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Separation of Turbulent Boundary Layer*

(Wall Pressure Distribution Near Separation)

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This paper treats the separation of a two-dimensional incompressible turbulent boundary layer caused by a strong adverse pressure gradient. From the equations of motion, we derive an equation which enables the stream function ψ_m at the point of maximum shear stress to be calculated. By using the equation, the streamwise variation of ψ_m is examined on strongly retarded flows. It is indicated that ψ_m and the total pressure on the stream line specified by ψ_m are kept constant through a region extending from a section within the log-law region to the separation point. With these results concerning the behavior of the inner layer, two expressions are deduced for describing the pressure distribution in the constant- ψ_m region. A method is also proposed for predicting the pressure coefficient at separation. These theoretical results are compared with various experimental results and satisfactory agreements are obtained.

1. Introduction

In this paper we consider the separation of a two-dimensional incompressible turbulent boundary layer. Development of a satisfactory method for the analysis of turbulent boundary layer is a challenging problem in fluid mechanics. Up to date many methods have been proposed. However, judging from the report⁽¹⁾ of the Stanford-conference held to evaluate these previous methods, no method has fully succeeded in predicting the separation point. The purpose of this paper is to calculate the wall pressure distribution near the separation point on the basis of the two-layer model attributed to Stratford and Townsend.

The total pressure P on a stream line is written as follows, as can be derived from the equations of motion:

$$P(x, \psi) = P(x_\infty, \psi) + \int_{x_\infty}^x (\partial \tau / \partial y) ds$$

where x , y are the distances along and normal to the wall, ψ the stream function, τ the shear stress and s the distance along the stream line specified by ψ on which the value of $\partial \tau / \partial y$ is calculated. Let us consider the case of a boundary layer which separates in a short distance owing to a strong

adverse pressure gradient. Near the outer edge of the layer, the above integral is small enough to be neglected because $\partial \tau / \partial y \neq 0$ always holds. Then it is thought to be a good approximation to estimate the integral by using the value of $\partial \tau / \partial y$ of the second comparison boundary layer which upstream of the beginning point of pressure rise x_∞ , has the same characteristics as those of the layer under consideration and is supposed to develop without a pressure gradient downstream. If we know $P(x_\infty, \psi)$ and the static pressure p , we can calculate the velocity in the outer region by using the velocity profile of the comparison layer. On the wall, the momentum equation is reduced to $\partial \tau / \partial y = \partial p / \partial x$, from which the shear stress can be obtained if $\partial p / \partial y = 0$. Then, based on the mixing length hypothesis the velocity profile near the wall can be calculated in terms of $\partial p / \partial x$ and the wall shear stress τ_w .

Emphasizing the different features of the flows in the outer and inner regions and treating them separately, Stratford⁽²⁾ obtained the half power law velocity profile for the inner layer at separation and proposed a separation criterion for the purpose of determining the separation point.

Townsend⁽³⁾ assumed that the rate of the advection of turbulent energy is negligible and the production $\tau \partial u / \partial y$ is balanced mainly with the dissipation ϵ near the wall. Connecting ϵ with τ by dimensional analysis, he derived the modified

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mixing length theory from the turbulent energy equation neglecting the advection terms. Using the velocity profile calculated from the modified mixing length theory for the inner (equilibrium) layer, Townsend⁽⁴⁾ refined the criterion proposed by Stratford.

The theoretical model attributed to Stratford and Townsend was checked experimentally by Bradshaw and Galea⁽⁵⁾ and proved to be valid. However, the present authors find that it is not a good approximation to estimate ψ at the joining point of two layers by using the comparison profile unless the adverse pressure gradient is very steep. Moreover, the criterion can not be used except for a flow with zero wall stress, although the concept of the model itself will be applicable to a flow having a non-zero wall stress.

The present authors deduce an equation which enables the stream function ψ_m at the point of maximum shear stress to be calculated. By using the equation, the streamwise variation of ψ_m is examined on various flows with strong adverse pressure gradients. It is indicated that ψ_m and the total pressure on the stream line specified by ψ_m are kept constant through a region extending from a section within the log-law region to the separation point. With these results concerning the behavior of the inner layer, two expressions are deduced for describing the pressure distribution in the constant- ψ_m region: One is applicable to the log-law region and the other to the half-power-law region near the separation. A method is also proposed for predicting the pressure coefficient at the separation point.

