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Reliability of Response Calculus in Irregular Vibration*

(In Relation to Aseismic Theory)

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This paper deals with the reliability of response calculus in random (irregular) vibration in relation to the problem of aseismic theory in mechanical engineering.

Some stochastic phenomena in a random vibration of structures in earthquake are dominated by two main factors. One is the uncertainty (randomness) of parameters in the system equations and the other is a disturbance (earthquake ground motion) to the system. Here truncated waves from stationary random inputs and responses subjected to them are studied theoretically and experimentally and in this case, statistical characteristics of the amplification factors considerably fluctuate around the values obtained by the analyses using stationary random inputs.

From these studies, it is found that the reliability analyses of response calculi are very important. Especially for lightly damped systems we should pay full attention to the confidence of the result from response calculation.

1. Introduction

Reliability of response calculi in an irregular vibration (random vibration) of mechanical structures is studied in relation to aseismic theory. In this case, mechanical equipments, pipings, etc. are considered as mechanical structures and ground motions as disturbances to them. There are two methods in response calculi of these systems subjected to ground motions. The first one is a step-by-step integrating method and the second one is a statistical method; statistical characteristics of responses of the system are calculated by means of statistical and probabilistic ways. In this case, inputs used to the system are described statistically. The former is usually used because of easiness of procedure, but we have to note the bias of response due to the individuality of each earthquake record. The latter is considered to be of advantage in this point. In this paper, the latter is mainly studied. In such a response calculus, an "idealized system", a system which is composed of deterministic spring constants and masses and to which ground motions are stationary random processes, has been treated. But it is more reasonable to consider that spring constants and masses are non-deterministic values rather than deterministic ones because of the production error and other uncertainty factors

of structural members. As for ground motions, it is more reasonable to consider non-stationary random processes rather than stationary ones. But in this case, statistical characteristics (mean, variance, and etc.) of responses show biases or variations from the ones which have been calculated in an idealized state. Then the estimation of these variations, the reliability of response calculus in a random vibration, is needed to analyze the reliability of structures in aseismic design. Here the "idealized state" means such a state that the parameters of the system are deterministic and/or the disturbances acting on the system are stationary random processes (infinite time length of duration). The study for an idealized state is relatively advanced in aseismic theory of mechanical structures^{(1)~(5)}. But in practical problems, the estimation of reliability of response calculus is needed by reason of the existence of the following factors;

- 1) There exists irregularity in structural systems.
- 2) There exists irregularity in disturbances acting on the structural systems.
- 3) The items 1) and 2) are involved simultaneously.

The aim of this paper is to study the reliability of statistical characteristics of responses and amplification factors originated by item 2) in relation to the aseismic theory. At first in Chapter 2, a survey of the studies related to the item 1) will be introduced. The studies related to the item 2) and conclusions

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will be given in Chapters 3 and 4.

2. Problems of structural systems with some irregularity

In the case that the irregularities (uncertainties) belong to the system itself, the parameters, initial conditions, and boundary conditions of differential equations describing the motions of the system have some variations, these values must be considered as random variables. Here the problem reduces to an random eigen value one, random impulse response one, etc., if we pay attention to the randomness in natural vibrations, impulse responses, etc. of the structures. But in either case, it is difficult to obtain exact solutions analytically and then the solutions are obtained approximately by various kinds of method. In the following two sections, the random eigen value problem and impulse response problem are briefly explained by some kinds of approximate method.

2.1 Random eigen value problem

The study of a random eigen value problem in mechanical engineering is relatively recent. R. P. N. Jones⁽⁶⁾ treated this problem in 1960. The variations of natural frequencies and corresponding modes of the system were studied when the system had small variations in its masses and spring constants. T. T. Soong-J. L. Bogdanoff⁽⁷⁾ also studied almost the same subject on an N -degrees-of-freedom linear mass-spring chain system in more detail. J. D. Collins-W. T. Thomson⁽⁸⁾ gave a general method to treat this problem formally. Recently, M. Miyamoto⁽⁹⁾ and F. Hara⁽¹⁰⁾ analyzed the second order distributed systems having some random parameters. As a text book of these problems, A. T. Bharucha-Reid's book⁽¹¹⁾ is recommended.

Here the random eigen value problem will be briefly explained on the basis of the reference (11). This problem is classified into the following three items;

- 1) There exists randomness (uncertainty) not in boundary conditions, but in the system itself.
- 2) There exists randomness not in the system, but in boundary conditions.
- 3) The items 1) and 2) are involved simultaneously.

This problem is analytically solved by the following approximate methods; a) variational method, b) perturbation method, c) kernel trace method, d) asymptotic method, e) iteration method, etc. In this section a concrete problem, i.e., a vibration of piping systems related to the item 1), is studied by the above methods instead of explaining the above various methods. Piping systems in vibration can be analyzed as linear beams constructed generally in space⁽¹²⁾. And the components of bending,

longitudinal, and torsional vibrations of them can be considered to be separable from each other. In general practical problems, it is not so rare that the masses, cross sectional areas, moduli of elasticity, moduli of rigidity, etc. in equations of motion have variations around their means, that is, these values are considered to be random variables. In this case natural frequencies (eigen values) and corresponding modal functions (eigen functions) can be obtained by the modal analysis method. Here eigen values of the bending vibration of a beam described as

$$\rho A_0 \frac{\partial^2}{\partial t^2} y + \frac{\partial^2}{\partial x^2} \left(E_0 I_0 \frac{\partial^2}{\partial x^2} y \right) = 0 \quad \dots\dots\dots (1)$$

is studied under the deterministic boundary conditions and parameters except the non-deterministic parameter ρ , where the density ρ is assumed to be given in terms of $\rho_0[1+\zeta(x)]$ and $\zeta(x)$ to be a random variable. After changing the dependent variable as $\xi = x/L_0$, we put $y = X(\xi)e^{i\omega t}$, Eq. (1) becomes

$$LX(\xi) - \lambda[1+\zeta(\xi)]X(\xi) = 0 \quad \dots\dots\dots (2)$$

by using the differential operator $L \equiv d^4/d\xi^4$ and $\lambda \equiv (\omega^2/a_0^2)L_0^4$, $a_0^2 \equiv E_0 I_0/\rho_0 A_0$, where L_0 is the total length of the beam. The values L , λ , a_0^2 corresponding to longitudinal, torsional, and shear vibrations⁽¹³⁾ are tabulated in Table 1 with bending vibration. Equation (2) is a self-adjoint linear differential equation. Here L is a deterministic differential operator and $\zeta(\xi)$ is assumed to be known only in its statistical characteristic. By means of the methods a)~e), variances of eigen value λ are described as follows;

$$\begin{aligned} \text{a) } \sigma^2\{\lambda_1^{-1}\} &\cong \mu_1^{-2} \int_0^1 \int_0^1 E\{\zeta(\xi)\zeta(\xi')\} \\ &\quad \times X_1^2(\xi)X_1^2(\xi')d\xi d\xi' \quad \dots\dots\dots (3) \end{aligned}$$

$$\begin{aligned} \text{b) } \sigma^2\{\lambda_k^{-1}\} &\cong \mu_k^{-2} \int_0^1 \int_0^1 E\{\zeta(\xi)\zeta(\xi')\} \\ &\quad \times X_k^2(\xi)X_k^2(\xi')d\xi d\xi' \quad \dots\dots\dots (4) \end{aligned}$$

