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A Numerical Method Considering the Bauschinger Effect for Large Deflection Analysis of Elastic-Plastic Circular Plates*

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In numerical analysis of elastic-plastic problems of various structures, the Prandtl-Reuss equations, which are expressed by the stress-strain matrix $[D^p]$ introduced by P. V. Marcal and Y. Yamada, have been used very often with the finite element method or the finite difference method. In these equations, however, the Bauschinger effect is not considered. In order to investigate the elastic-plastic behavior of any structure subjected to cyclic loading, the Bauschinger effect should be taken into account.

In this paper, the Bauschinger effect is considered with the use of Ziegler's modification of Prager's kinematic hardening rule, and the stress-strain relations are expressed using the matrix $[D^p]$. Application is made with the finite difference method to the large deflection analysis of elastic-plastic circular plates subjected to a cyclically varying uniform pressure. The results obtained are compared with those using the Prandtl-Reuss equations for a few cycles of reversed loading.

1. Introduction

With the advent of high speed digital computers, the elastic-plastic problems of various structures which have been until recently thought to be very difficult to solve analytically are analyzed numerically with fairly good accuracy by the finite element method, the finite difference method and so on using the incremental theory of plasticity. In these elastic-plastic analyses, it is essentially important to construct the incremental stress-strain relations⁽¹⁾. At present, the Prandtl-Reuss equations expressed by the stress-strain matrix $[D^p]$ introduced by P. V. Marcal^{(2) (3)} and Y. Yamada^{(1) (4)} are applied very often to the elastic-plastic analysis^{(2) ~ (4) (6) (7)}. Also the stress-strain relations based on Hill's orthogonally anisotropic yield condition⁽⁵⁾ are expressed by Y. Yamada⁽¹⁾ in the same fashion as above.

The stress-strain relations mentioned above, however, are based on the isotropic strain hardening concept and therefore they do not explain the Bauschinger effect which seems to be one of the most important characteristics in plasticity. Since

the Bauschinger effect seems to play a very important role in cyclic plasticity, it is necessary to obtain the stress-strain matrix which contains this effect, in order to investigate the elastic-plastic behaviors of the structures subjected to cyclic loading.

Although at present the stress-strain relations considering the Bauschinger effect are not yet completely made clear, Prager's kinematic hardening model^{(8) (9)} and Ziegler's modification^{(10) (11)} of this model seem to be interesting proposals in engineering, because they explain the Bauschinger effect in a simple way. These models have been applied frequently to plastic analysis of the structures made of rigid plastic material^{(12) ~ (15)}. Also Ziegler's modification has been used in the elastic-plastic analysis of a plane stress problem, and it is shown that the results obtained coincide well with the experimental ones⁽¹⁶⁾. Therefore in this research, the stress-strain relations based on Ziegler's modification are expressed with the use of the matrix $[D^p]$, so that they could be easily applied to the elastic-plastic analysis of a continuum in the generalized three-dimensional stress field. And by applying these constitutive equations to the numerical method for solving the elastic-plastic problems of shells of revolution by the finite difference method, which is reported in authors' previous

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paper⁽⁷⁾, the large deflection analysis of circular plates subjected to a cyclically varying uniform pressure is performed. The results obtained by the kinematic hardening are compared with those by the isotropic hardening. Also the elastic-plastic behaviors of the circular plates are shown, when the pressure varies in a few cycles.

2. Constitutive relations

When the material under consideration is initially isotropic, the Mises yield condition is assumed. Namely,

$$3\sigma_{ij}'\sigma_{ij}'/2=\sigma_0^2 \quad (1)$$

where σ_{ij}' denotes the deviatoric stress components and σ_0 the yield stress in uniaxial tension (or compression).

According to the kinematic hardening concept by Prager⁽⁸⁾ and Ziegler⁽¹⁰⁾, the yield surface translates only in the stress space without expansion and rotation. Therefore the subsequent yield surface after some plastic deformation is expressed in the following form, where α_{ij}' denotes the deviatoric components of the translation α_{ij} of the yield surface.

$$3(\sigma_{ij}'-\alpha_{ij}')(\sigma_{ij}'-\alpha_{ij}')/2=\sigma_0^2(\text{constant}) \quad (2)$$

Let

$$S_{ij}=\sigma_{ij}-\alpha_{ij}, \quad S_{ij}'=\sigma_{ij}'-\alpha_{ij}' \quad (3)$$

then Eq. (2) is rewritten as follows:

$$3S_{ij}'S_{ij}'/2=\sigma_0^2(\text{constant}) \quad (4)$$

By using the yield condition given in Eq. (2) or (4) as the plastic potential, the plastic strain increments $d\epsilon_{ij}^p$ are expressed by the associated flow law as follows:

$$d\epsilon_{ij}^p=d\lambda S_{ij}', \quad d\lambda>0 \quad (5)$$

i. e.,

$$\{d\epsilon^p\}=d\lambda\{S'\} \quad (6)$$

where

$$\begin{aligned} \{d\epsilon^p\}^T &= [d\epsilon_x^p \ d\epsilon_y^p \ d\epsilon_z^p \ d\gamma_{yz}^p \ d\gamma_{zx}^p \ d\gamma_{xy}^p] \\ \{S'\}^T &= [S_x' \ S_y' \ S_z' \ 2S_{yz}' \ 2S_{zx}' \ 2S_{xy}'] \end{aligned} \quad (7)^*$$

Here suffix T , $\{ \}$ and $[\]$ denote the transpose, column vector and row vector, respectively. By using the same notations as Yamada's⁽¹¹⁾⁽⁴⁾, Hooke's law is written in the incremental forms as follows:

$$\{d\epsilon^e\}=[C^e]\{d\sigma\} \quad (8)$$

where $[C^e]$ is a 6×6 symmetric square matrix, and

$$\{d\sigma\}^T=[d\sigma_x \ d\sigma_y \ d\sigma_z \ d\tau_{yz} \ d\tau_{zx} \ d\tau_{xy}] \quad (9)$$

Accordingly, substitution of Eqs. (6) and (8) into the equation

$$\{d\epsilon\}=\{d\epsilon^e\}+\{d\epsilon^p\} \quad (10)$$

leads to the following relation:

$$[C^e]\{d\sigma\}=\{d\epsilon\}-d\lambda\{S'\} \quad (11)$$

Let

$$[D^e]=[C^e]^{-1} \quad (12)$$

then we obtain by using Eq. (11) the following equation:

$$\{d\sigma\}=[D^e]\{d\epsilon\}-[D^e]\{S'\}d\lambda \quad (13)$$

In the isotropic strain hardening rule, the equivalent stress σ_e is introduced and by assuming

that this equivalent stress is a function of $\int d\bar{\epsilon}^p$,

which is the integrated value of the equivalent plastic strain increment $d\bar{\epsilon}^p$ along the deformation path, the following hardening rule is applied⁽⁵⁾.

