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On the Uniqueness and the Existence of Solution in Elastostatics for Inhomogeneous Materials*

By Rohzō FURUHASHI**

In the previous paper, an extension of the classical main results in the theory of elastic potential to the case of inhomogeneous materials and a reduction of the fundamental boundary value problems in elastostatics to Fredholm's system of singular integral equations were attempted. In the present paper, with intent to establish the theoretical foundations of the previous results, we constructed the uniqueness and the existence theorems of solutions.

1. Introduction

In the previous paper⁽¹⁾, we constructed the fundamental theory of 3-dimensional elastopotentials for inhomogeneous materials, and showed that the basic boundary value problems of elasticity can be reduced to a system of singular integral equations of Fredholm type. In this paper, based on the previous results, we will establish the uniqueness and existence theorems for the first to third boundary value problems of elasto-statics for inhomogeneous materials.

2. Definitions of boundary value problem

We assume $x=(x_1, x_2, x_3)$ is a point of a three dimensional euclidian space R^3 , Ω is an arbitrary subdomain of R^3 bounded by finite closed Liapunov surfaces $S_1, S_2, \dots, S_n$ (we will use the notation $\partial\Omega=S_1+S_2+\dots+S_n$), and $\lambda(x), \mu(x) \in C^\infty$ are Lamé's coefficients. (C^∞ is assumed to be a class of infinitely differentiable functions.)

In this paper, we will consider the next three kinds of problems of elasticity.

a) First boundary value problem: The displacements are prescribed on the boundary.

b) Second boundary value problem: The external forces are prescribed on the boundary.

c) Third boundary value problem: Linear combinations of displacements and external forces are prescribed on the boundary.

The above problems are equivalent to ones of seeking a displacement vector $u(x)$ by solving the partial differential equation in Ω

$$\begin{aligned} &\mu(x)\Delta u(x) + \{\lambda(x) + \mu(x)\} \operatorname{grad} \operatorname{div} u(x) \\ &+ \operatorname{grad} \lambda(x) \operatorname{div} u(x) + \operatorname{grad} \mu(x) \operatorname{grad} u(x) \\ &+ \operatorname{grad} \mu(x) \frac{du(x)}{dx} = 0 \dots\dots\dots (1) \end{aligned}$$

under the boundary conditions on $\partial\Omega$

$$\left. \begin{aligned} \text{a) } &u(x_0) = f(x_0) \\ \text{b) } &\lambda(x_0)n \operatorname{div} u(x_0) + 2\mu(x_0) \frac{\partial u(x_0)}{\partial n} \\ &\quad + \mu(x_0)(n \times \operatorname{curl}) u(x_0) = f(x_0) \\ \text{c) } &\lambda(x_0)n \operatorname{div} u(x_0) + 2\mu(x_0) \frac{\partial u(x_0)}{\partial n} \\ &\quad + \mu(x_0)(n \times \operatorname{curl}) u(x_0) + \Pi(x_0)u(x_0) \\ &\quad = f(x_0) \end{aligned} \right\} \dots\dots\dots (2)$$

where $f(x)$ is a known vector given on the boundary surface $\partial\Omega$, $\Pi(x_0) \in C^\infty$ is any 3×3 positive definite symmetric matrix, and n is the unit outward normal at a point $x_0 \in \partial\Omega$. For the external problems, we shall add the radiation condition at the infinity

$$\begin{aligned} |x| \rightarrow \infty; \quad &|u_i(x)| = O(|x|^{-1}) \\ &\left| \frac{\partial u_i(x)}{\partial x_j} \right| = O(|x|^{-2}) \\ &(i, j=1, 2, 3) \dots\dots\dots (3) \end{aligned}$$

We will call a solution of the boundary value problems regular if and only if $u(x)$ belongs to the class $C^2(\Omega) \cap C^1(\bar{\Omega})$, where $\bar{\Omega}$ is the closure of Ω , and for an external problem if it satisfies the radiation condition (3). For simplicity, we introduce the notations

$$\left. \begin{aligned} P\left(\frac{\partial}{\partial x}\right) &\equiv \mu(x)\Delta + \{\lambda(x) + \mu(x)\} \operatorname{grad} \cdot \operatorname{div} \\ &+ \operatorname{grad} \lambda(x) \operatorname{div} + \operatorname{grad} \mu(x) \operatorname{grad} \\ &+ \operatorname{grad} \mu(x) \frac{d}{dx} \\ D\left(\frac{\partial}{\partial x}\right) &\equiv \lambda(x)n \operatorname{div} + 2\mu(x) \frac{\partial}{\partial n} + \mu(n \times \operatorname{curl}) \end{aligned} \right\} \dots\dots (4)$$

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** Lecturer, Faculty of Engineering, Meiji University, Kawasaki.

