an orifice or a bell-mouth type fan guide an object of the study. Hence, in the theoretical model, the fan guide can be replaced by a thin, semi-infinite flat plate.

(4) The author admits that the accuracy of the present theory may deteriorate when the tip clearance is small. However, from the standpoint of engineering, since few investigations have been made about the blade characteristics of a fan with an orifice type fan guide, the present investigation will be significant.

(5) As pointed out, in the region $|y| \leq s_0$, the induced velocity changes from up wash to down wash at the position of the lifting line: there is not a flat plate in this region. In the region $|y| \geq s_0$, the first and the second term in Eq. (9) have no discontinuity.

The accuracy of the approximation depends on the rate of the change of the induced velocities in the x -direction. When ε_0 and H become small, the rate of change may be a little larger, and the accuracy lower. However, it is not caused by the bound vortex, but the trailing vortex which is located very close to the plate.