$$\sigma_e=H\left(\int d\bar{\epsilon}^p\right) \quad (14)$$

And the following value obtained from Eq. (14)

$$H'=d\sigma_e/d\bar{\epsilon}^p \quad (15)$$

is equated to the plastic tangent modulus of uniaxial tension (or compression). Although in the kinematic hardening rule the equivalent stress mentioned above is not applied directly, a similar relation to Eq. (15) is assumed between the increments⁽¹⁷⁾. Namely, as shown schematically in Fig. 1, let F denote the instantaneous yield surface, C its center and $\mathbf{n}_f$ the outward unit normal vector at the stress point P on F , then a similar relation to Eq. (15) is assumed between the component $d\sigma_f$ of the stress increment vector $d\sigma$ along $\mathbf{n}_f$ and the absolute value $d\bar{\epsilon}^p$ of the plastic strain increment vector $d\epsilon^p$.

$$C=d\sigma_f/d\bar{\epsilon}^p=d\sigma\mathbf{n}_f/|d\epsilon^p| \quad (16)$$

In this research, however, C is assumed to be a

Fig. 1 Stress and plastic-strain vectors on instantaneous yield surface (schematic)

Fig. 2 Stress-strain diagram of linear hardening material

* The shearing strains denoted by γ conform to the engineering definition.

constant, according to Prager's^{(8) (9)} and Ziegler's^{(10) (11)} kinematic hardening rules. This assumption implies that the stress-strain relation of uniaxial tension (or compression) is bilinear as shown in Fig. 2. Namely, the hardening model treated in this paper is called the linear kinematic hardening model⁽²⁰⁾.

On the other hand, from the definition of a unit vector

$$n_f = d\epsilon^p / |d\epsilon^p| = d\epsilon^p / d\epsilon^p \quad (17)$$

therefore, substitution of Eq. (17) into (16) yields

$$C = d\sigma d\epsilon^p / (d\epsilon^p)^2 \quad (18)$$

i. e.,

$$C d\epsilon_{ij}^p d\epsilon_{ij}^p = d\sigma_{ij} d\epsilon_{ij}^p \quad (19)$$

In the uniaxial tension along x direction

$$\left. \begin{aligned} d\epsilon_y^p &= d\epsilon_z^p = -d\epsilon_x^p/2 \quad (\text{the other } d\epsilon_{ij}^p = 0) \\ d\sigma_x &\neq 0 \quad (\text{the other } d\sigma_{ij} = 0) \end{aligned} \right\} \quad (20)$$

therefore

$$C = (2/3) d\sigma_x / d\epsilon_x^p = (2/3) H' \quad (21)$$

where H' is the plastic tangent modulus of the

stress-strain diagram in uniaxial tension (or compression).

If we substitute Eq. (21) into (19), then we obtain the following equation:

$$(2/3) H' d\epsilon_{ij}^p d\epsilon_{ij}^p = d\sigma_{ij} d\epsilon_{ij}^p \quad (22)$$

Substituting Eq. (5) into (22) and using Eq. (4), we obtain

$$d\lambda (4/9) H' \sigma_0^2 = S_{ij}' d\sigma_{ij} = \{S'\}^T \{d\sigma\} \quad (23)$$

The solution for $d\lambda$ obtained by substitution of Eq. (13) into (23) is

$$d\lambda = \{S'\}^T [D^e] \{d\epsilon\} / (4H' \sigma_0^2 / 9 + \{S'\}^T [D^e] \{S'\}) \quad (24)$$

If we substitute Eq. (24) into (13), then we can have the following relation between $\{d\sigma\}$ and $\{d\epsilon\}$

$$\{d\sigma\} = [D^p] \{d\epsilon\} \quad (25)$$

where

$$[D^p] = [D^p] - [D^e] \{S'\} \{S'\}^T [D^e] / (4H' \sigma_0^2 / 9 + \{S'\}^T [D^e] \{S'\}) \quad (26)$$

For an elastically isotropic body*, $[D^e]$ is expressed in the following form:

$$[D^e] = 2G \begin{pmatrix} (1-\nu)/(1-2\nu) & \nu/(1-2\nu) & \nu/(1-2\nu) & 0 & 0 & 0 \\ \nu/(1-2\nu) & (1-\nu)/(1-2\nu) & \nu/(1-2\nu) & 0 & 0 & 0 \\ \nu/(1-2\nu) & \nu/(1-2\nu) & (1-\nu)/(1-2\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix} \quad (\text{SYMMETRY}) \quad (27)$$

where G and ν denote the shear modulus and Poisson's ratio, respectively. Substituting Eq. (27) into the second term of the right hand side of Eq. (26), then we obtain the following expression:

$$[D^p] = [D^e] - \frac{2G}{S_0} \begin{pmatrix} S_x'^2 & S_x' S_y' & S_x' S_z' & S_x' S_{yz} & S_x' S_{zx} & S_x' S_{xy} \\ S_x' S_y' & S_y'^2 & S_y' S_z' & S_y' S_{yz} & S_y' S_{zx} & S_y' S_{xy} \\ S_x' S_z' & S_y' S_z' & S_z'^2 & S_z' S_{yz} & S_z' S_{zx} & S_z' S_{xy} \\ S_x' S_{yz} & S_y' S_{yz} & S_z' S_{yz} & S_{yz}^2 & S_{yz} S_{zx} & S_{yz} S_{xy} \\ S_x' S_{zx} & S_y' S_{zx} & S_z' S_{zx} & S_{yz} S_{zx} & S_{zx}^2 & S_{zx} S_{xy} \\ S_x' S_{xy} & S_y' S_{xy} & S_z' S_{xy} & S_{yz} S_{xy} & S_{zx} S_{xy} & S_{xy}^2 \end{pmatrix} \quad (\text{SYMMETRY}) \quad (28)$$

where

$$S_0 = (2\sigma_0^2/3)(H'/3G+1) \quad (29)$$

If we apply Eq. (28) to a plane stress problem in which

$$d\sigma_z = d\tau_{yz} = d\tau_{zx} = 0, \quad d\gamma_{yz} = d\gamma_{zx} = 0 \quad (30)$$

then we can get the following equation:

$$\begin{Bmatrix} d\sigma_x \\ d\sigma_y \\ d\tau_{xy} \end{Bmatrix} = \begin{bmatrix} E/(1-\nu^2) - S_1^2/S & \nu E/(1-\nu^2) - S_1 S_2/S & -S_1 S_6/S \\ \nu E/(1-\nu^2) - S_1 S_2/S & E/(1-\nu^2) - S_2^2/S & -S_2 S_6/S \\ -S_1 S_6/S & -S_2 S_6/S & E/2(1+\nu) - S_6^2/S \end{bmatrix} \begin{Bmatrix} d\epsilon_x \\ d\epsilon_y \\ d\gamma_{xy} \end{Bmatrix} \quad (\text{SYMMETRY}) \quad (31)$$

