Thermoelastic Effect Caused by Longitudinal Collision of Two Bars

By Kichinosuke Tanaka** and Tomoaki Kurokawa***

According to the one-dimensional elastic theory, it is well known that step waves of stress and strain propagate when two semi-infinite bars collide with each other. However, to describe precisely, the adiabatic deformation causes a thermal deviation along the bars. Consequently, the heat flow thus produced has an effect on the stress and strain waves. In this paper, the problem is formulated as a coupled transient phenomenon using the one-dimensional equation of motion, the thermal stress-strain relations and the equation of heat flow. The strain and temperature distributions are obtained.

It is found that the wave front propagates with velocity \( c_0 = \sqrt{E/\rho} \) (\( E \): isothermal Young's modulus), the strain is observed even in the region ahead of the wave front, and the magnitude of the strain behind the wave front is nearly constant but a little smaller compared with that of the ordinary elastic theory.

1. Introduction

The phenomenon in which the elastic vibration and the heat flow are coupled to cause internal friction to vibration was initiated and developed by Zener [1] [2]. A research of damped longitudinal vibration of rods was studied by Hori [3] in Japan. A thermoelastic effect is caused by the existence of a non-uniform field of strain. The strain gives rise to a temperature gradient and consequently a heat flow. This leads to a change in temperature distribution, causes a thermal stress and affects the states of stress and strain. It is interesting how this thermoelastic phenomenon affects a transient wave, not the stationary vibration. This subject has been studied by Boley [4] [5] and others [6] [7] [8]. In these studies, the problems for semi-infinite bodies were treated. The boundary conditions were given in terms of stress, strain and temperature. In this paper, analysis was made with regard to two elastic bars, each with semi-infinite length, coming into collision with each other with the relative velocity \( 2v \) (the boundary condition in this case is given in terms of velocity at the end face). According to the one-dimensional theory of elasticity, the strain wave is a step wave and propagates with the velocity of the one-dimensional longitudinal elastic wave \( c_0 \), where \( c_0 = \sqrt{E/\rho} \) (\( E \) is Young's modulus and \( \rho \) is the density). When the two bars are of the same quality and have the same diameter, the step wave has the magnitude of \(-v/c_0\).

That part of the rod through which the wave has passed is compressed adiabatically and develops a temperature rise which is a little higher than that part through which no wave has passed. Therefore, a temperature gradient and consequently a heat flow arises, and a strain is induced beyond the wave front. On the other hand, all the kinetic energy of the rods is not converted into strain energy, but some part of it is converted into thermal energy, and consequently, the magnitude of the strain wave is reduced a little comparing with that obtained by simple theory which has no consideration for thermoelastic effect. The value of the adiabatic Young's modulus \( E_0 \) differs slightly from that of the isothermal Young's modulus \( E_T \), making it necessary to examine which of these two values should be used for \( c_0 \). In this paper, one-dimensional analysis is made on the collision between the above-mentioned two bars as a transient phenomenon in the thermoelastic problem in which a thermal stress, an elastic wave and a heat flow are coupled with the aim of elucidating the propagation of the strain wave and the transition of the temperature distribution.

2. Nomenclature

\( t \) : time
\( x \) : distance along the axis
\( u \): axial displacement
\( \sigma \): axial stress
\( \epsilon \): axial strain
\( T \): temperature
\( T_0 \): temperature before collision
\( \theta = T - T_0 \)
\( \rho \): density
\( c_0 \): velocity of one-dimensional longitudinal elastic wave in the rod
\( E_r \): isothermal Young's modulus
\( E_s \): adiabatic Young's modulus
\( c_p \): specific heat at constant pressure per unit mass
\( D \): thermal diffusivity
\( \alpha_0 \): coefficient of linear thermal expansion
\( v \): velocity of the struck end

3. Basic equations

The physical quantities for describing the phenomenon consist of displacement \( u \), stress \( \sigma \), strain \( \epsilon \) and temperature \( T \). These quantities are determined by using an equation of motion for a one-dimensional longitudinal wave, an equation for a heat flow and an equation for thermal stress. The equation of motion for the longitudinal wave is written in the form

\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} \tag{1}
\]

The equation for the thermal stress is expressed as

\[
\sigma = \left( \frac{\partial \sigma}{\partial T} \right)_T \epsilon + \left( \frac{\partial \sigma}{\partial T} \right)_\theta \theta \tag{2}
\]

