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Buoyancy Effects in Forced-Convection Flow and Heat Transfer*

(2nd Report, Analysis by the Perturbation Method)

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An analysis is made of the effects of buoyancy forces in a laminar forced-convection flow on a vertical flat plate by using the perturbation method. The solutions are obtained up to the 3rd order approximation; accordingly, the range of applicability of the solution by the analysis is expanded. It is proved that solutions of the 1st report are applicable to a wider region of ξ for the fluids of $P_r \geq 1$ in the case of friction factor and of all Prandtl numbers as to heat transfer, by comparing with the results of this paper. Furthermore, the friction factor and heat transfer coefficient are calculated, and it is shown that a linear relation exists between the both values. Also it is proved that the solution of energy equation $\Theta'(\xi, 0)$ can be presented by a similar formula, $\Theta'(\xi, 0) = -(0.664 + 0.48\xi)P_r^{1/3}$, ($0 \leq \xi \leq 0.3$), which is formulated in a purely forced-convection heat transfer problem.

1. Introduction

Although the combined convective flow in a laminar boundary layer has been studied by Sparrow and Gregg⁽²⁾, Acrivos⁽³⁾, Mori⁽⁴⁾, Mabuchi⁽⁷⁾ and Fujii⁽⁶⁾ et al. under various conditions, the range of the applicability of these results is confined to the narrow region of ξ , and we have not yet obtained any theory which can satisfactorily explain the phenomena of combined forced and free convective flow. The present author and Nagai⁽¹⁾ have obtained the approximate similar solutions for the buoyancy effects in the boundary layer on a vertical flat plate with a view to acquiring the solution that can apply to as wide a region of ξ as possible. But the examinations for the range of applicability of the solutions have not been performed sufficiently for data on various Prandtl numbers. That is, since, in the 1st report, a few assumptions have been used in order to simplify the analysis, it is not sufficient to examine the condition of application.

Then the present study was begun in order to examine the results of the 1st report. Hitherto most of the analysis have been made only up to the 1st order approximation; therefore, the range of applicability of these solutions is limited to a narrow region. In this paper it is obtained up to the 3rd order approximation, and the effectiveness of perturbation analysis to this problem and of the assumption in the 1st report are examined.

2. Nomenclature

- u : velocity component of x -direction m/sec
 - v : velocity component of y -direction m/sec
 - g : gravitational acceleration m/sec²
 - c_p : specific heat at constant pressure kcal/kg°C
 - k : thermal conductivity kcal/m hr°C
 - T : absolute temperature °K
 - J : mechanical equivalent of heat kgm/kcal
 - P_r : Prandtl number ν/α
 - R_e : Reynolds number $u_\infty x/\nu$
 - N_u : Nusselt number hx/k
 - h : coefficient of heat transfer kcal/m hr°C
 - C_f : friction factor $\tau_w/(1/2)\rho u_\infty^2$
 - G_r : Grashof number $[g\beta(T_w - T_\infty)x^3/\nu^2]$
 - η : similarity variable $(y/2)\sqrt{u_\infty/\nu x}$
 - τ : shear stress at wall kg/m²
 - ϕ : stream function $[\sqrt{u_\infty \nu x} F(\xi, \eta)]$
 - ξ : dimensionless parameter G_r/R_e^2
 - ρ : density kg/m³
 - μ : viscosity kgsec/m²
 - ν : kinetic viscosity m²/sec
 - β : coefficient of volumetric expansion 1/°K
 - Θ : dimensionless temperature $(T - T_\infty)/(T_w - T_\infty)$
- Subscripts
- ∞ : main stream
 - w : wall condition.

3. Analysis

Consider a laminar flow over a vertical flat plate which is heated or cooled. The physical model and the co-ordinate system are illustrated in Fig. 1. Now under the assumption of the boundary layer

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approximation, the incompressibility other than the buoyancy term and of the negligible viscous dissipation, the basic governing equations for conservation of mass, momentum and thermal energy in the boundary layer flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (3)$$

The boundary conditions are

$$\left. \begin{aligned} u=v=0, \quad T=T_w \quad \text{at } y=0 \\ u=u_\infty, \quad T=T_\infty \quad \text{at } y=\infty \end{aligned} \right\} \quad (4)$$

