539.382 : 539.4.013

Stress Distributions in a Semi-Infinite Plate with a Row of Circular Holes*

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To investigate stress distributions in a plate under the tension through rivets, we consider a semi-infinite plate with equal spaced identical holes which are in a row parallel to the boundary of the plate and equally loaded at their edges.

Applying the theory of two-dimensional elasticity and deriving a series of periodic stress functions containing unfixed coefficients which have poles at the centers of the holes and satisfy the condition of free boundary at the straight boundary, we solve the problem by determining the coefficients in such manner that each hole is under given boundary conditions on the edge. Using these solutions, we calculate stress distributions in the plate for several cases.

1. Introduction

To investigate stresses in a plate under the tension through rivet joints, we consider a semi-infinite plate with equal-spaced identical circular holes which are in a row parallel to the straight boundary of the plate and equally loaded on their walls towards the boundary, and apply the theory of two-dimensional elasticity. Our semi-infinite plate is more similar to real states than an infinite plate with a row of circular holes as solved by R. C. J. Howland⁽¹⁾ and M. Ishida⁽²⁾, a semi-infinite plate with a row of loading points as solved by M. Sadowsky⁽³⁾, or the same plate with a circular hole as solved by E. Udoguchi⁽⁴⁾.

First, we find a general form of periodic stress function which has poles in the centers of the circular holes and satisfies the boundary condition that the straight edge is free from stress. By combining a series of the functions with stress functions which express a concentrated force in each hole and a uniform tension parallel to the straight edge, we obtain a stress function and determine unfixed coefficients contained in the function so that the displacement or the stress along the edge and the surface force on the wall of the hole may satisfy given boundary conditions. Using the function, we can calculate stress distributions anywhere in the plate.

2. Stress functions

As shown in Fig. 1, we denote the equal space

between the centers of the holes, the distance from the centers to the straight edge, the radius of the hole, the unbalanced force in the hole and the uniform tension in x -direction at infinity by b , a , c , P and T , respectively, in which $T=P/b$.

In x, y plane, putting

$$z = x + iy = re^{i\theta}$$

and taking, as dimensionless quantities,

$$\xi = x/b, \quad \eta = y/b, \quad \zeta = \xi + i\eta, \quad \alpha = a/b, \quad \rho = c/b$$

the straight edge and the wall of the hole are expressed by $\xi = -\alpha$ and $\zeta = \rho e^{i\theta}$, respectively.

We consider periodic functions of complex variable as follows:

$$\left. \begin{aligned} W_0 &= S_0 + iT_0 = \int \log \sinh \pi \zeta d\zeta \\ W_1 &= S_1 + iT_1 = \log \sinh \pi \zeta \\ W_2 &= S_2 + iT_2 = \pi \coth \pi \zeta \\ W_3 &= S_3 + iT_3 = -\pi^2 \operatorname{cosech}^2 \pi \zeta \\ &\vdots \\ W_n &= S_n + iT_n = \frac{d^{n-2}}{d\zeta^{n-2}} (\pi \coth \pi \zeta) \end{aligned} \right\} \dots\dots\dots (1)$$

Fig. 1

* Received 4th August, 1970.

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$$\left. \begin{aligned} w_0 &= s_0 + it_0 = \int \log \sinh \pi(\zeta + 2\alpha) d\zeta \\ w_1 &= s_1 + it_1 = \log \sinh \pi(\zeta + 2\alpha) \\ w_2 &= s_2 + it_2 = \pi \coth \pi(\zeta + 2\alpha) \\ w_3 &= s_3 + it_3 = -\pi^2 \operatorname{cosech}^2 \pi(\zeta + 2\alpha) \\ w_n &= s_n + it_n = \frac{d^{n-2}}{d\zeta^{n-2}} \{ \pi \coth \pi(\zeta + 2\alpha) \} \end{aligned} \right\} \dots (2)$$

In the above functions, it is shown that W_n has poles at the centers of the holes ($\eta = \dots, -2, -1, 0, 1, 2, \dots$) and the period $\eta=1$, and W_n is the one obtained by substituting $(\zeta + 2\alpha)$ for ζ into W_n .