3. The uniqueness theorem

In this chapter, we will establish the uniqueness theorems for the regular solutions of above problems. The method of proof is completely similar to that of the classical Kirchhoff-Clebsch's proof⁽²⁾ based on the Maxwell-Betti's reciprocal theorem for homogeneous materials. The generalization of Maxwell-Betti's theorem to the case of inhomogeneous materials is very easy, and the next lemma holds⁽¹⁾.

Lemma 1 For any regular solutions of partial differential Eq. (1) the equality

$$\int_{\Omega} \left\{ \lambda(x) \left(\sum_k \frac{\partial u_k}{\partial x_k} \right)^2 + \frac{1}{2} \mu(x) \sum_{j,k} \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right)^2 \right\} dx \\ = \int_{\partial\Omega} u D \left(\frac{\partial}{\partial x} \right) u d\sigma \geq 0 \quad \dots\dots\dots (5)$$

is valid.

Theorem 1 The solution of the interior first and the interior third problem is unique.

Proof. We put

$$e(u, u) = \int_{\partial\Omega} u D \left(\frac{\partial}{\partial x} \right) u d\sigma \geq 0 \quad \dots\dots\dots (6)$$

If $u(x)$ is a regular solution of the interior first problem with homogeneous boundary condition

$$u(x_0) = 0 \quad \dots\dots\dots (7)$$

we have from Eq. (5)

$$e(u, u) = 0 \quad \dots\dots\dots (8)$$

Therefore, we get

$$u(x) = a + b \times x \quad \dots\dots\dots (9)$$

where a and b are arbitrary constant vectors. From Eq. (7) we can conclude that the solution $u(x)$ must be identically zero in the whole domain $\Omega \cup \partial\Omega$.

For the interior third problem, we rewrite Eq. (5) as

$$e(u, u) + \int_{\partial\Omega} u (\Pi u) d\sigma \\ = \int_{\partial\Omega} u \left\{ D \left(\frac{\partial}{\partial x} \right) u + \Pi(x_0) u \right\} d\sigma \geq 0 \quad \dots\dots\dots (10)$$

If $u(x)$ is a regular solution of the interior third problem with homogeneous boundary condition, from this equality we get

$$e(u, u) + \int_{\partial\Omega} u \Pi u d\sigma = 0 \quad \dots\dots\dots (11)$$

Since Π is positive symmetric,

$$\int_{\partial\Omega} u \Pi u d\sigma \geq 0 \quad \dots\dots\dots (12)$$

Therefore,

$$e(u, u) = - \int_{\partial\Omega} u (\Pi u) d\sigma \leq 0 \quad \dots\dots\dots (13)$$

This relation and the inequality (6) will show that

$$e(u, u) = 0 \quad \dots\dots\dots (14)$$

Consequently $u(x)$ is given by Eq. (9), and the homogeneous boundary condition permits us only the solution $u(x) \equiv 0$ in $\bar{\Omega}$. Thus the theorem is proved.

Theorem 2 The regular solution of the second interior problem is uniquely determined as far as the additive vector (9).

Proof. The homogeneous boundary condition for the interior second problem and Eq. (5) will give the expression (9) as a regular solution of this problem. Since the vector (9) satisfies the homogeneous boundary condition, the theorem is obvious.

Theorem 3 The regular solution of the exterior first to third problem is unique.