$$\begin{aligned} \text{c) } \sigma^2\{\lambda_1^{-1}\} &\cong \int_0^1 \int_0^1 E\{\zeta(\xi)\zeta(\xi')\} \\ &\quad \times G^2(\xi, \xi')d\xi d\xi' \quad \dots\dots\dots (5) \end{aligned}$$

$$\begin{aligned} \text{d) } \sigma^2\{\lambda_k^{-1}\} &\cong (k^2\pi^2)^{-2n} \int_0^1 \int_0^1 E\{\zeta(\xi)\zeta(\xi')\} \\ &\quad \times d\xi d\xi' \quad \dots\dots\dots (6) \end{aligned}$$

$$\begin{aligned} \text{e) } \sigma^2\{\lambda_k^{-1}\} &\cong \mu_k^{-1} \int_0^1 \int_0^1 E\{\zeta(\xi)\zeta(\xi')\} \\ &\quad \times G(\xi, \xi')X_k(\xi)X_k(\xi')d\xi d\xi' \quad \dots\dots\dots (7) \end{aligned}$$

In all of these expressions μ_k , X_k , and $G(\xi, \xi')$ denote the eigen values, normalized eigen functions, and Green's function for the operator L of the "idealized system". By these methods several estimates have been obtained for the variance of an eigen value. Estimates based on Eqs. (3) and (5) are primarily useful in dealing with the smallest eigen value λ_1 , the estimate from Eq. (6) concerns λ_k

as $k \rightarrow \infty$, but the perturbation estimate [Eq. (4)] applies to any eigen value. On the other hand, the perturbation and asymptotic estimates are based on the assumption that $\zeta(\xi)$ is small, whereas the others do not. Numerical computations are done under a statistical assumption about $\zeta(\xi)$ in Eqs. (3)~(7). The simplest assumption about $\zeta(\xi)$ is that the mean of $\zeta(\xi)$ is zero, $\zeta(\xi)$ is stationary and $\zeta(\xi)$ and $\zeta(\xi')$ are not correlated (δ -correlation) if $\xi \neq \xi'$ (let spectral intensity be k_0). If we consider only the smallest eigen value λ_1 , then the asymptotic estimate does not apply, and the variational and perturbation results coincide. The following expressions are then obtained;

$$\sigma^2\{\lambda_1^{-1}\} \cong \frac{1}{2} k_0 \mu_1^{-2} \int_0^1 X_1^4(\xi) d\xi \dots\dots\dots (8)$$

$$\sigma^2\{\lambda_1^{-1}\} \cong \frac{1}{2} k_0 \int_0^1 G^2(\xi, \xi) d\xi \dots\dots\dots (9)$$

$$\sigma^2\{\lambda_1^{-1}\} \cong \frac{1}{2} k_0 \mu_1^{-1} \int_0^1 G(\xi, \xi) X_1^2(\xi) d\xi \dots\dots\dots (10)$$

These typical problems have been considered in more detail. They were selected partly because their Green's function, eigen values μ_k , and eigen functions X_k are quite simple. In more detail the relevant quantities are listed in Table 2. For each of these problems the variance of λ_1^{-1} has been computed according to each of Eqs. (8), (9), and (10).

Table 1 The values L , λ , and a_0^2 for vibration types

Vibration type	L	λ	a_0^2
Bending	$\frac{d^4}{d\xi^4}$	$\frac{\omega^2}{a_0^2} L_0^4$	$\frac{E_0 I_0}{\rho_0 A_0}$
Longitudinal	$\frac{d^2}{d\xi^2}$	$-\frac{\omega^2}{a_0^2} L_0^4$	$\frac{E_0}{\rho_0}$
Torsional			$\frac{G_0}{\rho_0}$
Shear			$\frac{k_s G_0}{\rho_0}$

k_s : correction coefficient

The results are given in Table 3. The results agree quite well for the fourth-order problem (III), but less well for the second-order problems, especially for the problem (I). Now let us be more realistic to the statistical assumption about $\zeta(\xi)$. Let $\zeta(\xi)$ be a stationary random function with zero mean; hence

$$E\{\zeta(\xi)\zeta(\xi')\} = \sigma_\zeta^2 \rho(\xi - \xi') \dots\dots\dots (11)$$

where $\sigma_\zeta^2 = E\{\zeta^2(\xi)\}$ and $\rho(\xi - \xi')$ is the correlation function. For the lowest eigen value the estimates given in Eqs. (3)~(7) and (11) reduce to

$$\sigma^2\{\lambda_1^{-1}\} \cong \sigma_\zeta^2 \mu_1^{-2} \int_0^1 \int_0^1 \rho(\xi - \xi') \times X_1^2(\xi) X_1^2(\xi') d\xi d\xi' \dots\dots\dots (12)$$

$$\sigma^2\{\lambda_1^{-1}\} \cong \sigma_\zeta^2 \int_0^1 \int_0^1 \rho(\xi - \xi') G^2(\xi, \xi') d\xi d\xi' \dots\dots\dots (13)$$

$$\sigma^2\{\lambda_1^{-1}\} \cong \sigma_\zeta^2 \mu_1^{-1} \int_0^1 \int_0^1 \rho(\xi - \xi') G(\xi, \xi') \times X_1(\xi) X_1(\xi') d\xi d\xi' \dots\dots\dots (14)$$

corresponding to Eqs. (3), (4); Eq. (5); Eq. (7) respectively. For illustrative purpose we will assume that

$$\rho(\xi - \xi') = \exp\{-a^2(\xi - \xi')^2\} \dots\dots\dots (15)$$

in Eq. (11) which is a typical form of a correlation function. The parameter a^2 governs the interval over which an appreciable correlation occurs, and thus it is related to the correlation length ξ_0 . This latter quantity is determined by the condition that $\rho(\xi - \xi')$ is negligible for $|\xi - \xi'| > \xi_0$. When $\rho(\xi_0) = 0.05$, the relation between a^2 and ξ_0 is as

$$a^2 \xi_0^2 = \ln 20 = 2.9957 \dots\dots\dots (16)$$

and from this $\sigma_\zeta^2 \sigma^2\{\lambda_1^{-1}\}$ is calculated by using Eqs. (12)~(14). The results by Eqs. (12)~(14) are tabulated in Table 4 for $\rho(\xi - \xi')$ given by Eq. (15). Several values of ξ_0 were used with a^2 then determined by Eq. (16). Now in these calculations, we are interested in the value itself of variation or relative dispersion of the lowest eigen value. For

Table 2 Examples for random eigen value problem

	I Fixed-Fixed Longi., Torgi., Shear	II Fixed-Free Longi., Torgi., Shear	III Simply supported Bending
LX	$-X''$	$-X''$	X^{IV}
Boundary conditions	$X(0)=0,$ $X(1)=0$	$X(0)=0,$ $X'(1)=0$	$X(0)=0, X''(0)=0,$ $X(1)=0, X''(1)=0$
X_k	$\sqrt{2} \sin k\pi\xi$	$\sqrt{2} \sin(k-1/2)\pi\xi$	$\sqrt{2} \sin k\pi\xi$
μ_k	$k^2\pi^2$	$(k-1/2)^2\pi^2$	$k^4\pi^4$
$G(\xi, \xi'); \xi < \xi'$	$\xi(1-\xi')$	ξ	$(1/6)\xi(1-\xi')(2\xi'-\xi'^2-\xi^2)$
μ_2/μ_1	4	9	16

the relative dispersion, it shows nearly $0.3\sigma_\zeta$ for $\xi_0=0.05$, and $0.8\sigma_\zeta$ for $\xi_0=0.4$. This suggests that even if the input of the system is considered to be deterministic the response of this system may have a variation around that of the idealized system. Therefore in section 2.2 we will study this problem briefly.