We define the separation point as the point where $\tau_w = 0$ and determine it in each experiment by extrapolation over the region where τ_w is less than a quarter of τ_w at x_∞ . All data are taken from Vol. 2 of reference 1 except those about Fig. 1, step flows and y_m of Table 1.

2. Separation induced by a step

Let us consider the separation of a two-dimensional incompressible turbulent boundary layer on a smooth flat plate. The adverse pressure gradient is supposed to be produced by a step on the plate. The external flow far upstream of the step is specified by the velocity U_∞ and the static pressure p_∞ . We assume that the wall pressure begins to rise at x_∞ and the boundary layer separates at x_s where the wall pressure is p_s . The logarithmic velocity profile is assumed for the inner layer at x_∞ ;

$$U = U_\tau (\ln \xi + A) / K \quad (1)$$

where $U_\tau = (\tau_{w\infty} / \rho)^{1/2}$, $\xi = U_\tau Y / \nu$, $A = 2.0$ and $K =$

0.41. $\tau_{w\infty}$ is τ_w at x_∞ , ρ the density of the fluid, ν the kinematic viscosity, Y the distance normal to the wall at x_∞ .

2.1 The velocity profile in the inner layer subjected to an adverse pressure gradient

The velocity profile was calculated by both Townsend⁽³⁾ and Mellor⁽⁶⁾. Mellor obtained it based on the eddy viscosity hypothesis. Here, we deduce it based on Prandtl's mixing length theory postulating the next assumptions:

(i) The turbulent shear stress profile is linear in the inner layer and $\partial \tau / \partial y = 0$ at the point of maximum shear stress, $y = y_m$.

(ii) The maximum shear stress τ_m is constant through the region from x_∞ to x_s ; $\tau_m = \text{constant} = \tau_{w\infty}$.

The assumption (i) has already been established⁽⁴⁾⁽⁵⁾. The assumption (ii) seems to be reasonable judging from Fig. 1 which shows the streamwise variation of $\tau_m / \tau_{w\infty}$ measured by Schubauer and Klebanoff⁽⁷⁾. The positions $x = 17.5$ ft and 25.4 ft correspond to x_∞ and slightly upstream position of x_s respectively.

The assumptions (i) and (ii) give

$$\tau / \rho = \alpha y + u_\tau^2 \quad (2)$$

and

$$\alpha = (U_\tau^2 - u_\tau^2) / y_m \quad (3)$$

where $u_\tau = (\tau_w / \rho)^{1/2}$. Prandtl's mixing length theory gives

$$\partial u / \partial y = (\tau / \rho)^{1/2} / (Ky) \quad (4)$$

Integrating Eq. (4) by using Eqs. (2) and (3), we get

$$u = \frac{U_\tau}{K} \left\{ 2\sqrt{(1-t^2)(y/y_m) + t^2} + t \ln \frac{\sqrt{(1-t^2)(y/y_m) + t^2} - t}{\sqrt{(1-t^2)(y/y_m) + t^2} + t} \right\} + \text{constant} \quad (5)$$

where $t = u_\tau / U_\tau$. The constant of integration is a function of only x .

Equation (5) can be reduced to the usual log-

Fig. 1 Streamwise variation of $\tau_m / \tau_{w\infty}$ in the experiment of Schubauer and Klebanoff

arithmetic profile when $(1-t^2)(y/y_m)$ approaches zero and to the half power law profile near the separation point where t^2 becomes vanishingly small. Then, in order to simplify the analysis, we divide the streamwise region into two parts, i. e. the upstream region and the region near separation. In the upstream region, we assume the logarithmic velocity profile,

$$u = u_\tau (\ln \eta + A) / K \quad (6)$$

where $\eta = u_\tau y / \nu$. In the region near separation, we use

$$u / U_\infty = u^* (y / y_m)^{1/2} + u_t(x) \quad (7)$$

where $u^* = 2U_\tau / (KU_\infty)$. u_t corresponds to the constant in Eq. (5). Equation (7) is the asymptotic form of Eq. (5) as t^2 approaches zero.