Proof. If the vector $u(x)$ is regular, the generalized Maxwell-Betti's theorem (5) is also valid for the exterior domain. Then, if $u(x)$ is a regular solution of the exterior first to third problem with homogeneous boundary condition

$$\left. \begin{aligned} a) \quad & u(x_0) = 0 \\ b) \quad & D \left(\frac{\partial}{\partial x_0} \right) u(x_0) = 0 \\ c) \quad & D \left(\frac{\partial}{\partial x_0} \right) u(x_0) + \Pi(x_0) u(x_0) = 0 \end{aligned} \right\} \quad \dots\dots\dots (15)$$

the general solution $u(x)$ is given by Eq. (9). However if we reflect upon the radiation condition (3) we must put $a=b=0$ in Eq. (9). Then we can conclude that $u(x)$ is identically zero in the whole exterior domain. Thus the theorem is proved.

4. Reduction of boundary value problem to integral equations

In this chapter, we will quote some basic results from the general theory of elastopotentials for inhomogeneous materials which has been established in author's previous paper⁽¹⁾.

Definition 1 For the scalar functions,

$$H_s^{(k)}(x, y) = \frac{\lambda(y) + 3\mu(y)}{2\mu(y) \{ \lambda(y) + \mu(y) \}} \frac{1}{|x-y|} \delta_s^k \\ + \frac{\lambda(y) + \mu(y)}{2\mu(y) \{ \lambda(y) + \mu(y) \}} \frac{(x_k - y_k)(x_s - y_s)}{|x-y|^3} \\ (k, s = 1, 2, 3) \quad \dots\dots\dots (16)$$

we construct the function $E_s^{(k)}(x, y)$ defined in any closed subregion of the bounded domain Ω , and we assume $E_s^{(k)}(x, y)$ satisfies the following conditions for some constant $h > 0$, i.e. the quantity

$$\left. \begin{aligned} |E_s^{(k)} - H_s^{(k)}| &= O(|x-y|^{-1+h}) \\ \left| \frac{\partial [E_s^{(k)} - H_s^{(k)}]}{\partial x_i} \right| &= O(|x-y|^{-2+h}) \\ \left| \frac{\partial^2 [E_s^{(k)} - H_s^{(k)}]}{\partial x_i \partial x_j} \right| &= O(|x-y|^{-3+h}) \end{aligned} \right\} \quad \dots\dots\dots (17)$$

($i, j, k, s = 1, 2, 3$)

where δ_s^k is Kronecker's delta. The 3×3 matrix $E(x, y)$ with elements $E_s^{(k)}$ is called the fundamental solution of the operator $P(\partial/\partial x)$ if it satisfies the differential equation

$$P \left(\frac{\partial}{\partial x} \right) E(x, y) = 0 \quad \dots\dots\dots (18)$$

at the point $x \in \Omega - y$.

We will extend the fundamental solution defined in the bounded domain Ω to R^3 as follows. We assume Σ is any bounded sphere strictly involving the finite region Ω , and in the exterior of Σ we assume that the elastic body is homogeneous. The extended fundamental solution as above will be expressed by $\Gamma(x, y)$.

Definition 2 The next vector is called the generalized potential of simple layer with density φ ,

$$V(x) = \frac{1}{2\pi} \int_{\partial\Omega} \Gamma(x, y) \varphi(y) d\sigma \dots\dots\dots (19)$$

Similarly, by the generalized potential of double layer with density φ we mean the vector

$$W(x) = \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x, y) \varphi(y) d\sigma \dots\dots\dots (20)$$

where $\Gamma_I(x, y)$ is the transposed matrix of $D(\partial/\partial y) \times [\Gamma(x, y)^t]$ and Γ^t is a transposed matrix of Γ .

Here, we can conclude as follows.

Lemma 2 The generalized potential of a simple layer is continuous in $\Omega \cup \partial\Omega$ if the density vector is continuous.