And the equation for the heat flow is defined as

\[
\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial T}{\partial x} \frac{\partial \epsilon}{\partial t} \tag{3}
\]

Equation (3) expresses a change of temperature per unit time. The first term on the right-hand side represents the ordinary thermal diffusion and the second term represents the contribution from the adiabatic change of volume. The latter term and Eq. (2) are combined together to express a phenomenon in which stress, strain and temperature are coupled. The suffixes \( T \), \( S \), \( \epsilon \) and \( \sigma \) to the differential coefficients in Eqs. (2) and (3) and in the equations in Appendix denote an isothermal change, an adiabatic change, a change at constant volume and a change at constant pressure, respectively. Thermodynamic consideration indicates that the differential coefficients are related to the Young's modulus, the coefficient of linear thermal expansion and the specific heat at constant pressure as

\[
\left( \frac{\partial \sigma}{\partial \epsilon} \right)_T = E_r \tag{4}
\]

\[
\left( \frac{\partial \sigma}{\partial \theta} \right)_T = -\alpha_0 E_r \tag{5}
\]

\[
\frac{\partial T}{\partial \epsilon} = E_s \frac{\alpha_0 T_0}{\rho c_p} \tag{6}
\]

[ref. Eqs. (33), (34) and (35)].

As the strain is defined as

\[
\epsilon = \frac{\partial u}{\partial x}
\]

the equation of motion for the longitudinal wave is derived from Eqs. (1), (2), (4) and (5) as

\[
\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2} - \frac{E_r c_0^2}{\rho} \frac{\partial \theta}{\partial x} \tag{7}
\]

where \( c_0 \) is the velocity of the longitudinal wave, that is,

\[
c_0 = \sqrt{\frac{E_r}{\rho}} \tag{8}
\]

The equation for the heat flow is obtained from Eqs. (3) and (6) as

\[
\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2} - \frac{E_s \alpha_0 T_0}{\rho c_p} \frac{\partial \epsilon}{\partial t} \tag{9}
\]

The above-mentioned equations of motion and of heat flow and relation between thermal stress and strain have been derived from one-dimensional treatment, while there is also a method by which these relations can be derived from three-dimensional thermoelasticity.

The collision between two bars, each having semi-infinite length, the same quality and the same diameter, at a relative velocity of \( 2v \) is equivalent to moving the end face \( (x=0) \) of a bar having semi-infinite length extending in the positive direction \( (x \geq 0) \) at a velocity of \( v \) at \( t \geq 0 \) and keeping the end face adiabatic. Therefore, the boundary conditions are written as, at \( x=0 \),

\[
\frac{\partial u}{\partial t} = v H(t), \quad \frac{\partial \theta}{\partial x} = 0 \tag{10}
\]

where \( H(t) \) is a Heaviside's step function. By introducing the variables

\[
\xi = \frac{x}{D/c_0}, \quad \tau = \frac{t}{D/c_0}, \quad U = \frac{u}{-vD/c_0^2}, \quad \epsilon = \frac{\epsilon}{v/c_0}, \quad \theta = \frac{\theta}{c_0 \alpha_0 T_0 E_r} \tag{11}
\]

the equations governing \( U \) and \( \theta \) and the boundary conditions become

\[
\frac{\partial^2 U}{\partial \xi^2} - \frac{\partial \theta}{\partial \xi} = \frac{E_s \alpha_0 T_0}{\rho c_p} \frac{\partial \theta}{\partial \xi} \tag{12}
\]

\[
\frac{\partial \theta}{\partial \xi} + \frac{\partial \xi}{\partial \theta} \frac{\partial \xi}{\partial \theta} = \frac{E_s \alpha_0 T_0}{\rho c_p} \frac{\partial \theta}{\partial \xi} \tag{12}
\]

\[
\xi = 0: \frac{\partial U}{\partial \tau} = H(\tau), \quad \frac{\partial \theta}{\partial \xi} = 0 \tag{13}
\]

The strain is given by \( \epsilon = \partial U / \partial \xi \).
4. Solution

The basic equations described in the preceding section are solved by using Laplace’s transform. Laplace’s transform

\[ \mathcal{L}[f(\tau)] = F(p) = \int_0^\infty f(\tau) e^{-p\tau} d\tau \]

is applied to Eqs. (12) and the boundary conditions (13). From Eqs. (12), the following ordinary differential equations for \( \tilde{U} \) and \( \tilde{\Theta} \) are obtained.