2.2 Random frequency response, impulse response, and step response problems

Natural frequencies and corresponding modal functions show the variations when the parameters of the equation of motion have uncertainty as described in section 2.1 and then the response of the system has also uncertainty even if the disturbance to the system is deterministic. The responses to sinusoidal, impulsive, and step disturbances are called the random frequency response, the random impulse response, and the random step response respectively. This problem is not clearly distinguished from that in section 2.1 but is thought to be a further problem as the responses are concerned. Recently the statistical characteristics of the dynamics of some physical systems with random parameter values have been the subject of a number of publications. F. Kozin⁽¹⁴⁾ developed a partial differential equation satisfied by the joint density function of the displacement and velocity of an undamped simple harmonic oscillator when the spring constant and initial conditions are regarded as

random variables. J. L. Bogdanoff-P. F. Chenea⁽¹⁵⁾ dealt with the frequency the impedance, and the impulsive response of a damped linear oscillator when the spring constant, the mass, and the damping constant are assumed to be random variables. They both reported in 1961. T. T. Soong-J. L. Bogdanoff⁽¹⁶⁾ studied almost the same subject for a linear mass-spring chain of an N -degrees-of-freedom system based on the perturbation method. M. Miyamoto-H. Shibata⁽¹⁷⁾ studied the step response of one-dimensional wave equation with random parameters also applying the perturbation method.

In order to show the significance of this problem, T. T. Soong's studies⁽⁷⁾⁽¹⁶⁾ are referred. Consider a linear mass-spring chain of an N -degrees-of-freedom system with one end fixed and the other free, as shown in Fig. 1. Here we concern ourselves with the impulsive response of a disordered linear chain as one having damping constants between every two masses which are deterministic and equal to c , the spring constants are also deterministic and identical value k , and the masses M_j ($j=1, 2, \dots, N$) are randoms defined by

$$M_j = m(1 + X_j) \dots \dots \dots (17)$$

where X_j are non-dimensional normalized random variables with zero mean and small standard deviations. We shall define the impulsive response $H_{ql}(t)$ of the system as the response at time t of the q th mass to a unit impulse* applied to the l th mass at $t=0$. On the other hand an impulsive response of the idealized system ($X_j=0, j=1, 2, \dots, N$) is denoted by $h_{ql}(t)$. Figure 2 shows $\sqrt{km} E\{H_{NN}(t)\}$, $\sqrt{km} h_{NN}(t)$, and $\sqrt{km} \sigma\{H_{NN}(t)\}$ as a function of $\omega_{01}t$ for $q=l=N=10$, damping $\zeta=0.2$, and $\sigma_x=0.05$, where ω_{01} is the mean first natural frequency of the undamped system. It is seen from Fig. 2 that there is no noticeable difference between $E\{H_{NN}(t)\}$ and $h_{NN}(t)$. This

Table 3 The value $2k_0^{-1}\sigma^2\{\lambda_1^{-1}\}$ for the smallest eigen value

Example	Variational, Perturbation	Kernel trace	Iteration
I	0.015 399 0	0.033 333 3	0.022 019 9
II	0.246 384	0.333 333	0.284 770
III	0.000 158 086	0.000 176 367	0.000 166 762

Table 4 Variances and relative dispersions of the smallest eigen value by various methods

ξ_0	Variational, Perturbation		Kernel trace		Iteration	
	$\sigma\zeta^{-2}\sigma^2\{\lambda_1^{-1}\}$	$\frac{\sigma\zeta^{-1}\sigma\{\lambda_1^{-1}\}}{E\{\lambda_1^{-1}\}}$	$\sigma\zeta^{-2}\sigma^2\{\lambda_1^{-1}\}$	$\frac{\sigma\zeta^{-1}\sigma\{\lambda_1^{-1}\}}{E\{\lambda_1^{-1}\}}$	$\sigma\zeta^{-2}\sigma^2\{\lambda_1^{-1}\}$	$\frac{\sigma\zeta^{-1}\sigma\{\lambda_1^{-1}\}}{E\{\lambda_1^{-1}\}}$
0.05	0.000 786	0.276 6	0.001 578	0.392 0	0.001 085	0.325 0
0.1	0.001 560	0.389 7	0.002 914	0.532 7	0.002 080	0.450 0
0.2	0.003 024	0.542 7	0.005 006	0.698 2	0.003 795	0.607 9
0.4	0.005 416	0.726 3	0.007 588	0.859 6	0.006 230	0.779 0

Fig. 1 N -degrees-of-freedom linear chain system

* Initial conditions of mass p are displacement=0 and acceleration= $\delta_{pi}/m(1+X_p)$. Note that it is not given in terms of velocity.

observation can be explained by the fact that, as shown in Fig. 3, only the top few natural frequencies are sensitive to parameter variations whereas the lowest few are quite stable. The impulsive response, although expressed as the sum of all modes of vibration, receives small contributions from higher modes because (the damping terms and) the amplitudes of successive normal modes decrease rapidly. The standard deviation of $\sqrt{km} H_{NN}(t)$ also has oscillatory forms, being approximately 90 degrees out of phase from the mean. The dispersion from the mean is quite small at extreme displacements but becomes large when the mean motion passes through equilibrium. This increase, however, can not continue indefinitely due to the fact that the motion is damped out eventually. The amplitude of $\sqrt{km} \sigma_{NN}(t)$ in Fig. 2 reaches a maximum value of 0.19 at approximately $\omega_0 t = 12\pi$ radians, then decreases gradually thereafter at a rather slow rate. The effect of the damping constant on the standard deviation of $H_{NN}(t)$ is shown in Fig. 4. It is clear that the dispersion from the mean for $H_{NN}(t)$ becomes substantial as ζ decreases. This observation suggests that for lightly damped systems little confidence can be placed in the response calculation if the parameter values are not precisely known. Therefore we can not neglect

the significance of this phenomenon.