where

$$\left. \begin{aligned} S &= 4H' \sigma_0^2 / 9 + S_x' S_1 + S_y' S_2 + 2S_{xy} S_6 \\ S_1 &= \{E/(1-\nu^2)\} (S_x' + \nu S_y'), \quad S_2 = \{E/(1-\nu^2)\} (\nu S_x' + S_y'), \quad S_6 = ES_{xy}/(1+\nu), \\ S_x' &= \{2(\sigma_x - \alpha_x) - (\sigma_y - \alpha_y)\}/3, \quad S_y' = \{-(\sigma_x - \alpha_x) + 2(\sigma_y - \alpha_y)\}/3, \\ S_{xy} &= \tau_{xy} - \alpha_{xy} \end{aligned} \right\} \quad (32)$$

Next, let us treat the translation $\{d\alpha\}$ of the yield surface.

W. Prager^{(8) (9)} has put

$$d\alpha_{ij} = (2H'/3) d\epsilon_{ij}^p \quad (33)$$

* In this research, elastic isotropy is assumed, because of lack of information on the relations between anisotropic elastic constants and plastic deformation⁽¹⁷⁾.

It was pointed out^{(9) (21)}, however, that the above hardening rule had a contradiction in the subspace where the zero stress components were not considered. And another hardening rule excluding the above contradiction is proposed by H. Ziegler⁽¹⁰⁾. Namely, Ziegler assumes the following relation:

$$d\alpha_{ij} = d\mu(\sigma_{ij} - \alpha_{ij}), \quad d\mu > 0 \quad (34)$$

In this paper, Ziegler's modification shown in Eq. (34) is employed.

$d\mu$ in Eq. (34) is determined by the condition that the stress point must stay on the yield surface even after the stress increment $\{d\sigma\}$ is produced. This condition is obtained by the differentiation of the yield condition Eq. (4) and is expressed as follows:

$$S_{ij}'(d\sigma_{ij} - d\alpha_{ij}) = 0 \quad (35)$$

If we substitute Eq. (34) into (35), then we have

$$S_{ij}'d\sigma_{ij} = S_{ij}'S_{ij}d\mu = S_{ij}'S_{ij}'d\mu = (2\sigma_0^2/3)d\mu \quad (36)$$

Therefore,

$$d\mu = S_{ij}'d\sigma_{ij}/(2\sigma_0^2/3) \quad (37)$$

$$= (3/2\sigma_0^2)\{S'\}^T\{d\sigma\} \quad (38)$$

Substitution of Eq. (38) into (34) leads to

$$\{d\alpha\} = (3/2\sigma_0^2)\{S'\}\{S'\}^T\{d\sigma\} \quad (39)$$

where

$$\begin{aligned} \{S'\}^T &= [S_x \ S_y \ S_z \ S_{yz} \ S_{zx} \ S_{xy}] \\ \{d\alpha\}^T &= [d\alpha_x \ d\alpha_y \ d\alpha_z \ d\alpha_{yz} \ d\alpha_{zx} \ d\alpha_{xy}] \end{aligned} \quad (40)$$

When plastic flow proceeds, $d\lambda$ shown in Eq. (24) must be positive. If

$$d\lambda \leq 0 \quad (41)$$

then unloading occurs and the stress-strain relations obey Hooke's law as follows:

$$\{d\sigma\} = [D^e]\{d\epsilon\} \quad (42)$$

3. An application to the large deflection analysis of elastic-plastic circular plates

The stress-strain relations derived in the previous chapter are applied to the large deflection analysis of elastic-plastic circular plates and the numerical results obtained by the kinematic hardening are compared with those by the isotropic hardening, when the plates are subjected to a few cycles of reversed loading. The notations used in this chapter are the same as in authors' previous report⁽⁷⁾ for the solution of axisymmetric shell problems, unless otherwise specified. Also the increment is denoted by a superimposed dot.

3.1 Governing differential equations

The basic differential equations of the large deflection problem of shells of revolution reported in the previous paper⁽⁷⁾ are derived using the coordinates and notations of the shell as shown in

Fig. 3. In these equation, if we put

$$\phi = 0, \quad r = \xi, \quad z = 0 \quad (43)$$

then we obtain the governing differential equations of the circular plates as shown in Figs. 4. In the following, the outline of this procedure is given.

The suffix ξ in Fig. 3 being replaced by r , the equilibrium equations, the interacting relations of resultant forces and the compatibility equations for the large deflection problem of elastic-plastic circular plates are expressed in the nondimensional incremental forms as follows. It is noted in the following that the nondimensional quantities are denoted by bar and the radius a of the plate is taken as the reference length.

Equilibrium equations:

$$\ddot{\bar{\Psi}}_v' + \bar{r}\ddot{\bar{p}}_v = 0 \quad (44)$$

$$\ddot{\bar{\Psi}}_H' - (a/h)\ddot{\bar{N}}_\theta + \bar{r}\ddot{\bar{p}}_H = 0 \quad (45)$$

$$(\bar{r}\ddot{\bar{M}}_r)' - \cos\beta\ddot{\bar{M}}_\theta + \ddot{\bar{M}}_\theta \sin\beta\dot{\beta} + \bar{r}\dot{\bar{Q}} = 0 \quad (46)$$

Interacting relations:

$$\begin{aligned} \bar{r}\dot{\bar{Q}} &= \ddot{\bar{\Psi}}_v \cos\beta - \ddot{\bar{\Psi}}_H \sin\beta - (\ddot{\bar{\Psi}}_v \sin\beta \\ &\quad + \ddot{\bar{\Psi}}_H \cos\beta)\dot{\beta} \end{aligned} \quad (47)$$

$$\begin{aligned} \bar{r}\dot{\bar{N}}_\theta &= (h/a)\{\ddot{\bar{\Psi}}_v \sin\beta + \ddot{\bar{\Psi}}_H \cos\beta \\ &\quad + (\ddot{\bar{\Psi}}_v \cos\beta - \ddot{\bar{\Psi}}_H \sin\beta)\dot{\beta}\} \end{aligned} \quad (48)$$

Compatibility equations:

$$(\bar{r}\dot{\bar{\epsilon}}_{\theta m})' - \dot{\bar{\epsilon}}_{rm} \cos\beta + (\bar{\epsilon}_{rm} + a^2/h^2) \sin\beta\dot{\beta} = 0 \quad (49)$$