From Eqs. (1) and (2), at the edge ($\xi = -\alpha$), we have the following relations:

$$\left. \begin{aligned} (S_n)_{\xi=-\alpha} &= -(-1)^n (s_n)_{\xi=-\alpha} \\ (T_n)_{\xi=-\alpha} &= (-1)^n (t_n)_{\xi=-\alpha} \end{aligned} \right\} \dots (3)$$

Now we consider Airy's stress function F_n which contains the above periodic functions and unfixed constants A_n , B_n and denote the stress components derived from F_n by σ_{zn} , σ_{yn} and τ_{zyn} . Then we have

$$\begin{aligned} F_n/b^2 &= A_n [S_n + (-1)^n \{s_n - 2(\alpha + \xi)s_{n+1}\} \\ &\quad + B_n [\xi S_{n+1} - (-1)^n \{\xi s_{n+1} + 2\alpha(\alpha + \xi)s_{n+2}\}]] \end{aligned} \dots (4)$$

$$\left. \begin{aligned} \sigma_{zn} &= \frac{\partial^2 F_n}{\partial y^2} = -A_n [S_{n+2} + (-1)^n \{s_{n+2} - 2(\alpha + \xi) \\ &\quad \times s_{n+3}\}] - B_n [\xi S_{n+2} - (-1)^n \{\xi s_{n+3} + 2\alpha \\ &\quad \times (\alpha + \xi)s_{n+4}\}] \\ \sigma_{yn} &= \frac{\partial^2 F_n}{\partial x^2} = A_n [S_{n+2} - (-1)^n \{3s_{n+2} + 2(\alpha + \xi) \\ &\quad \times s_{n+3}\}] + B_n [2S_{n+2} + \xi S_{n+3} - (-1)^n \{2s_{n+2} \\ &\quad + (\xi + 4\alpha)s_{n+3} + 2\alpha(\alpha + \xi)s_{n+4}\}] \\ \tau_{zyn} &= -\frac{\partial^2 F_n}{\partial x \partial y} = A_n [T_{n+2} - (-1)^n \{t_{n+2} + 2 \\ &\quad \times (\alpha + \xi)t_{n+3}\}] + B_n [T_{n+2} + \xi T_{n+3} - (-1)^n \\ &\quad \{t_{n+2} + (\xi + 2\alpha)t_{n+3} + 2\alpha(\alpha + \xi)t_{n+4}\}] \end{aligned} \right\} \dots (5)$$

and at the edge, from Eqs. (3) and (5),

Fig. 2

$$(\sigma_{zn})_{\xi=-\alpha} = 0, \quad (\tau_{zyn})_{\xi=-\alpha} = 0$$

Therefore we can conclude that F_n in Eq. (4) is a generalized stress function which has poles at the centers of the holes and satisfies the condition of free boundary at the straight edge.

Let us denote the distance between a loading point $y = nb$ (n is a integer) and a point $Q(x, y)$ by r_n and Poisson's ratio by ν , and the stress components σ_{x0} , σ_{y0} and τ_{xy0} due to the concentrated forces P in the holes are (see Fig. 2)

$$\left. \begin{aligned} \sigma_{x0} &= \frac{P}{2\pi} \left\{ \frac{(1-\nu)}{2} x \sum_{n=-\infty}^{\infty} \frac{1}{r_n^2} + (1+\nu) x^3 \sum_{n=-\infty}^{\infty} \frac{1}{r_n^4} \right\} \\ \sigma_{y0} &= \frac{P}{2\pi} \left\{ \frac{(1+3\nu)}{2} x \sum_{n=-\infty}^{\infty} \frac{1}{r_n^2} - (1+\nu) x^3 \sum_{n=-\infty}^{\infty} \frac{1}{r_n^4} \right\} \\ \tau_{xy0} &= \frac{P}{2\pi} \left\{ \frac{(1-\nu)}{2} \sum_{n=-\infty}^{\infty} \frac{y_n}{r_n^2} + (1+\nu) x^2 \sum_{n=-\infty}^{\infty} \frac{y_n}{r_n^4} \right\} \end{aligned} \right\} \dots (6)$$

where $y_n = y - nb$.