3. Problems of structural systems to non-stationary random inputs

When a time-dependent random disturbance is applied to a system, even if the system has no uncertainty in the parameters, initial conditions, and boundary conditions, the responses of the system show a time-dependent behaviour in their statistical characteristics. This problem is usually called the random response problem. In a random vibration, we have mainly treated the random responses due to the random disturbances and it has been the same matters as in aseismic studies. In an aseismic problem, the random disturbances to the idealized system (i.e. the structures having deterministic parameters) are ground motions. Random processes describing the ground motions probabilistically are so-called pseudo-earthquakes. Here in order to prepare the input signals used in the following studies we discuss the pseudo-earthquake based on the gaussian white noise signal^{(18)~(21)}. The pseudo-earthquake is classified as follows;

- 1) Stationary pseudo-earthquake (stationary random process)
- 2) Non-stationary pseudo-earthquake (non-stationary random process)

The item 2) is classified in more detail in its

Fig. 2 Impulse response (X_j : independent normal random variables)

Fig. 3 Probability density functions and standard deviations of natural frequencies (X_j : independent normal random variables)

Fig. 4 $\sigma\{H_{NN}(t)\}$ for damping values (X_j : independent normal random variables)

Fig. 5 Stationary pseudo-earthquake and truncated pseudo-earthquake

non-stationarity, and especially the truncated wave from stationary pseudo-earthquake is involved (see Fig. 5). In the present day, it is interpreted that the absolute acceleration** of the response of a single-degree-of-freedom second order system applied a stationary gaussian white noise describes analytically the stationary pseudo-earthquake in the simplest form. And in aseismic engineering, it is of interest to know the maximum response and amplification factor when this is applied to the structures. Because these values make it possible to estimate the maximum stresses in members or components of structures induced in earthquake motion. The amplification factor is particularly effective in the system where the elastic limit is regarded as the upper limit of the design, such as mechanical equipments and pipings in a nuclear power plant. H. Tajimi⁽²⁰⁾ estimated the amplification factor in terms of the ratio of standard deviation of output wave to that of input wave when the above stationary pseudo-earthquake is applied to a system. On the other hand, H. Sato⁽²³⁾ estimated the amplification factor from the probability density functions of upper extremes of output wave and input wave, and there he used the band limited white noise as input signal to the ground model to produce a stationary pseudo-earthquake. Both of the above studies are analyses in an idealized state for the disturbance. An earthquake motion affects mainly the structures in some part of time duration, which is called the main ground motion (S-wave) and this part of time duration lasts nearly ten or more seconds for each earthquake and the wave form of this part may be considered to be a truncated wave from stationary random processes. Then the first step analysis of structure, considering the non-stationarity on the ground motion, is the one subjected to the truncated wave from stationary random processes. In this case maxima of responses and amplification factors fluctuate around those as calculated in an idealized state. Therefore we will make clear the mechanism of these fluctuations and analyze these values quantitatively.

3.1 Variance and energy of truncated wave

The (temporal) variance of stationary random wave $I(t)$ with zero mean is defined as

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T I^2(t) dt \dots \dots \dots (18)$$

On the other hand the variance of truncated wave $I_T(t)$ of which the wave duration is T from the same stationary random wave is

$$P_I \equiv \frac{1}{T} \int_0^T I^2(t) dt = \frac{1}{T} \int_0^\infty I_T^2(t) dt \dots \dots \dots (19)$$

** By some data analyses of Matsushiro earthquake records, a velocity type model showed better agreement than an acceleration type model to their spectral density [see ref. (22)].

This value is also considered a power of the wave. Consequently from Eq. (19) energy ϵ_I of the wave becomes

$$\epsilon_I = P_I T \dots \dots \dots (20)$$

The square root value of Eq. (19) is considered to be the standard deviation of the wave of finite time duration. If the wave is of finite time duration and follows a certain probability distribution, there may be some proportional relation between the maximum value and standard deviation in its amplitude. Figure 6 shows the relation between maximum values and standard deviations (square root values of energies) for El Centro earthquakes and they have the proportional relation. Then in this section the probability distribution of energies instead of that of maxima itself will be briefly discussed in order to estimate the statistical characteristics of maxima and in section 3.2 the method of estimating the statistical characteristics of maxima will be theoretically given by means of the results obtained in this section and the results of numerical computations will be given in section 3.3.

D. Slepian⁽²⁴⁾ obtained the exact probability density functions for the responses of a single-degree-of-freedom second order system subjected to the gaussian white noise. And he explained that S. O. Rice's⁽²⁵⁾ approximate formula showed considerable agreement with the exact formula. M. Ohta⁽²⁶⁾ obtained an exact formula similar to the result after S. O. Rice, but here we investigate the amplification factor on the basis of the approximate formula by S. O. Rice. Because this formula is clear in its expression and easy to deal with.

S. O. Rice showed that if we define the normalized power (or normalized energy) as

$$x \equiv P / \left(\frac{\sigma_P^2}{m_P} \right) = \epsilon / \left(\frac{\sigma_\epsilon^2}{m_\epsilon} \right) \dots \dots \dots (21)$$

then the probability distribution function $f_1(x)dx$ of x has the following expression;

Fig. 6 Relation between maximum value and square root value of wave energy for El Centro earthquake

$$f_1(x)dx = \begin{cases} \frac{x^n}{\Gamma(n+1)} e^{-x} dx; & x \geq 0 \\ 0 & ; x < 0 \end{cases} \dots\dots\dots (22)$$

This is called the generalized χ^2 distribution (also called Γ distribution) function. The m_ϵ , σ_ϵ^2 and m_p , σ_p^2 in Eq. (21) are defined

$$\begin{aligned} m_\epsilon &= E\{\epsilon\} \\ \sigma_\epsilon^2 &= \sigma^2\{\epsilon\} = E\{\epsilon^2\} - m_\epsilon^2 \end{aligned} \dots\dots\dots (23)$$

and

$$\begin{aligned} m_p &= m_\epsilon/T \\ \sigma_p^2 &= \sigma_\epsilon^2/T^2 \end{aligned} \dots\dots\dots (24)$$

where $\Gamma(\cdot)$ and n denote a Gamma function and the degrees of freedom in χ^2 distribution respectively and n has the following relation between the statistical characteristics of the wave energy.

$$n+1 = \frac{m_\epsilon^2}{\sigma_\epsilon^2} = \frac{m_p^2}{\sigma_p^2} \dots\dots\dots (25)$$

This parameter value n is the measure of the dispersion for the distribution. It is known⁽²⁵⁾ that $f_1(x)dx$ approaches a Rayleigh or normal distribution as n is sufficiently small or large. In a random vibration the estimates of the mean and standard deviation are quite important and therefore we should note that the statistical characteristics of standard deviation of the amplitude for truncated waves vary with time length T as mentioned above. From a practical view point, there are two interpretations for the fluctuations of the standard deviation of wave amplitude of finite duration. First it is considered that the uncertainty is due to the finite time duration for estimating the phenomenon of stationary random vibration and second, the time length itself of the wave concerned is finite duration, then the fluctuation of the standard deviation is considered to be an inherent statistical phenomenon in the waves of finite time duration. The ground motions, disturbances in aseismic engineering, are considered as the second case. Therefore if we are based on the above interpretation, it is necessary to introduce the concept of confidence width in the traditional treatment of maxima of responses and then arises the necessity of considering them as random variables.

If we assume that the statistical characteristics of maxima of wave amplitudes can be estimated in terms of the standard deviations of them, the statistical characteristics of maxima are obtained from the equation

$$f_1(\chi)d\chi = \begin{cases} \frac{2\chi^{2n+1}}{\Gamma(n+1)} e^{-\chi^2} d\chi; & \chi \geq 0 \\ 0 & ; \chi < 0 \end{cases} \dots\dots\dots (26)$$

which is obtained after some calculation from Eq. (22) by changing the variable as $\chi = \sqrt{x}$. On the other hand, as for amplification factors, some values of input waves and those of output waves are needed.