In Fig. 2, Eq. (7) is compared with the measurement made by Schubauer and Klebanoff. Agreement is especially satisfactory near the separation point. We assume $u_t(x_s) = 0$ judging from Fig. 2. Townsend's velocity profile is of the same form as Eq. (5). Assuming that Eq. (6) always holds close to the wall when $u_\tau^3 \{ \nu \partial(\tau/\rho) / \partial y \}^{-1} > 20$, he determined u_t from the condition that Eqs. (5) and (6) join smoothly near the wall. Since $u_\tau^3 \{ \nu \partial(\tau/\rho) / \partial y \}^{-1}$ approaches zero in the region near separation, u_t determined in this manner may be inadequate. We determine u_t from the equations of motion. This is the reason why we use Eq. (7), that is, the simplified form of Eq. (5).

In the following analysis, we call the upstream region where we can use Eq. (6) the log-law region, and the region near separation where we use Eq. (7) the half-power-law region.

2.2 The log-law region

We derive an equation which enables us to estimate the stream function ψ_m at the point of the maximum shear stress from the equations of

Fig. 2 Comparisons between the half power law velocity profile and the measurement of Schubauer and Klebanoff

motion. Using the equation, we calculate the wall pressure distribution in the log-law region.

The turbulent boundary-layer momentum equations are

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \quad (8)$$

and

$$0 = -\frac{\partial p}{\partial y}, \quad y \leq y_m \quad (9)$$

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (10)$$

The boundary conditions are

$$u = v = 0 \quad \text{at} \quad y = 0 \quad (11)$$

Equation (6) gives

$$\frac{\partial u}{\partial x} = \frac{u_\tau'}{u_\tau} \left(u + \frac{u_\tau}{K} \right) \quad (12)$$

and

$$\frac{\partial u}{\partial y} = u_\tau / (Ky) \quad (13)$$

where prime means the differentiation with respect to x . Equations from (10) to (12) yield the y -component velocity,

$$v = -\frac{u_\tau'}{u_\tau} \left(U_\infty \psi + \frac{u_\tau}{K} y \right) = -\frac{u_\tau'}{u_\tau} u y \quad (14)$$

where

$$U_\infty \psi = \int_0^y u dy = \left(u - \frac{u_\tau}{K} \right) y \quad (15)$$

ψ is the stream function whose dimension is length. Equation (8) rearranged by using Eqs. (12) to (14) yields

$$\rho u^2 \frac{u_\tau'}{u_\tau} = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \quad (16)$$

From Eq. (16),

$$\rho u_m^2 \frac{u_\tau'}{u_\tau} = -\frac{\partial p}{\partial x}, \quad u_m = u(y_m) \quad (17)$$

since $\partial \tau / \partial y = 0$ at $y = y_m$. Equation (6) also gives

$$\int_0^y u^2 dy = (U_\infty \psi)^2 / y + (u_\tau / K)^2 y \quad (18)$$

Integrating Eq. (16) from 0 to y_m with respect to y by using Eqs. (15), (17) and (18), we get

$$-2 \frac{U_\infty \psi_m}{K} \frac{du_\tau}{dx} = (\tau_m / \rho) - u_\tau^2 \quad (19)$$

and putting $\tau_m = \tau_w$ in Eq. (19),

$$-2 \frac{U_\infty \psi_m}{KU_\tau} = (1-t^2) \left(\frac{dt}{dx} \right)^{-1} \quad (20)$$

Equations (19) and (20) give the fundamental basis for the present analysis.

It should be noted that ψ_m is zero at x_∞ and increases with x downstream. We can estimate the value of ψ_m from Eq. (20), knowing the values of U_∞ / U_τ , t and dt/dx in each section. Figure 3 (a) shows the variation of the right hand side of Eq. (20) estimated based on the measurement made by Bradshaw and Galea⁽⁵⁾ about a flow separation