Let us introduce the similarity variable η , dimensional parameter ξ , stream function ψ and so on.

$$\eta = \frac{y}{2} \sqrt{\frac{u_\infty}{\nu x}} \quad (5)$$

$$\xi = \frac{Gr}{R_s^2} \quad (6)$$

$$\psi = \sqrt{u_\infty \nu x} F(\xi, \eta) \quad (7)$$

$$\frac{T - T_\infty}{T_w - T_\infty} = \Theta(\xi, \eta) \quad (8)$$

From the conservation of mass, the following relations are obtained

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (9)$$

and using the equations from (5) to (9), Eq. (2) and (3) are expressed in a dimensionless form as follows

$$F''' + FF'' - 2\xi \left(F' \frac{\partial F'}{\partial \xi} - F'' \frac{\partial F}{\partial \xi} \right) + 8\xi \Theta = 0 \quad (10)$$

$$\Theta'' + Pr F \Theta' - 2Pr \xi \left(F' \frac{\partial \Theta}{\partial \xi} - \Theta' \frac{\partial F}{\partial \xi} \right) = 0 \quad (11)$$

The boundary conditions are given by

$$\left. \begin{aligned} F=F'=0, \quad \Theta=1 \quad \text{at } \eta=0 \\ F'=2, \quad \Theta=0 \quad \text{at } \eta=\infty \end{aligned} \right\} \quad (12)$$

where the primes denote differentiation with respect to η .

Since the dimensionless stream function and temperature $F(\xi, \eta)$ and $\Theta(\xi, \eta)$ are expanded into

Fig. 1 Physical model

$$\left. \begin{aligned} F(\xi, \eta) &= f_0(\eta) + \xi f_1(\eta) \\ &+ \xi^2 f_2(\eta) + \xi^3 f_3(\eta) + \dots \\ \Theta(\xi, \eta) &= \theta_0(\eta) + \xi \theta_1(\eta) \\ &+ \xi^2 \theta_2(\eta) + \xi^3 \theta_3(\eta) + \dots \end{aligned} \right\} \quad (13)$$

Substituting the Eq. (13) into Eqs. (10) and (11), and arranging with a power of nondimensional parameter ξ , we can obtain the ordinary differential equations as follows

$$f_0''' + f_0 f_0'' = 0 \quad (14.a)$$

$$\theta_0'' + Pr f_0 \theta_0' = 0 \quad (14.b)$$

$$f_1''' + f_0 f_1'' - 2f_0' f_1' + 3f_0'' f_1 + 8\theta_0 = 0 \quad (15.a)$$

$$\theta_1'' + Pr(f_0 \theta_1' - 2f_0' \theta_1 + 3\theta_0' f_1) = 0 \quad (15.b)$$

$$f_2''' + f_0 f_2'' - 4f_0' f_2' + 5f_0'' f_2 + 3f_1 f_1'' - 2f_1'^2 + 8\theta_1 = 0 \quad (16.a)$$

$$\theta_2'' + Pr(f_0 \theta_2' - 4f_0' \theta_2 + 3f_1 \theta_1' - 2f_1' \theta_1 + 5f_2 \theta_0') = 0 \quad (16.b)$$

$$f_3''' + f_0 f_3'' - 6f_0' f_3' + 7f_0'' f_3 + 3f_1 f_2'' - 6f_1' f_2' + 5f_1'' f_2 + 8\theta_2 = 0 \quad (17.a)$$

$$\theta_3'' + Pr(f_0 \theta_3' - 6f_0' \theta_3 + 3f_1 \theta_2' - 4f_1' \theta_2 + 5f_2 \theta_1' - 2f_2' \theta_1 + 7f_3 \theta_0') = 0 \quad (17.b)$$

The boundary conditions are given by

$$\left. \begin{aligned} f_0=f_1=f_2=f_3=f_0'=f_1'=f_2'=f_3'=0 \\ \theta_0=1, \quad \theta_1=\theta_2=\theta_3=0 \\ f_0'=2, \quad f_1'=f_2'=f_3'=0 \\ \theta_0=\theta_1=\theta_2=\theta_3=0 \end{aligned} \right\} \quad \text{at } \eta=0 \quad \dots (18)$$