Lemma 3 For any point in Ω , the generalized potentials of simple and double layers satisfy the basic differential equation of elasticity,

$$P\left(\frac{\partial}{\partial x}\right)V(x) = 0, \quad P\left(\frac{\partial}{\partial x}\right)W(x) = 0 \dots\dots\dots (21)$$

Lemma 4 The vectors $V(x)$ and $W(x)$ have the radiation property at infinity such as

$$\left. \begin{aligned} |V_j(x)| &= O(|x|^{-1}) \\ |W_j(x)| &= O(|x|^{-2}) \end{aligned} \right\} \dots\dots\dots (22) \quad (j=1, 2, 3)$$

Lemma 5 If the density φ is Hölder continuous, the following formulas hold for the generalized potential of a simple layer.

$$\left. \begin{aligned} D\left(\frac{\partial}{\partial x_0}\right)V(x_0)^+ &= \varphi(x_0) + \frac{1}{2\pi} \\ &\times \int_{\partial\Omega} \left[D\left(\frac{\partial}{\partial x_0}\right)\Gamma(x_0, y) \right] \varphi(y) d\sigma \\ D\left(\frac{\partial}{\partial x_0}\right)V(x_0)^- &= -\varphi(x_0) + \frac{1}{2\pi} \\ &\times \int_{\partial\Omega} \left[D\left(\frac{\partial}{\partial x_0}\right)\Gamma(x_0, y) \right] \varphi(y) d\sigma \end{aligned} \right\} \dots\dots\dots (23)$$

where we mean by (+) and (-) the interior and exterior limit from x to $x_0 \in \partial\Omega$, respectively. Similarly, if φ is Hölder continuous, the formulas

$$\left. \begin{aligned} W(x_0)^+ &= -\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x_0, y) \varphi(y) d\sigma \\ W(x_0)^- &= \varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x_0, y) \varphi(y) d\sigma \end{aligned} \right\} \dots\dots\dots (24)$$

are valid for the generalized potential of double layers.

Based on the above results, the first to third boundary value problems of elasticity for inhomogeneous materials can be easily reduced to the form of integral equations. If we seek a solution of the first and the second problems respectively in the form of generalized double and simple layer potentials, we have a system of singular integral equations of Fredholm type for the vector φ as follows.

a) For the first boundary value problems, the density φ satisfies the integral equation

$$\varphi(x_0) - \kappa \int_{\partial\Omega} \varphi(y) K(y, x_0) d\sigma = \dot{f}(x_0) \dots\dots\dots (25)$$

where,

$$K(x, y) = D\left(\frac{\partial}{\partial x}\right)\Gamma(x, y) \dots\dots\dots (26)$$

and we must put $\kappa = 1/2\pi$, $\dot{f} = -f$ for the interior problems, and $\kappa = -1/2\pi$, $\dot{f} = f$ for the exterior problems.

b) For the second boundary value problems, the density φ satisfies the integral equation,

$$\varphi(x_0) - \kappa \int_{\partial\Omega} K(x_0, y) \varphi(y) d\sigma = \dot{f}(x_0) \dots\dots\dots (27)$$

where $\kappa = -1/2\pi$, $\dot{f} = f$ for the internal problems, and $\kappa = 1/2\pi$, $\dot{f} = -f$ for the exterior problems.

c) Similarly, if we seek a solution of the third problems in a form of the generalized potential of a simple layer, we have the integral equation

$$\begin{aligned} \varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \left[D\left(\frac{\partial}{\partial x_0}\right)\Gamma(x_0, y) \right. \\ \left. + II(x_0)\Gamma(x_0, y) \right] \varphi(y) d\sigma = f(x_0) \dots\dots\dots (28) \end{aligned}$$

for the interior problems, and

$$\begin{aligned} -\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \left[D\left(\frac{\partial}{\partial x_0}\right)\Gamma(x_0, y) \right. \\ \left. + II(x_0)\Gamma(x_0, y) \right] \varphi(y) d\sigma = f(x_0) \dots\dots\dots (29) \end{aligned}$$

for the exterior problems.

5. On the regularization of singular integral equations

By using the definition of fundamental solution it is easy to verify that the kernel of singular integral equations constructed in the previous chapter can be expressed in the form

$$K(x, y) = \frac{\mu(x_0)}{\lambda(x_0) + 2\mu(x_0)} M(x_0, y) + N(x_0, y) \dots\dots\dots (30)$$