This problem will be discussed in section 3.2 in detail.

3.2 Theoretical estimations of statistical characteristics of amplification factors

The amplification factor λ in aseismic theory has been defined as

$$\lambda \equiv \frac{\text{maximum of output wave}}{\text{maximum of input wave}} \dots\dots\dots (27)$$

Same dimensions for output wave and input wave are usually dealt with, and if the absolute acceleration is taken notice of, we call it the amplification factor of absolute acceleration. H. Tajimi⁽²⁰⁾ gives an idea of how to estimate this value in connection with the ratio of the standard deviation of output wave amplitude to that of input one, that is, he estimates the amplification factor in terms of

$$\lambda \equiv \frac{\text{standard deviation of output wave}}{\text{standard deviation of input wave}} \dots\dots\dots (28)$$

under the assumption of earthquake records being gaussian distributions. In order to introduce the statistical properties due to the time length of duration T of wave, we define the amplification factor λ^* as

$$\begin{aligned} \lambda^* &= \frac{\left[\frac{\text{standard deviation of finite time}}{\text{duration of output wave}} \right]}{\left[\frac{\text{standard deviation of finite time}}{\text{duration of input wave}} \right]} \\ &= \left[\frac{\text{power of output wave}}{\text{power of input wave}} \right]^{1/2} \dots\dots\dots (29) \end{aligned}$$

and λ in Eq. (27) is estimated in terms of λ^* . It is found that Eq. (29) becomes Eq. (28) when the time length of duration T tends to infinity. By using Eqs. (19) and (20), Eq. (29) becomes

$$\lambda^* = \frac{\sqrt{\epsilon_0}}{\sqrt{\epsilon_i}} = \frac{\sqrt{P_0}}{\sqrt{P_i}} \dots\dots\dots (30)$$

and the probability distribution function $f_1(\lambda^*)d\lambda^*$ of λ^* can be obtained by Eq. (22) or (26).

3.2.1 For the case that energies of output waves are independent of those of input waves

There exists the relation $I_0^{(j)}(t) = T_L[I_i^{(j)}(t)]$ [(j) means a sample wave] between input waves and output waves in a time invariant linear system (T_L denotes a linear operator), so a similar relation exists between earthquake waves and responses of structures. Now when we study the amplification factor, the maxima of input waves and those of output waves must be considered. Then the probabilistic relations of those maximum values between before and after in linear transformation are needed, but it is very difficult to obtain them. So we introduce the energies of waves instead of the maxima of waves. That is, Eq. (30) becomes a basic equation to obtain the amplification factor in terms of wave energy and must be satisfied for each sample (j) , i.e. $\lambda^{*(j)} = \sqrt{P_0^{(j)}}/\sqrt{P_i^{(j)}}$. Therefore the two-dimensional joint probability density function $f_2(P_i, P_0)$ is needed in

order to obtain the probability density function of λ^* . So the requirements are not only one-dimensional probability density functions $f_1(P_i)$ and $f_1(P_0)$ for input waves and output waves, but also correlation property of P_i and P_0 . But if P_i and P_0 can be assumed to be independent*** of each other, the statistical characteristics of the amplification factors are easily obtained.

Suffices i and 0 designate the input quantity and output one. The joint probability distribution function of normalized powers of input waves and output waves can be obtained from Eq. (22) as

$$f_2(x_i, x_0) dx_i dx_0 = \begin{cases} \frac{x_i^{n_i} x_0^{n_0}}{\Gamma(n_i+1)\Gamma(n_0+1)} e^{-(x_i+x_0)} dx_i dx_0 & ; x_i, x_0 \geq 0 \\ 0 & ; \text{others} \end{cases} \quad (31)$$

Furthermore the joint probability distribution of normalized standard deviations of input waves and output waves can be obtained as

$$f_2(\chi_i, \chi_0) d\chi_i d\chi_0 = \begin{cases} \frac{4}{\Gamma(n_i+1)\Gamma(n_0+1)} \chi_i^{2n_i+1} \chi_0^{2n_0+1} e^{-(\chi_i^2+\chi_0^2)} \times d\chi_i d\chi_0 & ; \chi_i, \chi_0 \geq 0 \\ 0 & ; \text{others} \end{cases} \quad (32)$$

after changing the variables as $\chi_i = \sqrt{x_i}$ and $\chi_0 = \sqrt{x_0}$. Denoting the normalized amplification factor by η^* defined as

$$\eta^* \equiv \chi_0 / \chi_i \quad (33)$$

the probability distribution of η^* can be obtained by means of general procedures in statistics as

$$f_1(\eta^*) d\eta^* = \begin{cases} \frac{2}{B(n_i+1, n_0+1)} \frac{\eta^{*2n_0+1}}{(1+\eta^{*2})^{n_i+n_0+2}} \times d\eta^* & ; \eta^* \geq 0 \\ 0 & ; \eta^* < 0 \end{cases} \quad (34)$$

where $B(p, q)$ is a Beta function of arguments p and q . The λ^* in Eq. (30) is related to the η^* in Eq. (33) as follows;

$$\lambda^* = K\eta^* \quad (35)$$

Here K is a constant expressed by

$$K = \left(\frac{\sigma_{P_0}^2}{m_{P_0}} \right)^{1/2} / \left(\frac{\sigma_{P_i}^2}{m_i} \right)^{1/2} \quad (36)$$

If $m_{P_0} = m_{P_i}$, K becomes $\sigma_{P_0}/\sigma_{P_i}$, that is, it becomes the ratio of the standard deviation of power of output wave to that of input wave. On the other hand, if $\sigma_{P_0} = \sigma_{P_i}$, K becomes $\sqrt{m_{P_i}/m_{P_0}}$, that is, it becomes a square root value of the ratio of the mean power of input wave to that of output wave, and coincides with the reverse of the amplification factor obtained by H. Tajimi under the assumption that the input

waves are stationary random processes. The statistical characteristics of η^* are obtained from Eq. (34) and are related to those of actual amplification factor by Eq. (35) and K can be interpreted as the conversion coefficient from η^* into λ^* .

The k th moment of the normalized amplification factor α_k is calculated from the probability distribution of η^* . By its definition α_k becomes

$$\alpha_k = \frac{2}{B(n_i+1, n_0+1)} \int_0^\infty \frac{\eta^{*2n_0+1+k}}{(1+\eta^{*2})^{n_i+n_0+2}} d\eta^* \quad (37)$$

and after some calculation it reduces to

$$\alpha_k = \frac{B[n_i+1-(k/2), n_0+1+(k/2)]}{B(n_i+1, n_0+1)} \quad (38)$$

Especially the first three moments are

$$\left. \begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= B[n_i+(1/2), n_0+(3/2)]/B(n_i+1, n_0+1) \\ \alpha_2 &= (n_0+1)/n_i \end{aligned} \right\} \quad (39)$$

and consequently the mean value, standard deviation, and relative dispersion of the amplification factor become

$$\left. \begin{aligned} E\{\lambda^*\} &= K\alpha_1 \\ \sigma_{\lambda^*} &= K\sqrt{\alpha_2 - \alpha_1^2} \\ d_{\lambda^*} &= \sqrt{\alpha_2 - \alpha_1^2}/\alpha_1 \end{aligned} \right\} \quad (40)$$

3.2.2 For the case that energies of output waves are dependent on those of input waves

In general, earthquake waves and responses of the structures subjected to them are not statistically independent. Especially there exists a strong correlation property between earthquake wave and response in a rigid structure system. Therefore the theoretical analyses in section 3.2.1 are no more valid. In order to obtain the statistical characteristics of the amplification factor for this case, the equation corresponding to Eq. (31) involving the correlation property between input wave and output wave is discussed.