\[
\begin{align*}
\frac{d^4 \tilde{U}}{d\xi^4} - \left\{ p^2 + p \left( 1 + \frac{E_s \alpha_s^2 T_0}{\rho c_p} \right) \right\} \frac{d^2 \tilde{U}}{d\xi^2} + p^2 \tilde{U} = 0, \\
\frac{d^4 \tilde{\Theta}}{d\xi^4} - \left\{ p^2 + p \left( 1 + \frac{E_s \alpha_s^2 T_0}{\rho c_p} \right) \right\} \frac{d^2 \tilde{\Theta}}{d\xi^2} + p^2 \tilde{\Theta} = 0
\end{align*}
\]

On the other hand, the ratio \( E_s/E_r \), which is denoted by \( K \), is expressed by the following equation [ref. Eq. (36)].

\[ \frac{E_s}{E_r} = 1 + \frac{E_s \alpha_s^2 T_0}{\rho c_p} \equiv K > 1 \]

The solutions of Eqs. (14) are written as

\[ \tilde{U} = A_1 e^{-\eta_1 t} + B_1 e^{-\eta_2 t}, \quad \tilde{\Theta} = A_2 e^{-\eta_1 t} + B_2 e^{-\eta_2 t}, \]

where

\[ \eta_1 = \sqrt{\frac{p^2 + \alpha^2 + \beta^2}{\rho c_p}}, \quad \eta_2 = \sqrt{-\frac{p^2 + \alpha^2 + \beta^2}{\rho c_p}} \]

\[ \Re(\eta_1) > 0, \quad \Re(\eta_2) > 0 \]

\[ \xi = 2 - K, \quad \beta = 2 \sqrt{K - 1}, \quad K = \sqrt{\alpha^2 + \beta^2} = 2 - \alpha \]  \hspace{1cm} (18)

The boundary conditions, Eqs. (13), give the equations

\[ A_1 + B_1 = 1 / p, \quad A_2 \eta_1 + B_2 \eta_2 = 0 \]  \hspace{1cm} (19)

Substitution of Eqs. (16) into Eqs. (12) gives

\[ A_1 (y_1^2 - p) + A_2 \eta_1 p = 0, \quad B_1 (y_2^2 - p) + B_2 \eta_2 p = 0 \]  \hspace{1cm} (20)

\[ A_1, B_1, A_2 \text{ and } B_2 \text{ are determined by } Eqs. (19) \text{ and } (20). \text{ Then } \xi \text{ and } \eta \text{ are given as} \]

\[ \xi = y_1^2 y_2 (y_2^2 - p) - y_1^2 y_2 (y_1^2 - p) - y_1^2 e^{-\eta_1 t} + y_2^2 e^{-\eta_2 t} \]

\[ \eta = \frac{y_1 y_2^2}{p^2 (y_2^2 - y_1^2)} e^{-\eta_1 t} + \frac{y_1 y_2^2}{p^2 (y_1^2 - y_2^2)} e^{-\eta_2 t} \]  \hspace{1cm} (21)

It can be seen from Eqs. (17) that

\[ y_1 \sim p, \quad y_2 \sim \sqrt{p} \]

for \( |p| \to \infty \). Using the estimation, it is recognized that the first and second terms of Eqs. (21) and (22) have a connection with the propagation of the discontinuous wave front and the heat flow, respectively. With the aid of the relation shown in Eq. (18), inverse transforms of Eqs. (21) and (22) are obtained as follows [ref. Appendix],

\[ \varepsilon(\xi, \tau) = f_1(\xi, \tau) + f_2(\xi, \tau) + f_3(\xi, \tau): \xi > \tau, \]

\[ = -1/\sqrt{K - \theta}(\xi, \tau): \xi < \tau \]  \hspace{1cm} (23)

\[ \Theta(\xi, \tau) = \theta_1(\xi, \tau) + \theta_2(\xi, \tau) - \theta_3(\xi, \tau): \xi > \tau, \]

\[ = 1/\sqrt{K - \theta}(\xi, \tau): \xi < \tau \]  \hspace{1cm} (24)