If we use the equations

$$\left. \begin{aligned} x \sum_{n=-\infty}^{\infty} \frac{1}{r_n^2} &= \frac{1}{b} S_2, \quad x^3 \sum_{n=-\infty}^{\infty} \frac{1}{r_n^4} = \frac{1}{2b} (S_2 - \xi S_3) \\ \sum_{n=-\infty}^{\infty} \frac{y_n}{r_n^2} &= -\frac{1}{b} T_2, \quad x^2 \sum_{n=-\infty}^{\infty} \frac{y_n}{r_n^4} = \frac{1}{2b} \xi T_3 \end{aligned} \right\} \dots (A)$$

which are to be mentioned later, Eqs. (6) are transformed as follows⁽³⁾:

$$\begin{aligned} \sigma_{x0} &= \frac{P}{2\pi b} \left\{ S_2 - \frac{(1+\nu)}{2} \xi S_3 \right\} \\ \sigma_{y0} &= \frac{P}{2\pi b} \left\{ \nu S_2 + \frac{(1+\nu)}{2} \xi S_3 \right\} \\ \tau_{xy0} &= \frac{P}{2\pi b} \left\{ -\frac{(1-\nu)}{2} T_2 + \frac{(1+\nu)}{2} \xi T_3 \right\} \end{aligned}$$

It is easy to know that those stress components are derived from the following stress function:

$$\frac{Pb}{2\pi} \left\{ -S_0 + \frac{(1+\nu)}{2} \xi S_1 \right\}$$

The stress function F_0 having the concentrated forces P in a row on y -axis and the free boundary at the straight edge must be analogous to Eq. (4). Then we find

$$\begin{aligned} F_0 &= \frac{Pb}{2\pi} \left[-\{S_0 + s_0 - 2(\alpha + \xi)s_1\} + \frac{(1+\nu)}{2} \right. \\ &\quad \left. \times \{\xi S_1 - \xi s_1 - 2\alpha(\alpha + \xi)s_2\} \right] \end{aligned}$$

Finally, the stress distribution of the semi-infinite plate which has the parallel forces P in a row with the equal space b on y -axis and the free boundary, is given by the general function

$$\begin{aligned} F/b^2 &= \left(F_0 + \sum_{n=1}^{\infty} F_n + \frac{C}{2} x^2 \right) / b^2 = \sum_{n=0}^{\infty} A_n \\ &\quad \times [S_n + (-1)^n \{s_n - 2(\alpha + \xi)s_{n+1}\}] + \sum_{n=0}^{\infty} B_n \\ &\quad \times [\xi S_{n+1} - (-1)^n \{\xi s_{n+1} + 2\alpha(\alpha + \xi) \\ &\quad \times s_{n+2}\}] + \frac{1}{2} C \xi^2 \dots (7) \end{aligned}$$

in which the constants A_0 and B_0 are

$$A_0 = -\frac{P}{2\pi b} = -\frac{T}{2\pi}$$

$$B_0 = \frac{P}{2\pi b} \frac{(1+\nu)}{2} = \frac{T}{2\pi} \frac{(1+\nu)}{2}$$

and the unfixed constants $A_1, A_2, \dots, B_1, B_2, \dots$ are to be calculated from the given distribution of P on the wall of the hole as described later. The uniform tension C in y -direction is determined from the consideration of stress or displacement in the direction. For instance, when an object joined by rivets is made of a soft material and a plate is unconstrained in y -direction, the plate is free to deform and the stress at a long distance from the holes vanishes. On the contrary, when either the object is a rigid body or the plate is prevented from extension or contraction, where is to be no displacement in the direction. We will determine the value of C corresponding to each case.

(i) When the stress in y -direction vanishes at infinity, we have, from Eqs. (5) and (7),

$$\lim_{x \rightarrow \infty} \sigma_y = \lim_{\xi \rightarrow 0} \sum_{n=0}^{\infty} A_n [S_{n+2} - (-1)^n \{3s_{n+2} + 2(\alpha + \xi)s_{n+3}\}] + \lim_{\xi \rightarrow \infty} \sum_{n=0}^{\infty} B_n [2S_{n+2} + \xi S_{n+3} - (-1)^n \{2s_{n+2} + (\xi + 4\alpha)s_{n+3} + 2\alpha(\alpha + \xi)s_{n+4}\}] + C$$

and from Eqs. (1) and (2),

$$\lim_{\xi \rightarrow \infty} S_2 = \pi, \quad \lim_{\xi \rightarrow \infty} s_2 = \pi$$

and, for $n \geq 1$,

$$\lim_{\xi \rightarrow \infty} S_{n+2} = 0, \quad \lim_{\xi \rightarrow \infty} s_{n+2} = 0$$

$$\lim_{\xi \rightarrow \infty} (\xi S_{n+2}) = 0, \quad \lim_{\xi \rightarrow \infty} (\xi s_{n+2}) = 0$$