Then putting the heat flux from the plate to fluid per unit time by $q(x)$, the heat transfer coefficient is

$$h = \frac{q}{T_w - T_\infty} = -\frac{k}{2} \sqrt{\frac{u_\infty}{\nu x}} \{ \theta_0'(0) + \xi \theta_1'(0) + \xi^2 \theta_2'(0) + \xi^3 \theta_3'(0) + \dots \} \quad (19)$$

and local Nusselt number is written as

$$Nu = \frac{hx}{k} = -\frac{1}{2} \sqrt{R_s} \{ \theta_0'(0) + \xi \theta_1'(0) + \xi^2 \theta_2'(0) + \xi^3 \theta_3'(0) + \dots \} \quad (20)$$

Furthermore the ratio of the local Nusselt number, N_u in presence of buoyancy force, to N_{u0} in the absence of that, is given by

$$\frac{N_u}{N_{u0}} = 1 + \xi \frac{\theta_1'(0)}{\theta_0'(0)} + \xi^2 \frac{\theta_2'(0)}{\theta_0'(0)} + \xi^3 \frac{\theta_3'(0)}{\theta_0'(0)} + \dots \quad (21)$$

The local shearing stress at the plate is given by

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_w = \frac{\mu}{4} u_\infty \sqrt{\frac{u_\infty}{\nu x}} \{ f_0''(0) + \xi f_1''(0) + \xi^2 f_2''(0) + \xi^3 f_3''(0) + \dots \} \quad (22)$$

and the friction factor is

$$C_f \sqrt{R_s} = \frac{1}{2} \{ f_0''(0) + \xi f_1''(0) + \xi^2 f_2''(0) + \xi^3 f_3''(0) + \dots \} \quad (23)$$

The ratio of friction factor like the case of Nusselt number is given by

$$\frac{C_f}{C_{f0}} = 1 + \xi \frac{f_1''(0)}{f_0''(0)} + \xi^2 \frac{f_2''(0)}{f_0''(0)} + \xi^3 \frac{f_3''(0)}{f_0''(0)} + \dots \quad (24)$$

4. Numerical results and discussions

The equations from (14) to (17) are evaluated for Prandtl numbers of 0.1, 0.72, 1, 5, 10 and 15. These equations are a boundary value problem with two unknown initial values $f_n''(0)$ and $\theta_n'(0)$ which are found out by the trial and error method⁽⁸⁾. These ordinary differential equations are solved numerically by Runge-Kutta-Gill integration technique where the increment in the independent variable, $\Delta\eta$, is 0.005. The criterion of convergence is taken below 10^{-6} for the 0th order to 2nd order approximations, and below 10^{-4} for the 3rd order approximation, respectively. The criterion of convergence is used in order to judge whether the value calculated is convergence or not to the boundary condition of the present problem.

4.1 Remarks of the numerical solutions

The numerical results are shown in Table 1. The values of $F''(\xi, 0)$ and $\theta'(\xi, 0)$ are given in Figs. 2 and 3, where the solid and chain lines show values up to the 3rd and 1st order approximation, respectively. Then the dotted lines show Szewczyk's solutions up to the 2nd order solution, and the sign of circle denotes results of the 1st report.

Concerning $F''(\xi, 0)$ in Fig. 2, the solution up to the 3rd order approximation is smaller than that

up to the 1st order one, and the difference decreases as the Prandtl number increases. The fact shows that the convergence of the power series (Eq. 13) depends on the value of Prandtl number and the convergence of the series is improved with an increasing Prandtl number. Then it is considered that an almost exact solution is obtained in the region of ξ from 0 to 0.3 for $P_r \geq 1$, and from 0 to 0.2 for $P_r = 0.1$ and 0.72, where the range of applicability of the solution by the perturbation method is widened.

Certainly, the range of applicability by calculating up to the higher order approximation can be expanded, but it turns out that there are some questions in applying the perturbation method to this problem. Namely first, the convergence of numerical solution becomes delicate as the order of approximation progresses to a higher order; thus, it is difficult to find out the unknown initial value. For example, in order to obtain the 3rd order approximation which converges under the order of 10^{-4} to boundary condition, the solution from the 0th to 2nd order approximation is necessary to converge less than 10^{-6} . Therefore for the purpose of evaluation of the higher order approximation, it seems that the lower order approximation must be solved under a more severe criterion of convergence. Another of the questions is a convergence of the perturbation method. That is, the values of $f_3''(0)$ and $\theta_3'(0)$ become very large.