where

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\begin{align*}
&\xi = 2 - K, \quad \beta = 2 \sqrt{K - 1}, \quad K = \sqrt{\alpha^2 + \beta^2} = 2 - \alpha \\
&\Re(\eta_1) > 0, \quad \Re(\eta_2) > 0
\end{align*}
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The propagation of the strain and that of the temperature are shown in Figs. 1-4 in the cases of $K=1.01$ and $K=1.05$. There is a discontinuity of strain at $\xi = \tau$, that is, $x = c_o \tau$. It will be seen, therefore, that the wave front propagates at velocity $c_o$ of the longitudinal wave (Eq. (8)) which is calculated with isothermal Young's modulus $E_y$. It is noticeable that the strain is caused in the part beyond the wave front by the heat flow. The strain in this part cannot be seen when the thermoelastic effect is not taken into consideration. The extent of discontinuity of strain at the wave front decreases with the lapse of time. This kind of discontinuity does not exist in the distribution of temperature. Immediately after collision, the strain and the temperature are not distributed uniformly in the part behind the wave front (the feature of the distribution of the strain is not distinguished clearly in these figures), but $f_3$ and $\Theta_3$ approach zero with the lapse of time because the integrands of $f_3$ and $\Theta_3$ in Eqs. (26) and (28) contain $\exp (-\tau x^2)$ and consequently $\tau$ and $\Theta$ approach a uniform distribution. That is, the values of the strain and the temperature behind the wave front converge to a certain value. The value is $1/\sqrt{K}$ and is less than 1. That is, in the case of strain, it is smaller than $v/c_o$. In the case of actual metals, however, $E_s$ and $E_T$ differ by 1% at the most.

The above-mentioned may be summarized as follows:

(1) The elastic wave front propagates at the velocity of $c_o$, where $c_o$ is defined as $\sqrt{E_s/\rho}$ by using the isothermal Young's modulus $E_s$.

(2) The distribution of the strain reaches a certain value behind the wave front with the lapse of time. The value is $-(v/c_o) (E_s/E_T)^{1/2}$ and is smaller than $-v/c_o$.

(3) Strain is caused by the heat flow beyond the elastic wave front, but its domain where the strain is caused is extremely small ($D/c_o \sim 10^{-6}$ cm for metals).

(4) The temperature is caused to rise by the compression wave, and the temperature distribution shows almost the same tendency as the strain.

We wish to express our deepest gratitude to Chikashi Motoyama and Yoshio Iwahashi for their assistance in carrying out this study.

Appendix

1. Thermodynamic relations

Among four variables $\sigma$, $\epsilon$, $T$ and $S$, two of them are only independent variables from the standpoint of thermodynamics. By the definition,

$$\left( \frac{\partial \sigma}{\partial \epsilon} \right)_T = E_T, \quad \left( \frac{\partial \sigma}{\partial T} \right)_\epsilon = \alpha_0 \tag{33}$$

When $\sigma$ is $\sigma (\epsilon, T)$, in the process of constant pressure,

$$d\sigma = \left( \frac{\partial \sigma}{\partial \epsilon} \right)_T d\epsilon + \left( \frac{\partial \sigma}{\partial T} \right)_\epsilon dT = 0$$

Therefore,

$$\left( \frac{\partial \epsilon}{\partial T} \right)_S = -\left( \frac{\partial \sigma}{\partial T} \right)_\epsilon \left( \frac{\partial \sigma}{\partial \epsilon} \right)_T = -\alpha_0 E_T \quad \tag{34}$$

And

$$\left( \frac{\partial T}{\partial \sigma} \right)_S = \left( \frac{\partial T}{\partial \sigma} \right)_\epsilon \left( \frac{\partial \sigma}{\partial \epsilon} \right)_T = E_s \left( \frac{\partial \sigma}{\partial \epsilon} \right)_S \tag{35}$$

On the other hand, the condition $TdS-\epsilon d\sigma$ has an exact differential form is given by

$$\left( \frac{\partial T}{\partial \sigma} \right)_s = -\left( \frac{\partial \sigma}{\partial S} \right)_T \tag{36}$$

And
\[ \left( \frac{\partial \varepsilon}{\partial S} \right)_s = \left( \frac{\partial \varepsilon}{\partial T} \right)_s / \left( \frac{\partial S}{\partial T} \right)_s \]