induced by a step. The step is positioned at $x=0$. The separation point locates at $x=-2.5$ in as shown in Fig. 3 (b) which represents the original data about t and the pressure coefficient c_p . Equation (20) may not hold in two regions, i. e. one corresponding to $(0.95 < t \leq 1.0)$ where y_m seems to locate within the viscous sublayer and the other corresponding to $(0 \leq t < 0.50)$ near separation, since Eq. (6) cannot be used there. However, the variation of $(1-t^2)(-dt/dx)^{-1}$ in the region where $0.95 \geq t \geq 0.50$ is considered to correspond to that of ψ_m . It should be noted that there exists a wide region where ψ_m maintains an almost constant value. This fact means that the point of maximum shear stress in each section locates on the same stream line. Then, Fig. 3 (a) offers a firm evidence to the original authors' expectation that the outer edge of the inner (equilibrium) layer would coincide with a single stream line. The fact that the assumption (ii) is realized and there exists the constant- ψ_m region (where ψ_m keeps a constant value) suggests that τ_m is strongly affected by the upstream history and has a tendency to be maintained on a single stream line. Then, we can expect that, in a flow in which ψ_m begins to be kept constant at a certain section, it remains conserved through the region from the section to the separation point, as explained in Fig. 3 (c). The total pressure $P(x, \psi_m)$ also remains conserved in the constant- ψ_m region since $\partial\tau/\partial y = 0$ on the stream line specified by ψ_m , as can be seen from the discussion in section 1. In the step flow, since the constant- ψ_m region is rather wide, we assume that the total pressure on the stream line specified by the constant- ψ_m remains unchanged through the whole region from x_∞ to x_s , see Fig. 3 (c).

Equations (6) and (1) give

$$K \frac{U_\infty \psi_m}{\nu} = \eta_m (\ln \eta_m + A - 1) = \xi_m (\ln \xi_m + A - 1) \quad \dots\dots\dots (21)$$

From Eq. (21),

$$\eta_m = \xi_m, \text{ i. e. } u_\tau y_m = U_\tau Y_m \quad \dots\dots\dots (22)$$

where the quantities at x_∞ with subscript m mean the values corresponding to ψ_m , see Fig. 3 (c). From Eqs. (1), (6) and (22), we get

$$u_m/U_\infty = u_\tau/U_\tau \quad \dots\dots\dots (23)$$

Since the total pressure on the stream line specified by the constant- ψ_m is kept constant,

$$(U_m/U_\infty)^2 = c_p + (u_m/U_\infty)^2 = c_{ps} + u^{*2} \quad \dots\dots\dots (24)$$

where $c_p = 2(p - p_\infty)/(\rho U_\infty^2)$, $c_{ps} = c_p(x_s)$.

Equations (23) and (24) yield

$$\tau_w/\tau_{w\infty} = 1 - c_p/(c_{ps} + u^{*2}) \quad \dots\dots\dots (25)$$

or

$$c_p = (c_{ps} + u^{*2})(1 - t^2) \quad \dots\dots\dots (26)$$

Equation (25) is compared with measurements in Fig. 4. Head and Rechenberg's data⁽⁸⁾ are also about a flow with step induced separation. As can be seen from the process of its derivation, Eq. (25) may not hold close to separation. Judging both from Fig. 4 and from the fact that Eq. (20) holds in the region of $t \geq 0.50$, we suppose that Eq. (25) should be valid in the region where $\tau_w/\tau_{w\infty} \geq 0.25$. Hereafter we use $\hat{x}$ to specify the point where $\tau_w/\tau_{w\infty} = 0.25$.

(a) Streamwise variation of $(1-t^2)/(dt/dx)$

(b) Streamwise variation of t and c_p

(c) Explanation for $y_m(x)$, $\psi_m(x)$ and the stream line specified by the constant- ψ_m

Fig. 3

In the region between $\hat{x}$ and x_s , we assume that $\tau_w/\tau_{w\infty}$ varies linearly with c_p [see Eq. (45)];

$$\tau_w/\tau_{w\infty} = 0.25(c_{ps} - c_p)/(c_{ps} - \hat{c}_p) \quad \dots\dots\dots (27)$$

where $\hat{c}_p = c_p(\hat{x})$. Equation (27) is compared with experiments in Fig. 5.

In the constant- ψ_m region, Eq. (20) can be integrated, and we get

$$\ln \frac{1+t}{3(1-t)} = \frac{KU_r}{U_\infty \psi_m} (\hat{x} - x) \quad \dots\dots\dots (28)$$

Although the constant value of ψ_m can be determined from Eq. (20), it is more easily obtained if c_{ps} is known. Equations (6) and (24) give

$$\xi_m = \exp\{2(1 + c_{ps}/u^{*2})^{1/2} - A\} \quad \dots\dots\dots (29)$$

Substituting Eq. (29) into Eq. (21), we get

$$K \frac{U_\infty \psi_m}{\nu} = \{2(1 + c_{ps}/u^{*2})^{1/2} - 1\} \times \exp\{2(1 + c_{ps}/u^{*2})^{1/2} - A\} \quad \dots\dots\dots (30)$$

ψ_m can be calculated from Eq. (30).