In reference (22), it is found that the equation can be approximately obtained as

$$\begin{aligned} f_2(x_i, x_0) dx_i dx_0 &\cong \frac{x_i^{n_i} x_0^{n_0}}{\Gamma(n_i+1)\Gamma(n_0+1)} e^{-(x_i+x_0)} \\ &\times \left[1 + \frac{\mu_{i0}}{\sqrt{(n_i+1)(n_0+1)}} \{x_i - (n_i+1)\} \right. \\ &\times \{x_0 - (n_0+1)\} \left. \right] dx_i dx_0; x_i, x_0 \geq 0 \\ &= 0; \text{others} \end{aligned} \quad (41)$$

In Eq. (41), the first term of right hand side is the same as the right hand side of Eq. (31). The second term is the correlation one which is dominated by the correlation coefficient μ_{i0} of x_i and x_0 (ε_i and ε_0). Here

$$\mu_{i0} = \frac{\bar{x}_i \bar{x}_0 - \bar{x}_i^2 \bar{x}_0^2}{\sqrt{(\bar{x}_i^2 - \bar{x}_i^2)(\bar{x}_0^2 - \bar{x}_0^2)}} = \frac{\sigma^2\{\varepsilon_{i0}\}}{\sigma\{\varepsilon_i\}\sigma\{\varepsilon_0\}} \quad (42)$$

and

*** See sections 3.2.2 and 3.3.3 on the statistical independence.

$$\left. \begin{aligned} \sigma^2\{\varepsilon_{i0}\} &= 2 \int_0^T \int_0^T \phi_{0i}^2(\tau) dt_1 dt_2 \\ \sigma^2\{\varepsilon_i\} &= 2 \int_0^T \int_0^T \phi_i^2(\tau) dt_1 dt_2 \\ \sigma^2\{\varepsilon_0\} &= 2 \int_0^T \int_0^T \phi_0^2(\tau) dt_1 dt_2 \end{aligned} \right\}, \quad \tau = t_1 - t_2$$

.....(43)

where $\phi_{0i}(\tau)$, $\phi_i(\tau)$ and $\phi_0(\tau)$ are the cross-covariance function between output wave and input wave, auto-covariance functions of input wave and output wave respectively. Let us put the second term of Eq. (41) as $f_2(x_i, x_0) dx_i dx_0$. Then it is changed into

$$\begin{aligned} f_2(\chi_i, \chi_0) d\chi_i d\chi_0 &= \frac{4\mu_{i0}}{\sqrt{(n_i+1)(n_0+1)}} \\ &\times \{\chi_i^2 - (n_i+1)\} \{\chi_0^2 - (n_0+1)\} \\ &\times \frac{\chi_i^{2n_i+1} \chi_0^{2n_0+1} e^{-(\chi_i^2 + \chi_0^2)}}{\Gamma(n_i+1)\Gamma(n_0+1)} d\chi_i d\chi_0; \chi_i, \chi_0 \geq 0 \\ &= 0; \text{others} \end{aligned} \quad (44)$$

after some calculation similar to the procedures in section 3.2.1. And then the probability distribution of

$$\eta_r^* = \chi_0 / \chi_i \quad (45)$$

can be obtained as

$$\begin{aligned} f_1(\eta_r^*) d\eta_r^* &= \frac{2\mu_{i0}}{\sqrt{(n_i+1)(n_0+1)}} \\ &\times \frac{1}{\eta_r^{*2n_0+1}} \frac{1}{B(n_i+1, n_0+1)} \frac{1}{(1+\eta_r^{*2})^{n_i+n_0+2}} \\ &\times \left[(n_i+n_0+3)(n_i+n_0+2) \frac{\eta_r^{*2}}{(1+\eta_r^{*2})^2} \right. \\ &\quad - (n_0+1)(n_i+n_0+2) \frac{1}{1+\eta_r^{*2}} \\ &\quad - (n_i+1)(n_i+n_0+2) \frac{\eta_r^{*2}}{1+\eta_r^{*2}} \\ &\quad \left. + (n_i+1)(n_0+1) \right] \end{aligned} \quad (46)$$

The first three moments of η_r^* are

$$\left. \begin{aligned} \beta_0 &= 0 \\ \beta_1 &= -\frac{\mu_{i0}}{4\sqrt{(n_i+1)(n_0+1)}} \alpha_1 \\ \beta_2 &= -\frac{\mu_{i0}}{\sqrt{(n_i+1)(n_0+1)}} \frac{n_i+n_0+1}{n_i+n_0+2} \alpha_2 \end{aligned} \right\} \quad (47)$$

Then from Eqs. (44), (39) and (47), the k th moment κ_k of the normalized amplification factor η^* taking into consideration the correlation property between input wave and output wave becomes

$$\kappa_k = \alpha_k + \beta_k \quad (48)$$

And consequently the mean value, standard deviation, and relative dispersion of the amplification factor can be obtained as

$$\left. \begin{aligned} E\{\lambda^*\} &= K\kappa_1 \\ \sigma_{\lambda^*} &= K\sqrt{\kappa_2 - \kappa_1^2} \\ d_{\lambda^*} &= \sqrt{(\kappa_2 - \kappa_1^2)/\kappa_1} \end{aligned} \right\} \quad (49)$$

Here let us introduce ν_{i0} the ratio of the correlation

coefficient μ_{i0} to average value of degrees of input energy n_i and those of output one n_0 , defined by

$$\nu_{i0} \equiv \frac{\mu_{i0}}{\sqrt{(n_i+1)(n_0+1)}} \quad (50)$$

This value, shown also in Eq. (47), denotes the correlation coefficient per unit degree of fluctuation. Roughly speaking, n is dominated by the degrees of fluctuation of wave energies and μ_{i0} by the degrees of similarity between the forms of input waves and those of output waves. Then if there exist the high degrees of similarity and fluctuation (n ; small) of input waves and output waves, ν_{i0} becomes large. In this case, if the statistical characteristics of the amplification factor are calculated under the assumption that energies of output waves are independent of those of input waves ($\mu_{i0}=0$), they are estimated larger than the true values. Reversely, if there exist low degrees of similarity and fluctuation, ν_{i0} becomes small and consequently the calculation under the independence assumption has little error.

3.3 Results of calculations

In sections 3.1 and 3.2 we theoretically studied the fluctuations of variances and amplification factors of the truncated waves from stationary random processes. Now in this section numerical and analog computations will be given and compared with each other. From this study the theory obtained in previous sections will be shown to be satisfactory and the fluctuations of the amplification factor will be given quantitatively.

3.3.1 Results of numerical calculations

In sections 3.1 and 3.2 we theoretically studied the fluctuations of variances and amplification factors of the truncated waves from stationary random processes. Now in this section numerical and analog computations will be given and compared with each other. From this study the theory obtained in previous sections will be shown to be satisfactory and the fluctuations of the amplification factor will be given quantitatively.