Fig. 2 Solution of $F''(\xi, 0)$

Fig. 3 Solution of $\theta'(\xi, 0)$

Table 1 Numerical result

P_r	$f_0''(0)$	$f_1''(0)$	$f_2''(0)$	$f_3''(0)$	$\theta_0'(0)$	$\theta_1'(0)$	$\theta_2'(0)$	$\theta_3'(0)$
0.1	1.328 229	6.353 709	-6.556 753	21.243 993	-0.280 058	-0.239 939	0.544 635	-2.109 361
0.72	1.328 229	4.862 978	-4.000 845	10.035 917	-0.591 269	-0.496 701	0.893 776	-2.662 504
1	1.328 229	4.586 649	-3.591 707	8.527 575	-0.664 114	-0.542 154	0.925 407	-2.608 062
5	1.328 229	3.246 762	-1.880 932	3.227 629	-1.153 270	-0.740 449	0.920 373	-1.871 025
10	1.328 229	2.720 376	-1.341 502	1.946 284	-1.456 233	-0.805 575	0.847 868	-1.455 868
15	1.328 229	2.436 768	-1.084 749	1.416 476	-1.668 238	-0.837 164	0.793 403	-1.225 452

in the case of $P_r=0.72$ in comparison with the 0th 1st and 2nd order approximation. Such tendency suggests that the numerical solution may diverge in the higher order approximation, thereby, there seems to be a limitation to application of the perturbation method to this problem.

Also Sparrow and Gregg have considered the range of applicability of solution expressed by $\xi|f_1''(0)/f_0''(0)| \leq 0.05$. But it turns out that the solution up to the 1st order approximation can apply for a wider region than the condition reported by Sparrow Gregg.

There is a considerable difference between the approximate solution of the 1st report and the perturbation method in the case of $P_r=0.1$; thereby it seems that the assumption used in the 1st report is not effective with respect to the small Prandtl number. Meanwhile, since the solution of this paper is in good agreement with the one of previous paper for a large Prandtl number, it is proved that the solution of the 1st report is applicable for $P_r \geq 1$.

Fig. 4 Relation between $\theta'(\xi, 0)$ and Prandtl number

Fig. 5 Solution of the 0th order approximation f_0', θ_0

Fig. 6 Solution of the 1st order approximation f_1'

With regard to $\theta'(\xi, 0)$, the solution obtained by the perturbation method, which has a linear relation as is shown in Fig. 4, almost converges in range of $0 \leq \xi \leq 0.3$ for the case of $P_r=0.72$. The linear relation is expressed as follows

$$\theta'(\xi, 0) = -(0.664 + 0.48\xi)P_r^{1/3} \quad (0 \leq \xi \leq 0.3) \quad (25)$$

The approximate solution of the 1st report is in good agreement with the result of this paper. Accordingly, the method of the 1st report seems to be very effective for calculating the heat transfer coefficient.

There are described the quantities [$f_0'(\eta)$, $f_1'(\eta)$, $\theta_0(\eta)$, $\theta_1(\eta)$ etc.] which relate to velocity and temperature distribution respectively in Figs. 5 to 11; the maximum value of curved line becomes large with a progressing order of perturbation method, then

Fig. 7 Solution of the 1st order approximation θ_1

Fig. 8 Solution of the 2nd order approximation f_2'

Fig. 9 Solution of the 2nd order approximation θ_2

the fact suggests as stated before, that the solution obtained by the perturbation method would diverge in the higher order.