Equations (26) and (28) enable us to calculate the wall pressure distribution in the log-law region, knowing the values of U_∞/ν , U_r/U_∞ and c_{ps} .

2.3 The half-power-law region

We call the region between $\hat{x}$ and x_s the half-power-law region.

By using Eqs. (8) to (11), we get the momentum integral equation for the inner layer,

Fig. 4 Wall shear stress

Fig. 5 Wall shear stress (near separation)

$$\frac{d}{dx} \int_0^{y_m} \rho u^2 dy - u_m \frac{d}{dx} \int_0^{y_m} \rho u dy = -\frac{dp}{dx} y_m + \tau_m - \tau_w \quad \dots\dots\dots (31)$$

Dividing Eq. (31) by $\rho U_\infty^2/2$ and putting $\tau_m = \tau_{w\infty}$,

$$2 \frac{d}{dx} \int_0^{y_m} (u/U_\infty)^2 dy - 2(u_m/U_\infty) \frac{d}{dx} \int_0^{y_m} (u/U_\infty) dy = -\frac{dc_p}{dx} y_m + 2[(U_r/U_\infty)^2 - (u_r/U_\infty)^2] \quad \dots\dots\dots (32)$$

The second term in the left hand side of Eq. (32) equals zero since the integral, i. e. ψ_m is independent of x . Then Eq. (32) becomes

$$2 \frac{d}{dx} \int_0^{y_m} (u/U_\infty)^2 dy = -\frac{dc_p}{dx} y_m + 2 \left(\frac{U_r}{U_\infty} \right)^2 \left[1 - \left(\frac{u_r}{U_r} \right)^2 \right] \quad \dots\dots\dots (33)$$

By using Eq. (7),

$$\int_0^{y_m} (u/U_\infty)^2 dy = y_m \{u_i^2 + (4/3)u_i^* u_i + (1/2)u^{*2}\} \quad \dots\dots\dots (34)$$

and

$$\frac{d\psi_m}{dx} = \frac{d}{dx} [y_m \{u_i + (2/3)u^*\}] = 0 \quad \dots\dots\dots (35)$$

From Eqs. (7) and (24),

$$c_p = c_{ps} + u^{*2} - (u_i + u^*)^2 \quad \dots\dots\dots (36)$$

By substituting Eq. (27) and Eqs. (34) to (36) into Eq. (33), we get

Fig. 6 Wall pressure distribution (Bradshaw and Galea's experiment)

Fig. 7 Wall pressure distribution (Head and Rechenberg's experiment)

$$\frac{-\{(5/9)u^{*2} + (2/3)u^*u_t\}(du_t/dx)}{\{u_t + (2/3)u^*\}^2[4(c_{ps} - \hat{c}_p) + u^{*2} - (u_t + u^*)^2]} = \frac{(U_\tau/U_\infty)^2}{2(c_{ps} - \hat{c}_p)\psi_m} \quad (37)$$

The differential equation (37) determines u_t . Carrying out the integration and putting $u_t(x_s) = 0$, we get

$$\begin{aligned} & 2 \frac{(A^2 - 1)A^2}{(A^2 - 1/9)^2} \ln\{(3/2)\sigma + 1\} - \frac{A^2 - 1}{A} \\ & \times \frac{3(3A + 1/2)}{(3A + 1)^2} \ln\left(1 + \frac{\sigma}{1 + A}\right) \\ & + \frac{A^2 - 1}{A} \frac{3(3A - 1/2)}{(3A - 1)^2} \ln \frac{A - 1}{A - 1 - \sigma} \\ & + \frac{A^2 - 1}{A^2 - 1/9} \left(\frac{1}{2} - \frac{1}{3\sigma + 2}\right) = 3 \frac{KU_\tau}{U_\infty \psi_m} (x_s - x) \quad (38) \end{aligned}$$

where

$$\sigma = u_t/u^* \quad (39)$$

$$A^2 = 1 + 4(c_{ps} - \hat{c}_p)/u^{*2} \quad (40)$$

We can calculate the wall pressure distribution in the half-power-law region by using Eqs. (36) and (38).