Fig. 3 Coordinates and notations of axisymmetric shell⁽⁷⁾

Fig. 4 Circular plates subjected to uniform pressure

$$\dot{\bar{\kappa}}_r = (a/h)\dot{\beta}', \quad \dot{\bar{\kappa}}_\theta = (a/h)\dot{\beta} \cos \beta / \bar{r} \quad \dots\dots\dots (50)$$

where

$$(\quad)' = d(\quad)/d\bar{r} \quad \dots\dots\dots (51)$$

Since the axisymmetric deformation of the circular plate is treated, shearing stresses $\tau_{r\theta}$ and $\tau_{\zeta\theta}$ are zero. Also in the analysis of thin circular plates it is usual that the normal stress σ_ζ along ζ direction and the shearing stress $\tau_{r\zeta}$ are neglected as small quantities^{(6) (18) (19)}. So in this paper, the plane stress problem in which only σ_r and σ_θ are not zero is assumed. Using Eq. (31), the stress-strain relations are expressed in the nondimensional form as follows:

$$\begin{Bmatrix} \dot{\bar{\sigma}}_r \\ \dot{\bar{\sigma}}_\theta \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{12} & \bar{A}_{22} \end{bmatrix} \begin{Bmatrix} \dot{\bar{\epsilon}}_r \\ \dot{\bar{\epsilon}}_\theta \end{Bmatrix} = [\bar{D}^p] \begin{Bmatrix} \dot{\bar{\epsilon}}_r \\ \dot{\bar{\epsilon}}_\theta \end{Bmatrix} \quad \dots\dots\dots (52)$$

where

$$\left. \begin{aligned} \bar{A}_{11} &= 1/(1-\nu^2) - \bar{S}_1^2/\bar{S} \\ \bar{A}_{12} &= \nu/(1-\nu^2) - \bar{S}_1\bar{S}_2/\bar{S} \\ \bar{A}_{22} &= 1/(1-\nu^2) - \bar{S}_2^2/\bar{S} \\ \bar{S}_1 &= \{1/(1-\nu^2)\}(\bar{S}_r' + \nu\bar{S}_\theta') \\ \bar{S}_2 &= \{1/(1-\nu^2)\}(\nu\bar{S}_r' + \bar{S}_\theta') \\ \bar{S} &= 4\bar{H}\bar{\sigma}_0^2/9 + \bar{S}_r'\bar{S}_1 + \bar{S}_\theta'\bar{S}_2 \\ \bar{S}_r' &= (2\bar{S}_r - \bar{S}_\theta)/3 \\ \bar{S}_\theta' &= (-\bar{S}_r + 2\bar{S}_\theta)/3 \end{aligned} \right\} \quad \dots\dots\dots (53)$$

For plastic flow to proceed, $\dot{\lambda}$ shown in Eq. (24) must be positive, and if

$$\dot{\lambda} = E\dot{\lambda} = (\bar{S}_1\dot{\bar{\epsilon}}_r + \bar{S}_2\dot{\bar{\epsilon}}_\theta)/\bar{S} \leq 0 \quad \dots\dots\dots (54)$$

the stress-strain relations are given by Hooke's law, i. e., the coefficients of Eq. (52) are given as

$$\bar{A}_{11} = \bar{A}_{22} = 1/(1-\nu^2), \quad \bar{A}_{12} = \nu/(1-\nu^2) \quad \dots\dots\dots (55)$$

According to the Euler-Bernoulli hypothesis, the total strain increments of the point with the nondimensional distance $\bar{\zeta}$ from the middle surface

where

$$\{\dot{Z}\}^T = [\dot{\beta}\dot{\Psi}_H], \quad (\quad)' = d(\quad)/d\bar{r} \quad \dots\dots\dots (62)$$

$$\left. \begin{aligned} a_{11} &= (a/h)(\bar{r}/\Delta) \{ \bar{B}_{13}(\bar{B}_{12}\bar{B}_{14} - \bar{B}_{13}\bar{B}_{22}) + \bar{B}_{14}(\bar{B}_{12}\bar{B}_{13} - \bar{B}_{11}\bar{B}_{14}) + \Delta\bar{B}_{33} \} \\ a_{12} &= (h/a)(\bar{r}/\Delta) (\bar{B}_{11}\bar{B}_{14} - \bar{B}_{12}\bar{B}_{13}), \quad a_{21} = -(\Delta a^3/h^3)a_{12}, \quad a_{22} = \bar{r}\bar{B}_{11} \\ b_{11} &= (h/a) \{ \bar{B}_{13}\bar{B}_{22} - \bar{B}_{12}\bar{B}_{14} \} (\bar{\Psi}_V \cos \beta - \bar{\Psi}_H \sin \beta) + a_{11}' \\ b_{12} &= (h/a) [\cos \beta (\bar{B}_{13}\bar{B}_{22} - \bar{B}_{11}\bar{B}_{24})/\Delta + \{ \bar{r}(\bar{B}_{11}\bar{B}_{14} - \bar{B}_{12}\bar{B}_{13})/\Delta \}'] \\ b_{21} &= \bar{B}_{12}(\bar{\Psi}_H \sin \beta - \bar{\Psi}_V \cos \beta) - (a^2/h^2) [\{ \bar{r}(\bar{B}_{11}\bar{B}_{14} - \bar{B}_{12}\bar{B}_{13}) \}' \\ &\quad - (\Delta'\bar{r}/\Delta)(\bar{B}_{11}\bar{B}_{14} - \bar{B}_{12}\bar{B}_{13}) + \cos \beta (\bar{B}_{11}\bar{B}_{24} - \bar{B}_{13}\bar{B}_{22})] \\ b_{22} &= (1 - \bar{r}\Delta'/\Delta)\bar{B}_{11} + \bar{r}\bar{B}_{11}' \\ c_{11} &= (h/a) \{ \bar{B}_{13}\bar{B}_{22} - \bar{B}_{12}\bar{B}_{14} \} \{ (\bar{\Psi}_V' - \beta'\bar{\Psi}_H) \cos \beta - (\bar{\Psi}_H' + \beta'\bar{\Psi}_V) \sin \beta \} / \Delta \\ &\quad + (h/a) [\{ (\bar{B}_{13}\bar{B}_{22} - \bar{B}_{12}\bar{B}_{14})/\Delta \}' + \cos \beta (\bar{B}_{12}\bar{B}_{24} - \bar{B}_{14}\bar{B}_{22})/(\Delta\bar{r})] \\ &\quad \times (\bar{\Psi}_V \cos \beta - \bar{\Psi}_H \sin \beta) + (a/h) [\cos \beta \{ \bar{B}_{13}(\bar{B}_{12}\bar{B}_{24} - \bar{B}_{14}\bar{B}_{22}) + \bar{B}_{14}(\bar{B}_{12}\bar{B}_{14} \\ &\quad - \bar{B}_{11}\bar{B}_{24}) + \Delta\bar{B}_{34} \} / \Delta] + (\cos^2 \beta / \Delta\bar{r}) \{ \bar{B}_{14}(\bar{B}_{14}\bar{B}_{22} - \bar{B}_{12}\bar{B}_{24}) + \bar{B}_{24}(\bar{B}_{11}\bar{B}_{24} \\ &\quad - \bar{B}_{12}\bar{B}_{14}) - \Delta\bar{B}_{44} \}] - \bar{\Psi}_H \cos \beta + (\bar{M}_\theta - \bar{\Psi}_H) \sin \beta \\ c_{12} &= (h/a) [\beta' \sin \beta (\bar{B}_{12}\bar{B}_{14} - \bar{B}_{13}\bar{B}_{22})/\Delta + \{ (\bar{B}_{13}\bar{B}_{22} - \bar{B}_{12}\bar{B}_{14})/\Delta \}' \cos \beta \\ &\quad + \cos^2 \beta (\bar{B}_{12}\bar{B}_{24} - \bar{B}_{14}\bar{B}_{22})/(\Delta\bar{r})] - \sin \beta \\ c_{21} &= \bar{B}_{12} \{ (\beta'\bar{\Psi}_H - \bar{\Psi}_V') \cos \beta + (\bar{\Psi}_H' + \beta'\bar{\Psi}_V) \sin \beta \} + (\Delta'\bar{B}_{12}/\Delta - \bar{B}_{12}') \\ &\quad - \bar{B}_{22} \cos \beta / \bar{r} \{ \bar{\Psi}_V \cos \beta - \bar{\Psi}_H \sin \beta \} + (a^2/h^2) [\cos \beta (\bar{B}_{12}\bar{B}_{14} - \bar{B}_{11}\bar{B}_{24}) \}' \\ &\quad + (\Delta' \cos \beta / \Delta) (\bar{B}_{11}\bar{B}_{24} - \bar{B}_{12}\bar{B}_{14}) + \cos^2 \beta (\bar{B}_{14}\bar{B}_{22} - \bar{B}_{12}\bar{B}_{24})/\bar{r} + (\Delta h/a) (\bar{\epsilon}_{rm} + a^2/h^2) \sin \beta \end{aligned} \right\} \quad \dots\dots\dots (63)$$