Since $\lim_{x \rightarrow \infty} \sigma_y = 0$, the constant is

$$C = 2\pi A_0 = -T$$

(ii) When the plate has no stretching in y -direction, calculating the strain of an interval b , which must vanish, from Eqs. (5) and (7), we obtain

$$\frac{1}{b} \int_0^b (\sigma_y - \nu \sigma_x) dy = \sum_{n=0}^{\infty} A_n \int_0^1 [(1+\nu)S_{n+2} - (-1)^n \{(3-\nu)s_{n+2} + 2(1+\nu)(\alpha + \xi)s_{n+3}\}] d\eta$$

$$+ \sum_{n=0}^{\infty} B_n \int_0^1 [2S_{n+2} + (1+\nu)\xi S_{n+3} - (-1)^n \{2s_{n+2} + (1+\nu)\xi s_{n+3} + 4\alpha s_{n+3} + 2(1+\nu)(\alpha + \xi)s_{n+4}\}] d\eta + C$$

in which

$$\int_0^1 S_2 d\eta = \left| T_1 \right|_0^1 = \left| I_m \{ \log \sinh \pi \zeta \} \right|_0^1 = \left| \pi \eta \right|_0^1 = \pi$$

$$\int_0^1 s_2 d\eta = \left| t_1 \right|_0^1 = \left| I_m \{ \log \sinh \pi (\zeta + 2\alpha) \} \right|_0^1 = \pi$$

and, for $n \geq 1$,

$$\int_0^1 S_{n+2} d\eta = \left| T_{n+1} \right|_0^1 = 0, \quad \int_0^1 s_{n+2} d\eta = \left| t_{n+1} \right|_0^1 = 0$$

Then the constant in this case becomes

$$C = 2(1-\nu)\pi A_0 = -(1-\nu)T$$

Now the stress in x -direction at infinity is obtained from Eqs. (5) and (7), as follows:

$$\lim_{x \rightarrow \infty} \sigma_x = -\lim_{\xi \rightarrow \infty} \sum_{n=0}^{\infty} A_n [S_{n+2} + (-1)^n \{s_{n+2} - 2(\alpha + \xi)s_{n+3}\}] - \lim_{\xi \rightarrow \infty} \sum_{n=0}^{\infty} B_n [\xi S_{n+3} - (-1)^n \{\xi s_{n+3} + 2\alpha(\alpha + \xi)s_{n+4}\}] = -2\pi A_0 = T$$

from which it is seen that the plate undergoes a uniform tension T in x -direction at infinity.

3. Boundary conditions on the wall of the hole

The surface forces on the wall are given by stress components in polar coordinates derived from the stress function F in Eq. (7). First, we will express W_n and W_n in terms of coefficients of the exponent of ζ .

Differentiating the equation

$$W_2 = \pi \coth \pi \zeta = \frac{1}{\zeta} \left\{ 1 - \sum_{m=1}^{\infty} C_m \zeta^{2m} \right\}$$

where

$$C_m = (-1)^m \sum_{k=1}^{\infty} (1/k)^{2m}$$

we obtain

$$W_{n+2} = \frac{d^n}{d\zeta^{n+2}} (\pi \coth \pi \zeta) = \frac{1}{\zeta^{n+1}} \left\{ (-1)^n n! - \sum_{m=1}^{\infty} C_m (2m-1) \cdots (2m-n) \zeta^{2m} \right\}$$

and

$$w_n = \sum_{m=0}^{\infty} \frac{w_{n+m,0}}{m!} \zeta^m$$

in which

$$w_{n+m,0} = (w_{n+m})_{\zeta=0} = \left(\frac{d^m w_n}{d\zeta^m} \right)_{\zeta=0}$$

Next, we put $z = re^{i\theta}$ and calculate the stress components in polar coordinates from Eq. (7) by the above formulas and Eqs. (1), (2) and (7). By substituting $\zeta = \rho e^{i\theta}$ into the expression for the components of stress, we find the surface forces which are given by the following:

$$\begin{aligned}
(\sigma_r)_{r=c} &= \left(\frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} \right)_{r=c} \\
&= \sum_{n=0} A_n \left[\frac{1}{\rho^{n+1}} \{ (-1)^n n! \cos(n-1)\theta + \sum C_m (2m-1) \cdots (2m-n) \rho^{2m} \right. \\
&\quad \times \cos(2m-n+1)\theta \} + (-1)^n \sum_{m=0} \frac{\rho^m}{m!} \{ -2w_{n+m+2,0} \cos m\theta + (w_{n+m+2,0} + 2\alpha w_{n+m+3,0}) \\
&\quad \times \cos(m+2)\theta + \rho w_{n+m+3,0} (\cos \overline{m+1}\theta + \cos \overline{m+3}\theta) \} \Big] \\
&\quad + \sum_{n=0} B_n \left[\frac{1}{\rho^{n+1}} \left\{ (-1)^n n! \left(\frac{n-1}{2} \cos \overline{n-1}\theta + \frac{n+3}{2} \cos \overline{n+1}\theta \right) - \sum C_m (2m-1) \right. \right. \\
&\quad \times \cdots (2m-n) \rho^{2m} (\cos \overline{2m-n-1}\theta - \cos \overline{2m-n+1}\theta) + \frac{1}{2} \sum C_m (2m-1) \cdots \\
&\quad \times (2m-n-1) \rho^{2m} (\cos \overline{2m-n-1}\theta + \cos \overline{2m-n+1}\theta) \Big\} - (-1)^n \sum_{m=0} \frac{\rho^m}{m!} \{ (w_{n+m+2,0} \\
&\quad + 2\alpha w_{n+m+3,0}) (\cos m\theta - \cos \overline{m+2}\theta) - \left(\frac{\rho}{2} w_{n+m+3,0} + \alpha \rho w_{n+m+4,0} \right) \\
&\quad \times (\cos \overline{m+1}\theta + \cos \overline{m+3}\theta) - 2\alpha^2 w_{n+m+4,0} \cos(m+2)\theta \} \Big] + \frac{1}{2} C (1 - \cos 2\theta) \Big] \dots\dots\dots (8) \\
(\tau_{r\theta})_{r=c} &= \left\{ -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial F}{\partial \theta} \right) \right\}_{r=c} \\
&= \sum_{n=0} A_n \left[-\frac{1}{\rho^{n+1}} \{ (-1)^n n! \sin(n-1)\theta + \sum C_m (2m-1) \cdots (2m-n) \rho^{2m} \sin(2m-n \right. \\
&\quad + 1)\theta \} - (-1)^n \sum_{m=0} \frac{\rho^m}{m!} \{ \rho w_{n+m+3,0} (\sin \overline{m+1}\theta + \sin \overline{m+3}\theta) + (w_{n+m+2,0} \\
&\quad + 2\alpha w_{n+m+3,0}) \sin \overline{m+2}\theta \} \Big] \\
&\quad + \sum_{n=0} B_n \left[\frac{1}{\rho^{n+1}} \left\{ (-1)^n n! \left(\frac{n-1}{2} \sin \overline{n-1}\theta + \frac{n+1}{2} \sin \overline{n+1}\theta \right) \right. \right. \\
&\quad - \sum C_m (2m-1) \cdots (2m-n) \rho^{2m} \sin(2m-n+1)\theta - \frac{1}{2} \sum C_m (2m-1) \cdots \\
&\quad \times (2m-n-1) \rho^{2m} (\sin \overline{2m-n-1}\theta + \sin \overline{2m-n+1}\theta) \Big\} - (-1)^n \sum_{m=0} \frac{\rho^m}{m!} \left\{ \left(\frac{\rho}{2} \right. \right. \\
&\quad \times w_{n+m+3,0} + \alpha \rho w_{n+m+4,0} \Big) (\sin \overline{m+1}\theta + \sin \overline{m+3}\theta) + (w_{n+m+2,0} + 2\alpha w_{n+m+3,0}) \\
&\quad \left. \left. + 2\alpha^2 w_{n+m+4,0} \right) \sin(m+2)\theta \right\} \Big] + \frac{1}{2} C \sin 2\theta
\end{aligned}$$

We assume that the pressure of the rivet, P is distributed over the half circumference of the hole in proportion to $\cos \theta$ as shown in Fig. 3 and the frictional force between the rivet and the wall of the hole is negligible. The mean pressure on the half circumference is

$$p = \frac{P}{\pi c} = \frac{T}{\pi \rho}$$

Fig. 3

and then the conditions on the wall of the hole can be expressed by the following formulas:

$$\begin{aligned}
(\sigma_r)_{r=c} &= \frac{T}{\pi \rho} \left(-\frac{5}{8} + \cos \theta - \frac{3}{8} \cos 2\theta \right) \Big\} \dots\dots\dots (9) \\
(\tau_{r\theta})_{r=c} &= 0
\end{aligned}$$

The function F in Eq. (7) which has satisfied the condition of free boundary on the straight edge, must satisfy the remaining conditions, namely, Eqs. (9). Then, comparing the independent coefficient

Fig. 4

of θ and the coefficients of $\sin \theta$, $\cos \theta$, $\sin 2\theta$, $\cos 2\theta$, and so on in Eqs. (8) with those in Eqs. (9), we obtain simultaneous equations of the first degree which contain the unknown constants $A_1, A_2, \dots, B_1, B_2, \dots$. From the solution of the equations we can determine the constants.