From the relation \( T_0 dS = dQ = \rho c_p dT \), it gives, under constant stress,
\[ \left( \frac{\partial S}{\partial T} \right)_s = \frac{\rho c_p}{T_0} \]
Therefore,
\[ \left( \frac{\partial T}{\partial \varepsilon} \right)_s = -\frac{\alpha_c T_0}{\rho c_p} \]
Finally,
\[ \left( \frac{\partial T}{\partial \varepsilon} \right)_s = -\frac{E_s \alpha_c T_0}{\rho c_p} \]

Among four variables \( x_1, x_2, x_3, \) and \( x_4 \), only two are independent variables. It can be written as \( x_2 = x_2(x_1, x_3, x_4) \).
Therefore, using the relation
\[ 1 - \frac{\partial x_2}{\partial x_3} \left( \frac{\partial x_2}{\partial x_3} \right) = -\frac{\partial x_2}{\partial x_4} \left( \frac{\partial x_2}{\partial x_4} \right)_s \]
the following is obtained
\[ 1 - \frac{\partial x_1}{\partial x_3} \left( \frac{\partial x_1}{\partial x_3} \right) = -\frac{\partial x_1}{\partial x_4} \left( \frac{\partial x_1}{\partial x_4} \right)_s \]
Putting \( x_1 = \tau, \) \( x_2 = \varepsilon, \) \( x_3 = \sigma \), and \( x_4 = S \) and considering Eqs. (33) and (35), the following formula is obtained.
\[ \frac{E_s}{E_p} = 1 + \frac{E_s \alpha_c T_0}{\rho c_p} \]

\section*{2 Inverse transforms of Eqs. (21) and (22)}

To carry out the calculation of inverse transforms of Eqs. (21) and (22), the integral paths shown in Fig. 5 are adopted. When \( p \) is on the paths \( l_1, \) \( r_2, \) \( l_3, \) \( l_4, \) \( L_1, \) \( L_2, \) \( R_2 \) or \( L_1 \), the corresponding values of two complex roots given by Eqs. (17) are tabulated and illustrated in Table 1 in and Fig. 6 (b) and (c) respectively.

(i) Transform of \( \varepsilon \)
\[ \varepsilon_1 = -\frac{pe^{-\gamma_1}}{y_1^2 - y_2^2} \left( \frac{y_1}{p} - 1 \right) \]
\[ \varepsilon_2 = \frac{pe^{-\gamma_2}}{y_1^2 - y_2^2} \left( \frac{y_2}{p} - 1 \right) \]
(ii) The case \( \xi > \tau \)

The path shown in Fig. 5 (b) is adopted to get inverse transform of \( \varepsilon_1 \). Because \( y_1 \) tends to \( p \) when \( |p| \) tends to infinity, the following is obtained with the aid of Jordan's lemma.
\[ \frac{1}{2\pi i} \int_{B_r} \varepsilon_1 e^{p \tau} dp = 0 \]
Concerning the inverse transform of \( \varepsilon_2 \), the path given in Fig. 5 (a) is adopted.
\[ \frac{1}{2\pi i} \int_{B_r} \varepsilon_2 e^{p \tau} dp = -\frac{1}{2\pi i} (I_{l_1} + R_1 + I_{L_1} + L_1) \]
\[ + I_{r_2} + L_2 + I_{R_2} + L_2 + I_{l_1} + L_1 + I_{L_1} + L_1 + I_{L_1} + L_1 \]
\[ (I_{m_1 + M_1}) \]
Finally, using Eqs. (39) \( \sim (46) \) we obtain
\[ \frac{1}{2\pi i} \int_{B_r} \varepsilon e^{p \tau} dp = \frac{1}{2\pi i} \int_{B_r} (\xi_1 + \xi_2) e^{p \tau} dp \]
\[ = f_3(\xi, \tau) + f_4(\xi, \tau) + f_4(\xi, \tau) \]
\[ (i) \] The case \( \xi < \tau \)

The path shown in Fig. 5 (a) is taken to evaluate the inverse integrals of \( \varepsilon_1 \) and \( \varepsilon_2 \).
\[ \frac{1}{2\pi i} \int_{B_r} \varepsilon_1 e^{p \tau} dp = -\frac{1}{2\pi i} (I_{l_1} + R_1 + I_{L_1} + L_1) \]
\[ + I_{r_2} + L_2 + I_{R_2} + L_2 + I_{l_1} + L_1 + I_{L_1} + L_1 + I_{L_1} + L_1 \]
\[ (i) \] The case \( \xi < \tau \)