where,

$$\begin{aligned} M(x_0, y) &= \frac{1}{|x_0 - y|^3} \begin{pmatrix} n_1(y_1 - x_1^0) + n_2(y_2 - x_2^0) + n_3(y_3 - x_3^0) \\ n_2(y_1 - x_1^0) - n_1(y_2 - x_2^0) \\ n_3(y_1 - x_1^0) - n_1(y_3 - x_3^0), \\ n_1(y_2 - x_2^0) - n_2(y_1 - x_1^0) \\ n_1(y_1 - x_1^0) + n_2(y_2 - x_2^0) + n_3(y_3 - x_3^0) \\ n_3(y_2 - x_2^0) - n_2(y_3 - x_3^0), \\ n_1(y_3 - x_3^0) - n_3(y_1 - x_1^0) \\ n_3(y_2 - x_2^0) - n_2(y_3 - x_3^0) \\ n_1(y_1 - x_1^0) + n_2(y_2 - x_2^0) + n_3(y_3 - x_3^0) \end{pmatrix} \\ &\dots\dots\dots (31) \end{aligned}$$

and

$$N(x, y) = \frac{\mu(x_0)}{\lambda(x_0) + 2\mu(x_0)} \left(3[\lambda(x_0) + \mu(x_0)] \right. \\ \times \frac{(y_j - x_j^0)(y_k - x_k^0)}{|x_0 - y|^2} + 2\mu(x_0)\delta_{jk}, \quad \left. \begin{matrix} j \downarrow 1, 2, 3 \\ k \rightarrow 1, 2, 3 \end{matrix} \right) \\ \times \frac{\partial}{\partial n_y} \frac{1}{|x_0 - y|} + O(|x - y|^{2-h}), \quad h > 0 \quad (32)$$

The integral operator defined on $\partial\Omega$ is called the regularization operator for the singular equation

$$(I - \kappa K)\varphi = f \quad (33)$$

if the integral equation

$$(I + \kappa H)(I - \kappa K)\varphi = (I + \kappa H)f \quad (34)$$

is not singular. I means the 3×3 unit matrix.

The regularization problem for elastostatics of homogeneous materials was solved by Kupradze⁽⁸⁾ in a very simpler form than that of the general singular integral equations. In solving the regularization problem the next lemma is essential.

Lemma 5 If the vector φ is Hölder continuous, the formula

$$\int_{\partial\Omega} M(x_0, x') d\sigma_{x'} \int_{\partial\Omega} M(x', y) \varphi(y) d\sigma_y = 4\pi^2 \varphi(x_0) I \quad (35)$$

is valid.

The proof of this lemma is completely similar to that used to prove for homogeneous materials by Kupradze. According to the above lemma, the integral equation

$$(I + \kappa^* M)(I - \kappa^* M - \kappa^* N)\varphi = (I + \kappa^* M)f, \\ \kappa^* = \frac{\mu(x_0)}{\lambda(x_0) + 2\mu(x_0)} \kappa \quad (36)$$

can be rewritten in the form,

$$(I - 4\pi^2 \kappa^*)\varphi + \kappa^* H\varphi = (I + \kappa^* \mu)f \quad (37)$$

where H is

$$H = -N - \kappa^* M N \quad (38)$$

Since the integral operator N is completely continuous, then H is also completely continuous. Accordingly, we can conclude that the integral operator

$$M\varphi \equiv \frac{1}{2\pi} \int_{\partial\Omega} M(x_0, y) \varphi(y) d\sigma \quad (39)$$

defined by Eq. (31) is the desired regularization operator for singular integral Eq. (33).

7. On the existence theorem

Based on the previous results, we will construct the existence theorems for the boundary value problems of inhomogeneous materials. The method of proof is very similar to that used to prove the existence theorems for homogeneous materials⁽⁴⁾.

Theorem 4 If the boundary conditions are Hölder continuous, the interior first or exterior second problem has a uniquely determined solution, which is expressed in the form of generalized potential of a double or simple layer, respectively.