2.4 Comparison with experiments

The calculated results are compared with experiments in Figs. 6 and 7. In Fig. 6 $(x_s - x)/(x_s - \hat{x})$ is chosen as the abscissa. Then, it is independent of ψ_m . In Fig. 7, $\psi_m U_\infty/(3KU_\infty)$ is determined from both Eq. (20) (0.31in) and Eq. (30) (0.32in). The values are almost the same.

Agreements between the present theory and experiments are satisfactory. It is noted that Eq. (28) and Eq. (38) join smoothly at $\hat{x}$. The various results obtained in this chapter are derived by assuming that there exists a constant- ψ_m region where y_m in each section locates on a single stream line. All the results shown in Figs. 3(a), 4, 6 and 7 confirm that the assumption is valid through almost the whole pressure rising region in the flow with step induced separation. In this sense, it can be said that the flow with step induced separation shows the typical patterns about the behavior of the inner layer subjected to a strong adverse pressure gradient.

So far, we need not know the inviscid potential flow when calculating the wall pressure distribution. Therefore, the results obtained in this chapter can be expected to be applicable to the flow whose inner layer shows features similar to those of the step flow. In the next chapter, we examine these points about the flows on aerofoils or in diffusers, and discuss a method for predicting c_{ps} as well as the wall pressure distribution near separation.

3. The separation on aerofoil or in diffuser

3.1 The variation of ψ_m

Typical results calculated by using Eq. (20) are shown in Figs. 8 to 10. The data in Fig. 8 were measured by Schubauer and Klebanoff on an aerofoil. Each experiment in Figs. 9 and 10 was made in a flat plate diffuser by Schubauer and Spangenberg, and Ludwig and Tillmann respectively. The nondimensional number, $\pi_\delta^* = (\delta^*/\tau_{w\infty}) dp/dx$ may be a suitable parameter to represent the intensity of pressure gradient, where δ^* means the displacement thickness at x_∞ and dp/dx the maximum pressure gradient experienced by a flow upstream of separation. The value of π_δ^* is given

Fig. 8 Streamwise variation of ψ_m

Fig. 9 Streamwise variation of ψ_m

Fig. 10 Streamwise variation of ψ_m

in each figure. It is found from the examinations that in a flow in which $\pi_{\delta}^* > 1$, ϕ_m begins to have a constant value within the log-law region. In a flow in which $\pi_{\delta}^* < 1$, ϕ_m cannot have a constant value there. The values of t and the nondimensional number $\pi_{\phi} (= \phi_m(U_{\infty}/U_{\tau})dc_p/dx)$ at the section where ϕ_m begins to have a constant value are also given in each figure. In a flow having $\pi_{\delta}^* > 1$, there is a tendency that ϕ_m begins to have a constant value at the section where $\pi_{\phi} \approx 1$.

3.2 A method for predicting the pressure coefficient at separation

It has just been shown that in a flow subjected to a strong adverse pressure gradient ($\pi_{\delta}^* > 1$), ϕ_m becomes independent of x at the section where $\pi_{\phi} \approx 1$, but the constant- ϕ_m region is not generally so wide as that in the step flow. Then, instead of Eq. (24) we must use next equation in this chapter:

$$c_p + (u_m/U_{\infty})^2 = c_{ps} + u^{*2} \dots \dots \dots (41)$$

Let us examine Eq. (41) with the use of measurements. From Eq. (6),

$$(u_m/U_{\infty}) = U_{\tau}(\ln \eta_m + A)t/(KU_{\infty}) \dots \dots \dots (42)$$

From Eqs. (20) and (21),

$$\eta_m(\ln \eta_m + A - 1) = K^2 U_{\tau} (t^2 - 1) (dt/dx)^{-1} / (2\nu) \dots \dots \dots (43)$$

We use the following approximate expression in order to eliminate η_m from Eqs. (42) and (43):

$$\ln \eta + A = a \log \{ \eta (\ln \eta + A - 1) \} + b, \\ 10 \leq \eta (\ln \eta + A - 1) \leq 2 \times 10^2 ; a = 1.84, b = 1.59, \\ 2 \times 10^2 \leq \eta (\ln \eta + A - 1) \leq 5 \times 10^4 ; a = 1.98, b = 1.20$$