of the plate are written in the following expressions:

$$\begin{Bmatrix} \dot{\bar{\epsilon}}_r \\ \dot{\bar{\epsilon}}_\theta \end{Bmatrix} = \begin{Bmatrix} \dot{\bar{\epsilon}}_{rm} \\ \dot{\bar{\epsilon}}_{\theta m} \end{Bmatrix} + \bar{\zeta} \begin{Bmatrix} \dot{\bar{\kappa}}_r \\ \dot{\bar{\kappa}}_\theta \end{Bmatrix} = \{\dot{\bar{\epsilon}}_m\} + \bar{\zeta} \{\dot{\bar{\kappa}}\} \quad \dots\dots\dots (56)$$

Substitution of Eq. (56) into (52) yields

$$\begin{Bmatrix} \dot{\bar{\sigma}}_r \\ \dot{\bar{\sigma}}_\theta \end{Bmatrix} = \{\dot{\bar{\sigma}}\} = [\bar{D}^p] \{\dot{\bar{\epsilon}}_m\} + \bar{\zeta} [\bar{D}^p] \{\dot{\bar{\kappa}}\} \quad \dots\dots\dots (57)$$

Furthermore, the relations between the stress increments and the stress resultants per unit length are given as follows:

$$\left. \begin{aligned} \begin{Bmatrix} \dot{N}_r \\ \dot{N}_\theta \end{Bmatrix} &= \{\dot{N}\} = \int \{\dot{\bar{\sigma}}\} d\bar{\zeta} \\ \begin{Bmatrix} \dot{M}_r \\ \dot{M}_\theta \end{Bmatrix} &= \{\dot{M}\} = \int \{\dot{\bar{\sigma}}\} \bar{\zeta} d\bar{\zeta} \end{aligned} \right\} \quad \dots\dots\dots (58)^*$$

By substituting Eq. (57) into (58), the following relations are obtained.

$$\begin{Bmatrix} \dot{N} \\ \dot{M} \end{Bmatrix} = \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} & \bar{B}_{13} & \bar{B}_{14} \\ \bar{B}_{12} & \bar{B}_{22} & \bar{B}_{14} & \bar{B}_{24} \\ \bar{B}_{13} & \bar{B}_{14} & \bar{B}_{33} & \bar{B}_{34} \\ \bar{B}_{14} & \bar{B}_{24} & \bar{B}_{34} & \bar{B}_{44} \end{bmatrix} \begin{Bmatrix} \dot{\bar{\epsilon}}_m \\ \dot{\bar{\kappa}} \end{Bmatrix} \quad \dots\dots\dots (59)$$

where

$$\left. \begin{aligned} \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{12} & \bar{B}_{22} \end{bmatrix} &= \int [\bar{D}^p] d\bar{\zeta} \\ \begin{bmatrix} \bar{B}_{13} & \bar{B}_{14} \\ \bar{B}_{14} & \bar{B}_{24} \end{bmatrix} &= \int [\bar{D}^p] \bar{\zeta} d\bar{\zeta} \\ \begin{bmatrix} \bar{B}_{33} & \bar{B}_{34} \\ \bar{B}_{34} & \bar{B}_{44} \end{bmatrix} &= \int [\bar{D}^p] \bar{\zeta}^2 d\bar{\zeta} \end{aligned} \right\} \quad \dots\dots\dots (60)$$

Using Eqs. (45) through (50) and (59), the following differential equations are obtained through the same procedure as in authors' previous paper⁽⁷⁾.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \{\dot{Z}\}'' + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \{\dot{Z}\}' + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \{\dot{Z}\} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \quad \dots\dots\dots (61)$$

* The integration is performed along the plate thickness.