4.2 Friction factor and heat transfer coefficient

Friction factor and Nusselt number are presented in Figs. 12 and 13, particularly the change of the friction factor is very large comparing with the tendency in the Nusselt number; thereby it turns out how large the effects yielded by the heat flux are for the flow. Also, since in the case of the combined convective flow, the flow pattern is decided by the amount of the heat energy supplied, we take interest in the relation between heat fluxes and shear stress at wall. Then observing the relation between $F''(\xi, 0)$ and $\theta'(\xi, 0)$ as Fig. 14, both the values change linearly

Fig. 10 Solution of the 3rd order approximation f_3'

Fig. 11 Solution of the 3rd order approximation θ_3

in the region of $0 \leq \xi \leq 0.2$ and it is expressed as Eq. (26)

$$\theta'(\xi, 0) = -0.109 P_r^{0.41} F''(\xi, 0) \quad (P_r \geq 0.72) \quad (26)$$

4.3 On Szewczyk's solution

Szewczyk⁽⁵⁾ has previously presented up to the

Fig. 12 Result of friction factor

Fig. 13 Results of Nusselt number

Fig. 14 Relation between $F''(\xi, 0)$ and $\theta'(\xi, 0)$

Table 2 Numerical result reported by Szewczyk

Pr	$f_0''(0)$	$f_1''(0)$	$f_2''(0)$	$\theta_0'(0)$	$\theta_1'(0)$	$\theta_2'(0)$
0.72	1.328 230	4.829 637	1.625 673	-0.591 272	-0.496 701	0.204 304
1	1.328 230	4.553 310	1.858 074	-0.664 116	-0.542 154	0.150 154
5	1.328 230	3.232 969	2.300 771	-1.153 380	-0.740 358	-0.230 823
10	1.328 230	2.706 856	2.183 597	-1.456 286	-0.805 519	-0.407 354

Fig. 15 Relation between the slope of line in Fig. 14 and Prandtl number

2nd order approximations, one of which differs from author's solution. The Szewczyk's solutions are represented in Table 2 and compared with the author's one in Figs. 2 and 3. As is shown in Fig. 2, Szewczyk's solution up to the 2nd order approximation grows upward around $\xi \approx 0.1$ and the difference between Szewczyk's and author's one (which is calculated up to the 3rd order approximation) rapidly widens. Also the same tendency is seen in Fig. 3 except for the case of $Pr=0.72$. This is because the 1st and 2nd order approximations are in the same sign. Therefore, the 2nd order approximation can hardly modify the solution up to the 1st order approximation and the range of applicability does not widen.

Furthermore, the results of the present paper are in good agreement with those of the 1st report, in the region which the later one is considered to be valid, but the tendency of Szewczyk's solution differs from the result of the 1st report and the reason depends on the 2nd order approximation obtained by Szewczyk. Therefore, Szewczyk's 2nd order approximation seems to be rather not so valid as the result of this paper.

5. Conclusion

In this paper, an analysis made of a combined convective flow on a vertical flat plate by use of the

perturbation method which is carried out up to the 3rd order approximation, and the range of applicability of the solution obtained by the perturbation method is widened. Almost exact solutions are obtained in the region of $0 \leq \xi \leq 0.2$ and fairly accurate solutions are given in $0.2 \leq \xi \leq 0.3$. It is proved that the range of applicability of the solution up to the 1st order approximation is pretty wider than the condition of applicability defined by Sparrow and Gregg. Therefore, the solution of the 1st order approximation can apply to a pretty wide as an approximate solution.

Also the relation of $F''(\xi, 0)$ and $\theta'(\xi, 0)$ is linear in the region of this paper ($0 \leq \xi \leq 0.2 \sim 0.3$), and the approximate solutions of the 1st report are compared with the results of this paper. It is proved that the heat transfer coefficient is in close agreement with one of this paper. But the accuracy of friction factor grows worse in region of $Pr \leq 0.72$ and is improved as the Prandtl number increases more than $Pr \geq 1$.

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Discussion

Editorial Director: (1) Although the 5th term of the Eq. (16-a) is shown as $+3f_1f_1''$ in your paper, the same term in Szewczyk's paper (Eq. 9-a) is $+3f_1f_2''$. Please explain which is valid and if this is the reason for the difference in 2nd order approximation of the both papers. If it were so, how would you point out the thing in this paper?

Author's closure: (1) It seems that the Eq. (16-a) is right and the Szewczyk's equation would

probably be a misprint. Because, calculating the Eq. (9-a) in Szewczyk's paper, author has obtained a different solution from Szewczyk's result.

Example of the calculation

$$f_2''(0) = -1.206 07, \theta_2'(0) = 0.768 25 \text{ at } Pr = 1$$

Therefore, the difference between $3f_1f_1''$ and $3f_1f_2''$ would not seem to lead to the difference between the both papers.