4. Stress distributions

By using the determined constants $A_1, A_2, \dots, B_1, B_2, \dots$ and Eqs. (7) and (5), we can calculate the following stress distributions (see Fig. 4):

(i) along the straight edge,

$$(\sigma_y)_{x=-a} = 4 \left[\sum_{n=0}^{\infty} A_n (S_{n+2})_{\xi=-a} + \sum_{n=0}^{\infty} B_n \times \{ (S_{n+2})_{\xi=-a} - \alpha (S_{n+3})_{\xi=-a} \} \right] + C \quad (10)$$

(ii) on the line of centers of the holes,

$$(\sigma_x)_{x=0} = - \sum_{n=0}^{\infty} A_n [S_{n+2} + (-1)^n \{ S_{n+2} - 2\alpha S_{n+3} \}]_{\xi=0} + 2\alpha^2 \sum_{n=0}^{\infty} B_n (-1)^n (S_{n+4})_{\xi=0} \quad (11)$$

(iii) between the hole and the straight edge,

$$(\sigma_y)_{y=0} = C + \sum_{n=0}^{\infty} A_n [S_{n+2} - (-1)^n \{ 3S_{n+2} + 2(\alpha + \xi)S_{n+3} \}]_{\eta=0} + \sum_{n=0}^{\infty} B_n [2S_{n+2} + \xi S_{n+3} - (-1)^n \{ 2S_{n+2} + (4\alpha + \xi)S_{n+3} + 2\alpha(\alpha + \xi)S_{n+4} \}]_{\eta=0} \quad (12)$$

(iv) and along the wall of the hole,

$$(\sigma_\theta)_{r=c} = \{ (\sigma_x + \sigma_y) - \sigma_r \}_{r=c} = -4 \sum_{n=0}^{\infty} A_n (-1)^n \times (S_{n+2})_{\xi=\rho e i \theta} + 2 \sum_{n=0}^{\infty} B_n \{ S_{n+2} - (-1)^n \times (S_{n+2} + 2\alpha S_{n+3}) \}_{\xi=\rho e i \theta} + C - \frac{T}{\pi \rho} \times \left(-\frac{5}{8} + \cos \theta - \frac{3}{8} \cos 2\theta \right) \quad (13)$$

5. Results and discussions

Here, assuming that the plate is not free from extension or contraction in y -direction, that is, $C =$

$-(1-\nu)T$ and putting $\nu = 1/3$, we determine the constants $A_1, A_2, \dots, B_1, B_2, \dots$ and calculate the stresses.

5.1 The stress distributions

Denoting the diameter of the hole and the distance from the edge of the hole to the straight boundary by D and e , respectively (see Fig. 3), we have

$$\alpha = \rho(1 + 2e/D)$$

In Fig. 5, stress distributions calculated from Eqs. (10)~(13) for three cases, i.e.,

(b) Stress distribution in the plate of $c/b=0.15$ and $e/D=0.4$ ($\rho=0.15$ and $\alpha=0.27$)

(a) Stress distribution in the plate of $c/b=0.15$ and $e/D=0.8$ ($\rho=0.15$ and $\alpha=0.39$)

(c) Stress distribution in the plate of $c/b=0.10$ and $e/D=0.8$ ($\rho=0.10$ and $\alpha=0.26$)

Fig. 5

$c/b=0.15$, $e/D=0.8$ ($\rho=0.15$, $\alpha=0.39$)

$c/b=0.15$, $e/D=0.4$ ($\rho=0.15$, $\alpha=0.27$)

$c/b=0.10$, $e/D=0.8$ ($\rho=0.10$, $\alpha=0.26$)

are illustrated.

From the figures we can reach the following conclusions.

(1) $(\sigma_y)_{x=-a}$, the stress along the straight boundary is a tensile stress near the hole, but a compressive one distant from the hole, and the former increases, as c/b or e/D decreases; in other words, as the diameter of the hole is small or the hole approaches the boundary.

(2) $(\sigma_x)_{x=0}$, the stress on the line of centers of the holes (the narrowest cross section of the plate) is a tension which is not uniformly distributed and concentrated at the edge of the hole. The stress concentration increases as c/b decreases.