$$\begin{aligned}
c_{22} &= \beta' \bar{B}_{12} \sin \beta + \cos \beta (\Delta' \bar{B}_{12} / \Delta - \bar{B}'_{12} - \bar{B}_{22} \cos \beta / \bar{r}) \\
d_1 &= (h/a) (\sin \beta / \Delta) (\bar{B}_{12} \bar{B}_{14} - \bar{B}_{13} \bar{B}_{22}) \bar{\psi}'_v + (h/a) [\beta' \cos \beta (\bar{B}_{12} \bar{B}_{14} - \bar{B}_{13} \bar{B}_{22}) / \Delta \\
&\quad + \{(\bar{B}_{12} \bar{B}_{14} - \bar{B}_{13} \bar{B}_{22}) / \Delta\}' \sin \beta + (\sin \beta \cos \beta / \Delta \bar{r}) (\bar{B}_{14} \bar{B}_{22} - \bar{B}_{12} \bar{B}_{24}) - (a/h) \cos \beta] \bar{\psi}'_v \\
&\quad + (h/a) \{(\bar{B}_{12} \bar{B}_{13} - \bar{B}_{11} \bar{B}_{14}) / \Delta\} (\bar{r}^2 \bar{p}_H)' + (h/a) [\{(\bar{B}_{12} \bar{B}_{13} - \bar{B}_{11} \bar{B}_{14}) / \Delta\}' \\
&\quad + \cos \beta (\bar{B}_{11} \bar{B}_{24} - \bar{B}_{12} \bar{B}_{14}) / (\Delta \bar{r})] (\bar{r}^2 \bar{p}_H) \\
d_2 &= \bar{B}_{12} \sin \beta \bar{\psi}'_v + \{ \beta' \bar{B}_{12} \cos \beta - \sin \beta (\Delta' \bar{B}_{12} / \Delta - \bar{B}'_{12} - \bar{B}_{22} \cos \beta / \bar{r}) \} \bar{\psi}'_v \\
&\quad - \bar{B}_{11} (\bar{r}^2 \bar{p}_H)' - (\bar{B}_{11}' - \Delta' \bar{B}_{11} / \Delta + \bar{B}_{12} \cos \beta / \bar{r}) (\bar{r}^2 \bar{p}_H) \\
\Delta &= \bar{B}_{11} \bar{B}_{22} - \bar{B}_{12}^2
\end{aligned}$$

If the plate is subjected to uniform pressure p , the pressure increments are given as follows:

$$\dot{\bar{p}}_v = -\dot{\bar{p}} \cos \beta, \quad \dot{\bar{p}}_H = \dot{\bar{p}} \sin \beta \quad (64)^*$$

Next, let us consider the boundary conditions of various plates as shown in Figs. 4. In this research, the three cases shown in Figs. 4 are treated. Then, the respective boundary conditions at the edge of the plate are given as follows.

In Fig. 4 (a),

$$\dot{\bar{M}}_r = 0, \quad \dot{\bar{\psi}}_H = 0 \quad (65)$$

In Fig. 4 (b),

$$\dot{\bar{M}}_r = 0, \quad \dot{\bar{u}} = 0 \quad (66)$$

In Fig. 4 (c),

$$\dot{\beta} = 0, \quad \dot{\bar{u}} = 0 \quad (67)$$

The conditions at the center of the plate are all the same for the cases shown in Figs. 4, regardless of the edge conditions, i. e.,

$$\dot{\beta} = 0, \quad \dot{\bar{\psi}}_H = 0 \quad (68)$$

3.2 The numerical method of solution

In order to solve numerically the differential equations given in Eqs. (61), the finite difference method is employed. Though the details of the finite difference approximations are not mentioned here, the finite difference relations of Eqs. (61) are expressed compactly in the following expressions, if the same approximate relations for $\{\dot{Z}\}''$ and $\{\dot{Z}\}'$ as in the previous paper⁽⁷⁾ and numbers O and N are given to the points of $\bar{r}=0$ and $\bar{r}=1$, respectively, when the region $0 \leq \bar{r} \leq 1$ is divided equally into N segments.

For $j=1 \sim N-1$

$$[AF]_j \{\dot{Z}\}_{j-1} + [BF]_j \{\dot{Z}\}_j + [CF]_j \{\dot{Z}\}_{j+1} = \{DF\}_j \quad (69)$$

Moreover, the boundary conditions are expressed as follows:

$$\{\dot{Z}\}_0 = 0 \quad (70)$$

$$[AF]_N \{\dot{Z}\}_{N-2} + [BF]_N \{\dot{Z}\}_{N-1} + [CF]_N \{\dot{Z}\}_N = \{DF\}_N \quad (71)$$

In the above equations, $[AF]_j$, $[BF]_j$ and $[CF]_j$ are 2×2 matrices and $\{DF\}_j$ is a 2×1 matrix. Equations (69) through (71) are solved directly for $\{\dot{Z}\}_j$ with fairly good accuracy by the elimina-

tion procedure with the forward and backward sweepings, if the following relations:

$$\{\dot{Z}\}_j = \{X\}_j + [Y]_j \{\dot{Z}\}_{j+1} \quad (72)$$

are assumed.

If the increments are obtained, the final states of the deformed plate are determined by the integration of these increments. A modified Euler method is applied to the integration of the increments in the same fashion as in the previous paper⁽⁷⁾.

3.3 Numerical results and considerations

The authors have constructed earlier⁽⁷⁾ a numerical method for solving elastic-plastic problem of axisymmetric shells by using the Prandtl-Reuss equations. And as referred to in the above, if Eqs. (43) and (64) are used in this method, the large deflection problem of elastic-plastic circular plates subjected to uniform pressure can be analyzed for the case of the isotropic hardening. All we have to do in this research is to replace the stress-strain relations by Eqs. (52) and (53). Therefore, the computer program of the procedure could be constructed without difficulty. In this paper, however, whether the computer program for the kinematic hardening has failures or not is checked by the following consideration. Namely, if we take the particular case of $H'=0$ in the stress-strain relations based on the isotropic hardening and the kinematic hardening, then these stress-strain relations must be the same as the ones of the elastic-

Fig. 5 Pressure-deflection characteristics (elastic-perfectly-plastic material, $a/h=12.5$)

* The terms due to the total pressure change induced by the infinitesimal increment β of rotation are neglected.

perfectly-plastic material. So numerical computations for the circular plate movably supported at the edge are performed using the relation of $H' = 0$ and the same material constants as in the experiment by Y. Ohashi and S. Murakami⁽¹⁸⁾. The results obtained by the two methods for the load-deflection characteristics coincide well with each other as shown in Fig. 5. In the figure, the analytical results according to the deformation theory given by Ohashi and Murakami⁽¹⁸⁾ exist between $0 \leq \bar{p} \leq 0.4$ and a difference from the present solutions does not appear on the figure. Moreover, both analytical results explain fairly well the experimental ones.

It might be concluded from the above consideration that the computer program used in this research follows completely the theory employing the stress-strain relations mentioned above. In the numerical computations, the division number along the radial direction is taken as $N=20$, and the plate thickness is also divided equally into 20, in order to integrate numerically Eqs. (60) and so on by using the Simpson rule. Moreover, the pressure increment is chosen as $1/20$ of the first yield pressure* estimated by linear elastic solution⁽¹⁹⁾ at which a certain point of the plate tends to the elastic-plastic region for the first time. Though the division number of $N=20$ for the finite difference approximation is recommended by A. Shindo and others⁽⁶⁾ in the elastic-plastic analysis of circular plates with the Prandtl-Reuss equations, the appropriateness to choose N as 20 is also ascertained in this research by performing computations for other cases. Namely, the numerical results obtained by choosing N as 20, 50 and 100 are slightly different in the third figure of the effective number. So in this paper, the division number is taken as $N=20$, allowing for the computation time. The division number along the plate thickness is chosen as 20 for the same reason as in the previous paper⁽⁷⁾. Furthermore, as to the pressure increment, computations were performed for three cases of $1/10$, $1/20$ and $1/40$ of the first yield pressure. The results obtained for the three cases are slightly different in the third figure of the effective number for deflection, strains and stresses**. So in this research, $1/20$ is used by considering that numerical computations are performed for a few pressure cyclings.