3.3.1 Results of numerical calculations

We consider a simple structure described by a single-degree-of-freedom second order system as shown in Fig. 7. For the stationary pseudo-earthquake used

Fig. 7 Single-degree-of-freedom mass-spring system

here the power spectral density $A_0(\omega)$ is assumed such that

$$A_0(\omega) = |H_g(i\omega)|^2 \cdot k_0 \quad \dots\dots\dots(51)$$

and the earthquake wave is assumed to be a gaussian random process. The transfer function of absolute acceleration of the ground $H_g(i\omega)$, which we use here, is

$$H_g(i\omega) = \frac{2\zeta_g \omega_g(i\omega) + \omega_g^2}{(i\omega)^2 + 2\zeta_g \omega_g(i\omega) + \omega_g^2} \quad \dots\dots\dots(52)$$

Here ω_g , ζ_g and k_0 designate the predominant frequency, damping ratio, and the intensity of the ground model. Power spectral density of absolute acceleration of the structure becomes

$$S_{\ddot{z}_b}(\omega) = \frac{4\zeta_b^2 \omega_b^2 \omega^2 + \omega_b^4}{\omega^4 - 2(1 - 2\zeta_b^2) \omega_b^2 \omega^2 + \omega_b^4} A_0(\omega) \quad \dots\dots\dots(53)$$

Auto-covariance functions corresponding to Eqs. (51) and (53), and cross-covariance function between absolute acceleration of structure and earthquake are denoted by $\phi_g(\tau)$ and $\phi_{\ddot{z}_b}(\tau)$, and $\phi_{0i}(\tau)$ respectively. Here $\phi_g(\tau)$ can be easily obtained as

$$\phi_g(\tau) = e^{-\zeta_g \omega_g' |\tau|} (\gamma_g \cos \omega_g' |\tau| + \delta_g \sin \omega_g' |\tau|) \quad \dots\dots\dots(54)$$

where

$$\omega_g' = \sqrt{1 - \zeta_g^2} \omega_g, \quad \gamma_g = [(1 + 4\zeta_g^2)/8\zeta_g] \omega_g$$

and

(a) Normalized auto-covariance function of stationary pseudo-earthquake

(b) Normalized auto-covariance functions of absolute acceleration for single-freedom system subjected to stationary inputs

Fig. 8

Fig. 9 Cross-covariance functions between absolute accelerations for single-freedom system and stationary inputs

$$\delta_g = [(1 - 4\zeta_g^2)/8\sqrt{1 - \zeta_g^2}] \omega_g.$$

On the other hand, it is difficult to obtain $\phi_{\ddot{z}_b}(\tau)$ and $\phi_{0i}(\tau)$ analytically, so we calculate them numerically by using a digital computer. In reference (22), some calculi for obtaining covariance functions of an N -degrees-of-freedom linear system are given.

It has been known that Eqs. (23) and (43) become

$$m_e = T\phi(0) \\ \sigma_e^2 = 4 \int_0^T (T - \tau) \phi^2(\tau) d\tau \quad \dots\dots\dots(55)$$

for stationary gaussian random processes. Then we can calculate m_{ei} , σ_{ei}^2 , m_{e0} , and σ_{e0}^2 by substituting the covariance functions obtained above into Eq. (55), and consequently n_i and n_0 can be obtained by substituting these values into Eq. (25). For numerical calculations, the parameters of pseudo-earthquake and those of a structure used here are as follows; $T_g = 1.0$ sec, $\zeta_g = 0.3$, $T_b = 0.2 \sim 3.0$ sec, and $\zeta_b = 0.07$. Figures 8 and 9 show the normalized auto-covariance functions of pseudo-earthquake and responses, and cross-covariance functions between pseudo-earthquake and responses. Figure 10 shows

Fig. 10 Relation between degrees-of-freedom of energy distribution and time length for stationary pseudo-earthquake

Fig. 11 Relation between degrees-of-freedom of energy distribution and time length for absolute acceleration of single-freedom system subjected to stationary inputs

the degrees of freedom of distributions for input waves and output waves. These values n_i and n_0 are almost proportional to T/T_g and increase with the damping ratio of the ground model and the structure. Figure 12 shows the probability density functions $f_1(\lambda^*)$ of the amplification factor λ^* obtained from Eqs. (34) and (35). This $f_1(\lambda^*)$ is strongly influenced by the time length of wave duration similar to the degrees of freedom of distribution. Figure 13 shows the μ_{i0} obtained from Eqs. (43) and (42). This value is almost equal to 1.0 when $T_b/T_g \approx 0.2$ and there are a maximum and a minimum at the points $T_b/T_g \approx 1.0$ and 0.7 in this curve. Qualitative interpretation of the change for this curve can be given as follows: Firstly the correlation property depends on the power spectral density of input wave, and this spectral density shows a peak at $T_g = 1.0$ sec and decreases on both sides of it, secondly the correlation property depends on the similarity between input wave and output wave, and this similarity is dependent on the natural frequency of a structure; if the structure is a rigid one ($T_b/T_g < 1.0$) the degree of this similarity is high

Fig. 12 Probability density functions of amplification factor of absolute acceleration

Fig. 13 Correlation coefficient between ϵ_i and ϵ_0 of absolute acceleration for single-freedom system subjected to stationary inputs

and if the structure becomes a soft one ($T_b/T_g > 1.0$), this similarity disappears. Then the shape of the curve may be as shown in Fig. 13.

3.3.2 Results of analog simulations

The power spectral density function of stationary

Fig. 14 Response waves of single-freedom system obtained by analog computation

Fig. 15 Histogram of amplification factors of absolute acceleration for single-freedom system subjected to stationary inputs (by analog computations)

pseudo-earthquake used in analog simulations $A(\omega)^{****}$ is as follows;

$$A(\omega) = |H_f(i\omega)|^2 A_0(\omega) \dots\dots\dots (56)$$

Here the transfer function $H_f(i\omega)$ is the band limited filter. By using an analog computer, time historical waves of pseudo-earthquake were produced as follows; The signal from a (low frequency) white noise generator was put into the band limited filter and then

Fig. 16 Mean value of amplification factor of absolute acceleration for single-freedom system (by means of energy method)

Fig. 17 Mean value of amplification factor of absolute acceleration for single-freedom system (by means of maximum value method)

Fig. 18 Relative dispersion of amplification factor of absolute acceleration for single-freedom system (by means of energy method)

**** The power spectral density function of stationary pseudo-earthquake used here differs from the one used in numerical calculation in section 3.3.1, because the pseudo-earthquake used here was also used for an analog computation for another purpose. But the differences of the response due to H_f may not appear in these sets of parameters.

the output signal of it was also put into a single-degree-of-freedom second order system which simulated the ground model. The output signal of this single-degree-of-freedom system was considered as a stationary pseudo-earthquake. The predominant period, damping ratio, upper and lower cut-off frequencies, and time length of duration of pseudo-earthquake used here are $T_g = 10$ sec, $\zeta_g = 0.3$, $\psi_1 = 3.0$ sec/rad, $\psi_2 = 0.08$ sec/rad, and $T = 20$ sec respectively. By the same operational procedures, we obtained fifty samples of pseudo-earthquake and recorded them on magnetic tapes in a tape recorder. The parameters of the structures used here are $\zeta_b = 0.07$ and nine points for natural frequencies within $T_b/T_g = 0.2 \sim 3.0$. Figure 14 shows an example of the responses. The lower part of each row shows the energy of response

Fig. 19 Relative dispersion of amplification factor of absolute acceleration for single-freedom system (by means of maximum value method)

Fig. 20 Mean value and standard deviation of amplification factor of absolute acceleration for single-freedom system

Fig. 21 Two-degrees-of-freedom mass-spring system

corresponding to the response wave shown in upper part. Figure 15 shows a histogram of the amplification factors. In Fig. 13, the μ_{i0} obtained from an analog simulation is also shown together with the result from theoretical calculation. The two curves are qualitatively similar but there exists some quantitative difference. This difference may come from the difference of initial conditions between an analog simulation and theoretical calculation.