Proof. According to the results of Chapter 4,

the integral equation

$$\varphi(x_0) - \frac{1}{2\pi} \int_{\partial\Omega} \varphi(y) K(y, x_0) d\sigma = f(x_0) \quad (40)$$

for the interior first problems, and

$$\varphi(x_0) - \frac{1}{2\pi} \int_{\partial\Omega} K(x_0, y) \varphi(y) d\sigma = f(x_0) \quad (41)$$

for the exterior second problems form the adjoint systems. Consider the homogeneous equation

$$\varphi(x_0) - \frac{1}{2\pi} \int_{\partial\Omega} \varphi(y) K(y, x_0) d\sigma = 0 \quad (40)^\circ$$

$$\varphi(x_0) - \frac{1}{2\pi} \int_{\partial\Omega} K(x_0, y) \varphi(y) d\sigma = 0 \quad (41)^\circ$$

It is easy to prove that the above homogeneous equations have only trivial solutions. Assuming conversely that $\varphi(x_0)$ is a non-trivial solution of Eq. (41) we can construct a generalized potential of a simple layer $V(x, \varphi)$ which satisfies the homogeneous condition on $\partial\Omega$

$$D\left(\frac{\partial}{\partial x_0}\right) V(x_0, \varphi)^- = 0 \quad (42)$$

and the partial differential Eq. (1) in the exterior domain Ω^- . However, according to the uniqueness theorem for exterior domain (Th. 3), $V(x, \varphi)$ must be identically zero in Ω^- , and from the continuity of the generalized potential of simple layer, the interior limit of $V(x, \varphi)$ on the boundary is zero. Then, the uniqueness theorem for the interior domain (Th. 1) shows that $V(x, \varphi)$ is identically zero in the interior domain Ω^+ . Accordingly, we get

$$D\left(\frac{\partial}{\partial x_0}\right) V(x_0, \varphi)^+ = 0 \quad (43)$$

If we discriminate Eq. (42) from (43) and use the formula (23), we get $\varphi(x_0) = 0$. This contradicts the assumption that $\varphi(x_0)$ is not trivial. According to the second Fredholm's theorem, Eq. (40)[°] has no non-trivial solution. Then, the third Fredholm's theorem shows that the integral Eqs. (40), (41) are soluble.

Similarly, we get:

Theorem 5 If the boundary condition is Hölder continuous, there exists a unique solution for the interior and exterior third problems, and this solution is represented in the form of generalized potential of a simple layer.

Next, we will consider the interior second problem. According to the third Fredholm's theorem, it is necessary and sufficient for the integral equation in this problem to be soluble that

$$\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} K(x_0, y) \varphi(y) d\sigma = f(x_0) \quad (44)$$

if the known vector $f(x_0)$ satisfies the conditions

$$\int_{\partial\Omega} f \varphi_*^{(k)} d\sigma = 0, \quad (k = 1, 2, \dots, n) \quad (45)$$

where $\varphi_*^{(k)}$ is a complete system of the fundamental solutions for the homogeneous adjoint system of Eq.

(44). It is easy to verify that the complete system $\varphi_*^{(k)}$ is given by

$$\varphi_{*j}^{(k)}(x) = \begin{cases} A^{(k)}(x), & x \in S_j \quad (j=1, 2, \dots, n) \\ 0, & x \notin S_j \end{cases} \quad \dots (46)$$

where $A^{(k)}(x)$ is constructed from the vector $a + b \times x$ by substituting the value zero for each element of a and b with the exception of only one element. Obviously, the system (46) is linearly independent. From this, Eq. (45) can be represented in the form

$$\int_{S_j} f(x_0) d\sigma = 0, \quad \int_{S_j} x_0 \times f(x_0) d\sigma = 0 \quad (j=1, 2, \dots, n) \quad \dots (47)$$

These conditions show respectively the sum of the external forces and moments on the boundary S_j must be statically equilibrium. The condition (47) can be easily deduced from the generalized Maxwell-Betti's reciprocal theorem (5). Then, we get:

Theorem 6 If the boundary condition $f(x_0)$ is Hölder continuous and if it satisfies the condition (47), there exists a solution of interior second problems of elasticity, and its solution is uniquely determined by the generalized potential of a simple layer as far as the additive vector (9).

Similarly, we can prove the next theorem.