Next, let us present the analytical results of such plates as shown in Figs. 4 obtained by

* In the following, this pressure is denoted by $\bar{p}_1$.

** Fluctuations of the results were larger for stresses, strains and deflection in the order mentioned.

assuming the linear hardening material as shown in Fig. 2 and the following material constants and configuration parameter.

$$E/\sigma_0 = 600, \quad \nu = 0.3, \quad H' = 0.1E, \quad a/h = 12.5 \quad \dots\dots\dots (73)$$

In Figs. 6 and 7 are shown the pressure-deflection characteristics of the movably supported and the immovably supported plates, respectively, when

Fig. 6 Pressure-deflection characteristics (movably supported)

Fig. 7 Pressure-deflection characteristics (immovably supported)

Fig. 8 Stress-strain behaviors of extremely outside points at center of movably supported plate

the pressure varies in three cycles between zero and the maximum value. And the results by the kinematic hardening are compared with those by the Prandtl-Reuss equations. From these figures, in the initial loading the differences between the results by the kinematic hardening and the isotropic hardening are comparatively small. This fact would mean that, for the elastic-plastic analysis of the loading processes without unloadings, the results obtained by the two hardening rules differ slightly from each other. Moreover, the deflection component due to the isotropic hardening seems to move on the same curve in unloading and reloading, and the influences of the Bauschinger effect are not so remarkable. Take for example, however, the stress-strain behaviors of the points ① and ①' at the center on the surfaces of the plate for all cycles, then we get Fig. 8. Namely, though the Bauschinger effect influences the stress-strain behaviors at certain portions, this effect on the deflection is so small that a difference does not appear on the figure.

In Fig. 9 are shown the results of the clamped plate. Though the differences between the results by the isotropic hardening and the kinematic hardening in the initial loading are small, a remarkable difference appears in the unloading and the reloading cycles, that is, a small hysteresis loop is formed in the case of the kinematic hardening as shown in the figure, compared with the case of the isotropic hardening where the pressure-deflection characteristics are entirely elastic. To investigate such behaviors in more details, let us draw the stress loci of the representative points on the surfaces of the plate as shown in Figs. 10. From Figs. 10, in the case of the kinematic hardening the Bauschinger effect is recognized clearly at the edge points, but in the case of the isotropic hardening the stress points move along the same loci between the start point of the first unloading and the finish point of the first loading-unloading cycle. Therefore it might be concluded that the pressure-deflection characteristics are entirely elastic in the case of the isotropic hardening.

Next, let us show the three-cycle pressure-deflection characteristics for completely reversed loading.

In Figs. 11(a) and (b) are shown the results of the movably supported plate by the two hardening rules. In the case of the isotropic hardening, the nonlinearities which would be caused by plastic deformation decrease within 2 or 3 cycles and the state of deformation shakes down gradually to the

Fig. 9 Pressure-deflection characteristics (clamped)

(a) Kinematic (b) Isotropic
Fig. 10 Stress loci of representative points

(a) Kinematic (b) Isotropic
Fig. 11 Three-cycle pressure-deflection characteristics
(at $r/a=0$, movably supported)

elastic state, but in the case of the kinematic hardening the characteristics tends to nearly one steady hysteresis loop.

In Figs. 12(a) and (b) are shown similar results of the immovably supported plate.

In Figs. 13(a) and (b) are shown the results of the clamped plate. In this case, for the kinematic hardening the second cycle overlaps nearly the third cycle. But in the isotropic hardening the behaviors are the same as in the other cases.

Numerical computations have been performed using the computing facilities at Data Processing Center of Kyoto University. For the case where the pressure varies between $0 \leq \bar{p} \leq 3\bar{p}_t$, it takes about 9 minutes in the isotropic hardening and about 9.5 minutes in the kinematic hardening, to complete three cycles, and for the case where the pressure varies between $-2\bar{p}_t \leq \bar{p} \leq 2\bar{p}_t$, it takes about 10 minutes in the isotropic hardening and about 11 minutes in the kinematic hardening.

4. Conclusions

In the present research, the stress-strain relations considering the Bauschinger effect are expressed in terms of the stress-strain matrix $[D^p]$, so that they could be easily applied to elastic-plastic analysis of a continuum in the generalized three-dimensional stress field. And they are employed in the numerical analysis of a large deflection problem of elastic-plastic circular plates which is considered to be one of plane stress problems. The results by the kinematic hardening are compared with those by the isotropic hardening, when the circular plates are subjected to a few pressure cyclings.

Although at present the constitutive equations which take account of the Bauschinger effect are not yet completely established, in this paper Ziegler's modification⁽¹⁰⁾⁽¹¹⁾ of Prager's kinematic hardening⁽⁸⁾⁽⁹⁾ is adopted. Since these models explain the Bauschinger effect in a simple way, they might be interesting proposals in engineering. And they have been

frequently applied to plastic analysis of structures. The authors conclude that since the stress-strain relations based on Ziegler's model are expressed with the use of the matrix $[D^p]$, the applicable field of this model would be broadened.

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(a) Kinematic (b) Isotropic
Fig. 12 Three-cycle pressure deflection characteristics
(at $r/a=0$, immovably supported)

(a) Kinematic (b) Isotropic
Fig. 13 Three-cycle pressure-deflection characteristics
(at $r/a=0$, clamped)

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Discussion

Y. SEGUCHI (Kobe University):

(1) Is it impossible as described below Eq. (16) to apply this model to the case of an arbitrary stress-strain curve?

(2) Are numerical computations iterated at every load step?

The discussor and others have reported in reference (6) that even in the initial loading numerical results are erroneous if iterations are not performed. Did the authors check the computation errors which would be comparatively large after a few pressure cyclings?

(3) The discussor would like to know the numerical treatment at the broken point shown in Fig. 2.

(4) Does reloading not happen locally at the other points of Fig. 10 even in the unloading process of the clamped plate with the isotropic hardening?

T. SEKIYA (University of Osaka Prefecture):

(5) Analytical results are compared in Fig. 5 with experimental ones by Ohashi and others for monotonous loading and fairly good coincidence is approved. For cyclic loading, however, which is particularly stressed in this paper, only theoretical solutions are given. What do the authors think about the agreement of theoretical results by the kinematic hardening with experimental ones?