3.3.3 Comparisons between the results by analog simulations and theoretical calculations

The comparisons between the results from analog simulations and theoretical analyses are done by treating the fifty samples of the amplification factor obtained in section 3.3.2 statistically and by substituting the expressions given in section 3.3.1 into Eq. (49). Figures 16 and 17 show the mean value of λ_a^* and that of λ_a corresponding to the definitions (30) and (28). In Fig. 16, we see that the result of theoretical analysis is slightly larger than that of an analog simulation. For $\bar{\lambda}_a^*$ and $\bar{\lambda}_a$, these values almost coincide with each other, but there exists a

slight difference. When $T_b/T_g \geq 1.0$, $\bar{\lambda}_a^* < \bar{\lambda}_a$, on the other hand when $T_b/T_g < 1.0$, $\bar{\lambda}_a^* < \bar{\lambda}_a$. Figures 18 and 19 show the relative dispersion of the amplification factor of absolute acceleration for a single-degree-of-freedom system and these correspond to λ_a^* and λ_a respectively. As we have seen in section 3.3.1, the standard deviation is influenced by μ_{i0} more than the mean value, so if we assume that $\mu_{i0} = 0$, the estimates of d_{λ}^* in a rigid system will contain a considerably large error. In particular there exists a large difference for $T_b/T_g \leq 0.6$ in Fig. 18. On the other hand in a soft system such that $T_b/T_g \geq 1.5$, both estimates show good agreement excepting small variations. Here, when we consider μ_{i0} in computation (Fig. 18), almost correct estimations are made for all T_b/T_g . But in analog simulation, the result rapidly increases and decreases for $T_b/T_g \approx 0.8$ and 1.0. The difference between the results of theoretical analysis and analog simulation for $T_b/T_g \approx 0.8$ is due to the difference in estimation of μ_{i0} as we have seen in Fig. 13. Figure 20 shows the comparison of response spectra between the results obtained here and

Table 5 Statistical characteristics of amplification factor of absolute acceleration (two-degrees-freedom system, stationary inputs)

T_m/T_g	$\bar{\lambda}^*$	σ_{λ}^*	d_{λ}^*	μ_{i0}	n_0	κ_1	κ_2	K	ν_{i0}
0.2	3.300	0.4678	0.1417	0.6836	6.018	0.4095	0.2167	7.160	0.04568
	3.338	0.7054	0.2113	0.0	6.018	0.4662	0.2271	7.160	0.0
0.4	3.769	0.5338	0.1416	0.6785	6.086	0.4632	0.2189	8.136	0.04512
	3.812	0.8204	0.2104	0.0	6.086	0.4685	0.2293	8.136	0.0
0.6	5.233	0.6640	0.1268	0.6762	7.607	0.5127	0.2671	10.20	0.04080
	5.287	1.028	0.1944	0.0	7.607	0.5180	0.2785	10.20	0.0
0.8	11.99	1.820	0.1517	0.5376	6.729	0.4859	0.2415	24.68	0.03423
	12.09	2.456	0.2030	0.0	6.729	0.4901	0.2501	24.68	0.0
0.9	23.36	5.580	0.2388	0.3954	2.776	0.3338	0.1178	69.97	0.03602
	23.57	6.537	0.2773	0.0	2.776	0.3368	0.1222	69.97	0.0
1.0	38.89	12.97	0.3336	0.2880	1.148	0.2457	0.06708	158.2	0.03479
	39.23	14.22	0.3625	0.0	1.148	0.2478	0.06950	158.2	0.0
1.1	27.89	8.259	0.2960	0.3581	1.578	0.2714	0.08015	102.7	0.03948
	28.17	9.350	0.3318	0.0	1.578	0.2741	0.08344	102.7	0.0
1.2	17.83	4.862	0.2726	0.3785	1.972	0.2933	0.09243	60.80	0.03887
	18.00	5.585	0.3101	0.0	1.972	0.2962	0.09617	60.80	0.0
1.5	7.764	2.239	0.2883	0.2694	1.886	0.2895	0.09078	26.81	0.02807
	7.819	2.458	0.3144	0.0	1.886	0.2915	0.09341	26.81	0.0
2.0	4.019	1.346	0.3349	0.1536	1.312	0.2570	0.07349	15.63	0.01788
	4.037	1.412	0.3497	0.0	1.312	0.2582	0.07483	15.63	0.0
3.0	2.308	0.9124	0.3951	0.0778	0.7345	0.2191	0.05554	10.53	0.01045
	2.315	0.9326	0.4028	0.0	0.7345	0.2197	0.05612	10.53	0.0

$T/T_g=20$, $\zeta_m=0.002$, $\zeta_g=0.3$, $\zeta_b=0.07$, $T_b/T_g=1.0$, ($n_i=30.899$)

the results given by H. Tajimi and H. Sato. Solid-line and thin-dotted-line show the mean value and the standard deviation obtained on the theory discussed here for $T/T_0=20.0$ and $\mu_{i0}=0$. On the other hand, the dotted-line shows the results by H. Tajimi or H. Sato [$p(y) \rightarrow 0$]. An example of numerical computations applying the above theory to a two-degrees-of-freedom system (Fig. 21) is shown in Table 5. The system used in numerical computations is interpreted as a building-machine structure system and the upper mass as a machine structure and the lower mass as a building structure. We assumed that the direction of force transmission was one directional from building to machine. Relative dispersion of amplification factor in the machine structure is seen to reach nearly 0.3~0.4.

4. Conclusions and acknowledgements

In response calculi in random vibration, the uncertainty of response due to the randomness of members consisting of the structure and fluctuation of amplification factor due to the non-stationarity of disturbance acting on the structure are studied theoretically and experimentally by means of statistical ways. From these studies the importance of the qualitative analysis for the fluctuations of responses and amplification factors, which have not received attention was made clear. Especially, for lightly damped systems, the reliability of calculated values decreases rapidly and some risks accompany in the use of the calculated values for an idealized system instead of those for a real one.

Practical systems there are not so few where the above two factors are involved simultaneously and in such a case we should pay enough attention to the confidence width of response calculation. This problem still remains unsolved and furthermore in aseismic engineering a more realistic analysis considering the non-stationarity in amplitudes and frequency distributions of a ground motion is now being expected.

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