Theorem 7 If the boundary condition $f(x_0)$ is Hölder continuous, there exists a unique solution for exterior first problems, and the solution can be represented by the sum of generalized potentials of double and simple layers,

$$u(x) = W(x, \varphi) + \sum_{n=1}^{6n} C_n V(x, \psi^{(n)}) \quad \dots (48)$$

where $\psi^{(n)}$ is a complete system of the fundamental solutions of homogeneous integral equation,

$$\psi^{(n)}(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} K(x_0, y) \psi^{(n)}(y) d\sigma = 0 \quad \dots (49)$$

C_n is determined from the algebraic equation,

$$\int_{\partial\Omega} \left[f - \sum_{m=1}^{6n} C_m V(x, \psi^{(m)}) \right] \psi^{(k)} d\sigma = 0 \quad (k=1, 2, \dots, 6n) \quad \dots (50)$$

and the functional equation for φ is given by

$$\begin{aligned} \varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \varphi(y) K(y, x_0) d\sigma \\ = f(x_0) - \sum_{m=1}^{6n} C_m V(x_0, \psi^{(m)}) \quad \dots (51) \end{aligned}$$

For the proof of this theorem, it is sufficient to repeat the well known scheme for the homogeneous materials⁽⁴⁾.

The author of this paper thanks Miss K. Takeda for her many advices.

References

- (1) Furuhashi, R., and Kataoka, M., *Bull. JSME*, Vol. 11, No. 48 (1968), p. 972.
- (2) Todhunter, I., and Pearson, K., *A history of the theory of elasticity*, Vol. II, No. 2, (1965), p. 11, Dover Pub.
- (3) Куирадзе, В.Д., Дифф. Урав. III-5 (1967), p. 707.
- (4) Куирадзе, В.Д., Методы потенциала в теории упругости, (1962) p. 166, Физико-Мат.

Discussion

M. KATAOKA (Meiji University):

(1) It is considered that the object of the present study to extend the classical results in the theory of elasticity to the case of inhomogeneous materials was almost accomplished by this paper. I expect the results of the study will be applied to various problems. Although all the boundary value problems that were considered heretofore are treated in outline, it seems that there remains a possibility of future study of practical boundary conditions. In the case of inhomogeneous materials specially, it seems that there are many problems to be considered. Please show me your opinion and your future plan if any, as to the above matter.

Editorial Director

(2) Reading your paper, I cannot understand clearly whether a solution exists without a condition or with conditions attached. I think that there are many persons who cannot understand your paper clearly among those who use elastic theories. Therefore, I wish you would state in the conclusion of

your paper under what conditions the existence of a solution is proved by engineering expressions avoiding expressions by formulae as far as possible, so that we can use the results of your study depending on the contents thereof if we cannot understand them. Please examine.

Author's closure

(1) Provided that a problem of small deformation is the object, it can not be considered that a way of giving conditions in the case of inhomogeneous materials differs markedly from the one in the case of homogeneous materials. However, it is necessary in future to study the following problems concerning the boundary conditions which are treated in the present paper.

(a) For the region with infinite boundary surface.

(b) When the singular boundary conditions such as a concentrated force are given.

(c) For the external problems, the radiation

condition at infinity is essential. However, in practical cases, there are important solutions which do not satisfy this condition such as the solution of uniform extension problem of infinite body, then it is desirable to establish the theoretical foundations for such elastic problems.

(2) In the present paper, the author considered a problem of uniqueness of the solution in the former part and a problem of existence of the solution in the latter part. Those two problems are related to each other as follows. The uniqueness theorem shows that "if a solution exists" this solution is limited to one, and at this stage the existence of solutions is not definite. Therefore, it is possible that there is no solution at all. To clarify the above point, we cannot neglect the demonstration of existence of solution. The theory is not well established until the above both demonstrations are made. As the result of the above consideration, the following conclusions were reached in the present paper.

1) When the displacement on the boundary

is given, there exists a unique solution to this problem. If the displacement decreases to zero at the infinity in order of the inverse of distance, there exists a unique solution to the external problem.

2) When the external force is given on the boundary, the next two cases arise.

a) If the displacement decreases to zero at the infinity in order of the inverse of distance, there exists a unique solution.

b) For the internal problems, if the external forces and the moments on the boundary are in equilibrium, a solution to this problem exists and it is uniquely determined as far as the additive vector of rigid displacement. Here, since any rigid displacement gives no stress and strain, we can conclude that there exists a unique stress solution to this problem.

3) When a linear combination of displacement and external force on the boundary is given, there exists a uniquely determined solution.