Then, we get

$$\left(\frac{u_m}{U_{\infty}} \right)^2 = \left(\frac{U_{\tau}}{KU_{\infty}} \right)^2 t^2 \left[a \log \left\{ \frac{K^2}{2} \frac{U_{\tau}}{\nu} (t^2 - 1) \right\} \right. \\ \left. \times \left(\frac{dt}{dx} \right)^{-1} + b \right]^2 \dots \dots \dots (44)$$

Table 1 shows the streamwise variation of

Table 1 Examination of validity of Eq. (41)

Station x ft		22	22.5	23	23.5	24
Data for calculation	c_p	0.303	0.341	0.375	0.411	0.442
	t	0.646	0.590	0.556	0.502	0.422
	$-dt/dx$ (ft) $^{-1}$	0.100	0.109	0.111	0.121	0.129
Calculated results	$(u_m/U_{\infty})^2$	0.259	0.217	0.193	0.156	0.113
	$c_p + (u_m/U_{\infty})^2$	0.562	0.558	0.568	0.567	0.555
	c_{ps}	0.535	0.531	0.541	0.540	0.528
	y_m in	0.97	1.13	1.20	1.33	1.59
y_m (Exp.) in		0.9~1.1	Not clear	1.2~1.4	1.3~1.5	1.5~1.7
$U_{\infty}/\nu = 10^6$ (ft) $^{-1}$, $U_{\tau}/U_{\infty} = 3.40 \times 10^{-2}$, c_{ps} (Exp.) = 0.52						

Table 2 Separation pressure coefficient calculated from present method by using characteristics at $\hat{x}$

Data Source name of datum	Kind of flow Pressure gradient π_{δ}^*	U_{∞}/ν	U_{τ}/U_{∞}	c_p	$-dt/dx$	c_{ps}			
		$x = x_{\infty}$	$x = \hat{x}$			-50%	0%	+50%	Experiment
Schubauer -Klebanoff	Aerofoil	1.00×10^6 ft $^{-1}$	3.40×10^{-2}	0.411	0.12 ft $^{-1}$	0.56	0.54	0.53	0.52
	3.8								
Newman	Aerofoil	9.87×10^6 ft $^{-1}$	2.95×10^{-2}	0.496	0.21 ft $^{-1}$	0.60	0.58	0.57	0.57
	3.2								
Ludwig -Tillmann	Flat wall diffuser	2.14×10^6 m $^{-1}$	3.75×10^{-2}	0.464	0.30 m $^{-1}$	0.63	0.61	0.59	0.55
	1.3								
Schubauer -Spangenberg	Flat wall diffuser	4.78×10^5 ft $^{-1}$	3.97×10^{-2}	0.384	0.23 ft $^{-1}$	0.54	0.52	0.51	0.54
	Flow A, 2.0								
Schubauer -Spangenberg	Flat wall diffuser	4.86×10^5 ft $^{-1}$	3.94×10^{-2}	0.448	0.088 ft $^{-1}$	0.64	0.62	0.60	0.62
	Flow B, 1.2								
Schubauer -Spangenberg	Flat wall diffuser	4.86×10^5 ft $^{-1}$	4.10×10^{-2}	0.474	0.047 ft $^{-1}$	0.71	0.68	0.67	0.69
	Flow E, 0.52								
Bradshaw -Galea	Step flow	1.60×10^4 in $^{-1}$	4.06×10^{-2}	0.294	0.17 in $^{-1}$	0.36	0.34	0.33	0.35
	14.4								
Head -Rechenberg	Step flow	4.32×10^4 in $^{-1}$	4.59×10^{-2}	0.344	0.37 in $^{-1}$	0.44	0.41	0.40	0.41
	8.2								
Moses case 3	Axisymmetric diffuser	4.67×10^5 ft $^{-1}$	4.87×10^{-2}	0.500	0.63 ft $^{-1}$	0.70	0.66	0.64	0.70
	1.8								
Fraser flow B	Conical diffuser	1.12×10^6 ft $^{-1}$	4.02×10^{-2}	0.430	0.54 ft $^{-1}$	0.59	0.57	0.55	0.69
	5.4								