(3) $(\sigma_y)_{y=0}$, the stress on the space between the hole and the straight boundary is a tension which becomes maximum, adjacent to the boundary.

(4) $(\sigma_\theta)_{r=c}$, the hoop stress is a tensile one along the edge of the hole, except in the neighborhood of the most distant place from the boundary ($\theta=0$). The maximum hoop stress, which occurs at $\theta=90^\circ \sim 135^\circ$, is the largest among the above-mentioned stresses.

5.2 Stress along the wall of the hole

From the obtained results, it is concluded that we may take up only the hoop stress to research the strength of a plate joined by rivets. Then, for the examples, $c/b=\rho=0.5$, 0.15 and 0.25, the variations of the hoop stresses measured by the mean pressure p are shown in Fig. 6, from which it is understood that the point of the maximum hoop stress is at $\theta=90^\circ \sim 135^\circ$ (in the range of $c/b=0.05 \sim 0.25$, i.e., $e/D=0.2 \sim 1.0$), and approaches $\theta=90^\circ$ as e/D increases. Further we see that $(\sigma_\theta)_{r=c, \max}$ (the maximum

stress) increases with c/b , but it is stationary near $e/D=1.0$ and $(\sigma_\theta)_{r=c, \max}=(1.85 \sim 2.65)p$. Then we may conclude that the value of e/D larger than 1.0, namely, the standard proportion in machine design, is not very effective to strengthen the plate, because the maximum pressure on the wall of the hole is equal to $2p$ (see Fig. 3), and the effect of c/b has to be considered.

(b) Stress along the wall of the hole in the case of $c/b=\rho=0.15$

(a) Stress along the wall of the hole in the case of $c/b=\rho=0.05$

(c) Stress along the wall of the hole in the case of $c/b=\rho=0.25$

Fig. 6

6. Appendix

6.1 Demonstration of Eq. (A)

Since $\sum_{n=-\infty}^{\infty} (1/r_n^2)$ has the period $y=b$ (see Fig. 2), it must be expressed in Fourier's series as follows:

$$\sum_{n=-\infty}^{\infty} \frac{1}{r_n^2} = b_0 + \sum_{m=1}^{\infty} b_m \cos(2m\pi y/b)$$

in which the coefficients b_0, b_m are

$$\begin{aligned} b_0 &= \frac{1}{b} \int_0^b \left(\sum_{n=-\infty}^{\infty} \frac{1}{r_n^2} \right) dy = \frac{1}{b} \sum_{n=-\infty}^{\infty} \int_0^b \frac{dy}{x^2 + (y-nb)^2} \\ &= \frac{1}{b^2} \sum_{n=-\infty}^{\infty} \int_{-n}^{-n+1} \frac{d\eta}{\xi^2 + \eta^2} \\ &= \frac{1}{b^2} \int_{-\infty}^{\infty} \frac{d\eta}{\xi^2 + \eta^2} = \frac{1}{b^2} \frac{\pi}{\xi} \\ b_m &= \frac{2}{b} \int_0^b \left(\sum_{n=-\infty}^{\infty} \frac{1}{r_n^2} \right) \cos(2m\pi y/b) dy \\ &= \frac{2}{b^2} \int_{-\infty}^{\infty} \frac{\cos 2m\pi \eta}{\xi^2 + \eta^2} d\eta = \frac{2}{b^2} \frac{\pi}{\xi} e^{-2m\pi \xi} \end{aligned}$$

Then we have

$$\begin{aligned} x \sum_{n=-\infty}^{\infty} \frac{1}{r_n^2} &= \frac{\pi}{b} \left(1 + 2 \sum_{m=1}^{\infty} e^{-2m\pi \xi} \cos 2m\pi \eta \right) \\ &= \frac{1}{b} R_e \{ \pi \coth \pi \xi \} = \frac{1}{b} S_2 \end{aligned}$$

and, by differentiating the both sides,

$$x^3 \sum_{n=-\infty}^{\infty} \frac{1}{r_n^4} = \frac{1}{2b} (S_2 - \xi S_3)$$

In the same manner, putting

$$\sum_{n=-\infty}^{\infty} \frac{y_n}{r_n^2} = \sum_{m=1}^{\infty} a_m \sin(2m\pi y/b)$$