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Path & \( p \) & \( y_1 \) & \( y_2 \) \\
\hline
\( l_1 \) & \(-x\) & \(2(x + \alpha^2) + \beta^2\) & \(\alpha^2 + \beta^2 - x\) \\
\(-x\) & \(2(x - \alpha^2) + \beta^2\) & \(\alpha^2 + \beta^2 + x\) \\
\hline
\( L_1 \) & \(-x\) & \(2(x + \alpha^2) + \beta^2\) & \(\alpha^2 + \beta^2 - x\) \\
\(-x\) & \(2(x - \alpha^2) + \beta^2\) & \(\alpha^2 + \beta^2 + x\) \\
\hline
\( l_1 \) & \(-x\) & \(2(x + \alpha^2) + \beta^2\) & \(\alpha^2 + \beta^2 - x\) \\
\(-x\) & \(2(x - \alpha^2) + \beta^2\) & \(\alpha^2 + \beta^2 + x\) \\
\hline
\end{tabular}
\end{table}

Fig. 5
\[ \tilde{T}_1 = -\frac{p}{(y_1^2 - y_2^2)} e^{-\tau \xi} \quad \text{(56)} \]
\[ \tilde{T}_2 = \frac{p}{(y_1^2 - y_2^2)} e^{-\tau \xi} \quad \text{(57)} \]
(i) The case \( \xi \geq \tau \)

To transform \( \tilde{T}_1 \) and \( \tilde{T}_2 \), the path shown in Fig. 5 (b) and one shown in Fig. 5 (a) are taken respectively. By Jordan's lemma, it can be written as

\[ \frac{1}{2\pi i} \int_{B_r} \tilde{T}_1 e^{\nu p} dp = 0, \quad (\xi \geq \tau) \quad \text{(58)} \]

On the other hand, the following are obtained.

\[ \frac{1}{2\pi i} \int_{B_r} \tilde{T}_2 e^{\nu p} dp = -\frac{1}{2\pi i} (I_{r_1 + R_1} + I_{l_1 + L_1}) \quad \text{(59)} \]

\[ |p| \to \infty : I_{r_1 + R_1} \to -1 \quad \text{(60)} \]
\[ |p| \to 0 : I_{r_2 + R_1} \to -1 \quad \text{(61)} \]

\[ I_{l_1 + L_1} = 0 \quad \text{(62)} \]
\[ I_{l_1 + L_2} = 2\pi i \Theta_2 (\xi, \tau) \quad \text{(63)} \]
\[ I_{l_2 + L_1} = -2\pi i \Theta_1 (\xi, \tau) \quad \text{(64)} \]
\[ I_{l_2 + L_2} = -2\pi i \Theta_2 (\xi, \tau) \quad \text{(65)} \]

\[ \frac{1}{2\pi i} \int_{B_r} \tilde{T}_1 e^{\nu p} dp = \frac{1}{2\pi i} (\tilde{T}_1 + \tilde{T}_2) e^{\nu p} dp \quad \text{(66)} \]

(ii) The case \( \xi < \tau \)

The path shown in Fig. 5 (a) is adopted to obtain the inverse transform of \( \tilde{T}_1 \) and \( \tilde{T}_2 \).

\[ \frac{1}{2\pi i} \int_{B_r} \tilde{T}_1 e^{\nu p} dp = -\frac{1}{2\pi i} (I_{r_1 + R_1} + I_{l_1 + L_1}) \quad \text{(67)} \]

\[ |p| \to \infty : I_{r_1 + R_1} \to 0 \quad \text{(68)} \]
\[ I_{l_1 + L_1} = 0 \quad \text{(69)} \]
\[ I_{l_1 + L_2} = 0 \quad \text{(70)} \]
\[ I_{l_2 + L_1} = -I_{l_1 + L_1} \quad \text{(71)} \]
\[ I_{l_2 + L_2} = -I_{l_1 + L_1} \quad \text{(72)} \]

\[ \frac{2\pi i}{\sqrt{\alpha^2 + \beta^2}} = \frac{2\pi i}{\sqrt{K}} \quad \text{(73)} \]

\[ = \frac{1}{2\pi i} (I_{r_1 + R_1} + I_{l_1 + L_1}) \quad \text{(74)} \]

References