$c_p + (u_m/U_\infty)^2$ calculated from Eq. (44) in the constant ψ_m region of the flow measured by Schubauer and Klebanoff. It is noted that $c_p + (u_m/U_\infty)^2$ has almost constant values through the section examined. The calculated values of c_{ps} from Eq. (41) agree well with experimental values. The values of y_m obtained from Eq. (42) also agree with those of the experiment. The examinations made in the light of measurements carried out by Schubauer and Spangenberg (flow A) and by Ludwig and Tillmann show results similar to those in Table 1. Then, by using Eqs. (41) and (44), we can predict c_{ps} if we know the values of $c_p(x)$ and $t(x)$ in the constant- ψ_m region. Even in a flow having $\pi_\delta^* < 1$, $d\psi_m/dx$ tends to zero near $\hat{x}$, see Fig. 9. Therefore, we can expect that if we use Eq. (44) at $\hat{x}$, c_{ps} can be well predicted even in a case of flow having $\pi_\delta^* < 1$. The values of c_{ps} calculated at $\hat{x}$ are compared with those of experiments in Table 2. The calculated values in the column of 0% are obtained by using experimental values of dt/dx . The values labeled $\pm 50\%$ are obtained by using the values of dt/dx having $\pm 50\%$ relative errors with respect to the corresponding experimental values. Agreement is as good even in the case

Fig. 11 Examination of Eq. (25) in the case of flows on aerofoil or in diffuser

Fig. 12 Wall pressure distribution near separation

of $\pi_\delta^* < 1$ as in the case of $\pi_\delta^* > 1$.

It should be emphasized that by the present method we can predict c_{ps} without knowing the pressure distribution in the region between $\hat{x}$ and x_s .

3.3 Wall pressure distribution near separation

From Eqs. (41) and (42), we get

$$t^2 = 0.25(c_{ps} + u^{*2} - c_p) / (c_{ps} + u^{*2} - \hat{c}_p) \dots (45)$$

If putting $\hat{c}_p = 0.75(c_{ps} + u^{*2})$, Eq. (45) becomes Eq. (25) and when neglecting u^{*2} , Eq. (45) becomes Eq. (27). As shown in Fig. 11, Eq. (25) seems to be reasonable also for the flows in this chapter. Then, we use Eq. (25) instead of Eq. (45) and use Eq. (27) in the half-power-law region.

If we obtain ψ_m and c_{ps} from the method described in section 3.2, we can calculate the wall pressure distribution in the constant- ψ_m region by using Eqs. (28) and (38). The calculated results are compared with experimental results in Figs. 12 to 14. In each figure, $\hat{x}_{cal}$ indicates the section corresponding to the calculated value of $\hat{c}_p$. Equations (28) and (38) are joined at $\hat{x}_{cal}$. Each com-

Fig. 13 Wall pressure distribution near separator

Fig. 14 Wall pressure distribution near separation

parison shows good agreement in the half-power-law region.

By differentiating Eq. (25), we get

$$(dt/dx)_{x=\hat{x}} = -(c_{ps} + u^{*2})^{-1} (dc_p/dx)_{x=\hat{x}} \quad \dots (46)$$

With the use of Eq. (46), we get

$$\Psi \equiv \phi_m U_\infty / (3KU_s) = (c_{ps} + u^{*2}) / \{8(dc_p/dx)_{x=\hat{x}}\} \quad \dots (47)$$

The values of Ψ calculated from both Eqs. (20) and (47) are given in each figure. We often assume the values of c_{ps} and dc_p/dx when designing a diffuser, and in such a case, Eq. (47) may be useful.

4. Conclusions

(1) By assuming the linear shear stress profile, the velocity profile in the inner layer is derived on the basis of Prandtl's mixing length hypothesis.

(2) Equation (19) which enables us to estimate ϕ_m , the stream function at the point of maximum shear stress, is deduced from the equations of motion.

(3) By using Eq. (20), the streamwise variation of ϕ_m is examined on various flows with strong adverse pressure gradients $[(\delta^*/\tau_{w\infty})dp/dx > 1]$. It is pointed out that ϕ_m and the total

pressure on the stream line specified by ϕ_m are kept constant through the region extending from a section within the log-law region to the separation point.

(4) Equations (28) and (38) are deduced from the equations of motion for calculating the wall pressure distribution near separation.

(5) A method for predicting the pressure coefficient at separation is proposed.

Comparisons between these theoretical results and those of experiments show good agreement.

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