Y. TOZAWA (Nagoya University):

(6) Since the kinematic hardening model takes account of the Bauschinger effect in a simple way, this model could be easily applied to stress analysis. On the other hand, however, this model does not exactly explain the behaviors of real materials. What knowledge would be furnished by the analysis based on this model?

Y. OHASHI and K. KAWASHIMA (Nagoya University):

(7) Since there is no fundamental restriction

to the numerical method of solution using the electronic computer in taking account of the complication in constitutive equations, it might not have essential meaning to find only the numerical results by means of such simple models as used in this paper.

The discussors think that the main object of numerical calculation should be to find the useful results in engineering based on the constitutive equations which may approximate the actual behavior of material as close as possible.

Though the kinematic hardening model is equivalent to the Masing model for uniaxial stressing, the latter has been shown to be inadequate to express precisely the behavior of a mild steel by the discussors^(*). In the results of deformation analysis in the unloading^(*) or cyclic loading^(*) of a plastically bent circular plate by using the actual stress-strain relation obtained from a mild steel and the corresponding Masing model, it has been shown that the result by the actual relation agrees well with the corresponding experimental result while the result by the Masing model does not.

(8) In the relation mentioned above, is it more significant to show the limitations of isotropic and kinematic models by comparing the numerical results obtained by using these models with the corresponding experimental ones than to present the results of numerical solution only?

K. OHJI (Osaka University):

(9) The authors presented in this paper a stress-strain relation based on the Prager-Ziegler kinematic hardening model, which includes the Bauschinger effect in it, expressed in terms of the

(*) Ohashi, Y. and Kawashima, K., *Bull. JSME*, Vol. 13, No. 56 (1970), p. 215.

(*) Ohashi, Y. and Kawashima, K., *ZAMM*, Bd. 50(1970), S. 467.

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stress-strain matrix so that it could be easily applied to stress analysis. As shown in this paper, the elastic-plastic behavior of elements under cyclic loading is one of the problems to be solved by using this type of constitutive equations. Therefore it might be essentially important to express the constitutive equations in a simplified mathematical form easily applicable to the analysis, by abstracting the essential part of the complicated cyclic behaviors of real materials. In this respect, the Prager-Ziegler model is considered to be an abstraction of the cyclic behavior at its steady-state stage, and is somewhat different from the usual isotropic hardening model. If the Prager-Ziegler model is taken as a basis of the discussion, for example, expansion of the plastic potential should be taken into account together with its translation, as a next step. Since a considerable amount of data on stress-strain or stress amplitude-strain amplitude behaviors under cyclic loading is available in the literature, the discussor hopes further improvement of the present model and its matrix representation.

Authors' closure

(1) Equation (28) corresponds to the linear kinematic hardening model as described below Eq. (16). By introducing nonlinearity to C or H' shown in Eq. (21), however, it is possible to adapt this model to an arbitrary uniaxial stress-strain relation as already suggested by H. Ziegler^{(10) (11)}.

(2) Though numerical computations are not iterated at every load step, the present method adopts a solution technique similar to the mean stiffness coefficients method developed by P. V. Marcal^{(2) (3)}, namely, the mean values of $[D^p]$ predicted by the variables at the load step under consideration and at the next step are used. Certainly, this method has a defect that the results obtained might be influenced by the magnitude of the external load increment. Therefore in this paper numerical computations are performed for various pressure increments, i. e., 1/10, 1/20 and 1/40 of the first yield pressure, and 1/20 is employed in consideration of accuracy of solutions and computation time.

(3) As far as the authors know, the following two methods have been proposed for numerical treatment of the yield point, which is a transition point from elastic state to elastic-plastic state, and both of them are successfully applied to elastic-plastic analysis of various structures. Namely, the one technique by Y. Yamada and others⁽⁴⁾ in

which the load increment is controlled so that the stress point passes through the above transition point, and the other by P. V. Marcal^{(2) (3)}, so-called the mean stiffness coefficients method which keeps always the magnitude of load increment constant as outlined in item (2) of authors' closure. The present technique belongs to the latter, therefore the stress point does not pass through the transition point. So in this paper, numerical computations are performed, as described in item (2) of authors' closure, accuracy of solutions being checked for various pressure increments.

(4) It is concluded by examining the numerical results for all the other points of Fig. 10 that the unloading process is entirely elastic.

(5) Though comparison of theoretical results with experimental ones is desired as Dr. Sekiya points out, there have been presented few experiments to investigate the incipient elastic-plastic behaviors of the circular plates subjected to cyclic pressure.

Since the kinematic hardening model adopted in this paper takes account of the Bauschinger effect in an extremely simple way, analytical results obtained by this model do not probably coincide well with experimental ones. In cyclic loading, however, this model may give better results than the isotropic hardening one, because it considers the Bauschinger effect which plays an essentially important role in cyclic plasticity. The authors aim to establish a hardening model which expresses more precisely the elastic-plastic behaviors of real materials under cyclic loading and to construct a corresponding solution technique.

(6) Since an idealized Bauschinger effect is considered in the kinematic hardening model, it could not precisely explain the complicated behaviors of real materials under cyclic loading. Therefore it is expected at present to establish a better hardening model than the kinematic one. The authors hope that this research would be interpreted as the first step to investigate analytically the elastic-plastic behaviors of plates and shells subjected to cyclic loading.

(7) Since the elastic-plastic behaviors of real materials are much complicated as Dr. Ohashi and Dr. Kawashima point out, the researches intended to express such behaviors of materials in multi-axial stress field in a mathematical form easily applicable to stress analysis might be still developing, although there has been already presented a considerable amount of such approaches as above. On the other hand, with the development of computation techniques associated with high speed

digital computers; numerical methods are being investigated at a fast pace. Therefore closer connection might be necessary between the two aspects mentioned above.

Though at present there are proposed considerably many numerical methods of solution adopting isotropic hardening models, such models neglect the Bauschinger effect. So in this paper, a numerical method is treated with the use of the simplest model considering this effect. The authors hope that the present method of solution would be interpreted as the first step to a further improved method adopting a more suitable model for stress-strain behaviors under cyclic loading.

(8) Though the object of this research is described in item (7) of authors' closure, the results by the kinematic hardening model which

takes account of the Bauschinger effect are compared with those by the isotropic hardening model which does not consider this effect, in order to show quantitatively the differences under cyclic loading between the results by the two hardening models. The authors have a plan to compare these results with those by a more suitable model for real materials.

(9) The authors wish to express their cordial thanks to Dr. Ohji for his useful discussion. Though the authors have been able to give in this paper only a numerical method of solution, they intend to develop this method to a solution technique employing a more suitable model for real materials under cyclic loading, referring to Dr. Ohji's suggestion and so on.