we get the coefficient

$$\begin{aligned} a_m &= \frac{2}{b} \int_0^b \left(\sum_{n=-\infty}^{\infty} \frac{y_n}{r_n^2} \right) \sin(2m\pi y/b) dy \\ &= \frac{2}{b} \int_{-\infty}^{\infty} \frac{\eta \sin 2m\pi \eta}{\xi^2 + \eta^2} d\eta = \frac{2\pi}{b} e^{-2m\pi \xi} \end{aligned}$$

from which the above formula becomes

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{y_n}{r_n^2} &= \frac{2\pi}{b} \sum_{m=1}^{\infty} e^{-2m\pi \xi} \sin 2m\pi \eta \\ &= -\frac{1}{b} I_m \{ \pi \coth \pi \xi \} = -\frac{1}{b} T_2 \end{aligned}$$

and, by differentiation, the following equation is derived:

$$x^2 \sum_{n=-\infty}^{\infty} \frac{y_n}{r_n^4} = \frac{1}{2b} \xi T_3$$

Thus, Eqs. (A) which have been used to transform Eqs. (6) are obtained.

6.2 Calculations of W_n and w_n

The value of W_n is calculated by one of the following formulas:

$$\begin{aligned} W_{n+2} &= \frac{(-1)^n n!}{\xi^{n+1}} \left\{ 1 - (-1)^n \sum C_m \right. \\ &\quad \times \frac{(2m-1) \cdots (2m-n)}{n!} \xi^{2m} \left. \right\} \dots \dots \dots (14) \\ W_{n+2} &= (-1)^n (2\pi)^{n+1} \sum_{m=1}^{\infty} m^n e^{-2m\pi \xi} \end{aligned}$$

In consideration of the convergence, we use either the former or the latter corresponding to the magnitude of $|\xi|$.

The value of w_n is obtained by substituting $(\xi + 2\alpha)$ for ξ in Eqs. (14) and that of $w_{n,0}$ is given by putting 2α for ξ in the equation.

The values of S_n and s_n in Eqs. (10)~(13) are found by

$$S_n = R_e[W_n], \quad s_n = R_e[w_n]$$

6.3 Practical calculations

Since W_n and w_n have too great dimensions and accordingly A_n and B_n become too small for calculation, we put

$$\begin{aligned} U_{n+2} &= (-1)^n \frac{\rho^{n+1}}{n!} W_{n+2}, \quad u_{n+2} = (-1)^n \frac{\rho^{n+1}}{n!} w_{n+2} \\ A_n' &= (-1)^n \frac{n!}{\rho^{n+1}} A_n, \quad B_n' = (-1)^n \frac{n!}{\rho^{n+1}} B_n \end{aligned}$$

and substituted U_n, u_n, A_n' and B_n' instead of W_n, w_n, A_n and B_n into Eqs. (8) and (10)~(14), respectively.

We determined twenty constants which consist of $A_1' \sim A_{11}'$ and $B_1' \sim B_9'$.

All the above calculations were made by use of FACOM 270-20.

References

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Discussion

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(1) If the condition is considered that the semi-infinite plate is free from stress at the straight boundary and the equation $(\sigma_{xn})_{\xi=-\alpha}=0$, can it be concluded that $(\sigma_x)_{\xi=-\alpha}=0$?

(2) The magnification by $n!/\rho^{n+1}$, i.e., that of

more than 10^{10} times, was used in the practical calculations as described in the section 6.3. Does it make the convergence of the series difficult?

Author's closure

(1) The stress components derived from the

stress function F_n are

$$(\sigma_{zn})_{\xi=-\alpha}=0, \quad (\tau_{zyn})_{\xi=-\alpha}=0$$

and the function is, from Eq. (7),

$$F = \sum_{n=0} F_n + \frac{C}{2} x^2$$

The stress components originated in the function F are

$$\sigma_x = \sum_{n=0} \sigma_{xn}, \quad \sigma_y = \sum_{n=0} \sigma_{yn} + C, \quad \tau_{zy} = \sum_{n=0} \tau_{zyn}$$

Then we have the following results:

$$(\sigma_z)_{\xi=-\alpha}=0, \quad (\tau_{zy})_{\xi=-\alpha}=0$$

(2) It is shown in Eqs. (14) that W_{n+2} has the order of $n!/\rho^{n+1}$. If the coefficients in the simultaneous equations of the first degree are calculated by W_{n+2} they have the same order and diverge. Then, putting

$$U_{n+2} = (-1)^n (\rho^{n+1}/u!) W_{n+2}$$

and using U_n instead of W_n , we find they are not divergent.

This is the same as to w_n .