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Deformation of a Supported Long Cylindrical Shell of Aluminum Alloy under Internal Pressure*

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In the previous paper, the deformation under internal pressure was analysed for a supported short cylinder of aluminum alloy as the material having non-linear stress-strain relation. On the other hand, it is well known that the mode of elastic deformation of thin cylindrical shells of finite length under internal pressure is different according to the ratio $l/\sqrt{ha}$, where $2l$, $2h$ and a are length, wall thickness and radius of the cylinder, respectively.

Such a trend may be expected to be modified for the above mentioned material. In the present paper, a long cylinder with the same size as that mentioned above except its length was analysed in the same way as in the previous paper, in order to investigate the trend of deformation mode in such materials.

As the result of analysis, it may be concluded that the mode of the long cylinder varies gradually with an increase of internal pressure and approaches that of a short cylinder, in the non-linear range of stress-strain relation.

1. Introduction

In the previous papers, fundamental equations for analysing the deformation under internal pressure of thin cylindrical shells made of material having the non-linear stress-strain relation were derived according to the deformation theory⁽¹⁾⁽³⁾. The result of theoretical analysis of the deformation of a supported short cylinder of aluminum alloy subjected to internal pressure by means of these equations was confirmed to have a sufficient accuracy for approximating the result of corresponding experiment⁽²⁾⁽³⁾.

It is well known that the mode of elastic deformation of cylindrical shells of finite length under internal pressure varies according to the value of ratio $l/\sqrt{ha}$, where $2l$, $2h$ and a denote the length, wall thickness and radius of the cylinder, respectively⁽⁴⁾. That is, though the deflection of cylinder decreases monotonously from the central portion to the ends for the ratio less than a certain value, for larger values of the ratio than that, the maximum deflection appears at a location near the end and the deflection decreases slightly towards the center with a point of inflection and monotonously towards the ends.

Such a deformation mode of a long cylinder in the state of linear elastic deformation is expected to be modified for materials having non-linear stress-strain relations in such a way that the location of maximum deflection near the end approaches the center with an increase of internal pressure. Therefore, in order to investigate the variations of deformation mode, distributions of bending moment and membrane force and others, the deformation under internal pressure of the same cylinder as in the former case except its length is analysed by the proposed equations and examined in comparison with the corresponding experimental result.

2. Derivations of fundamental equations and a system of non-linear differential equations of first order

Since the derivation of the fundamental equations was performed in the previous paper⁽¹⁾, it is only described as short as possible. As shown in Fig. 1, a coordinate system x , θ and z is established for a

Fig. 1 Cylindrical shell and its co-ordinate system

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cylindrical shell in which the origin lies in the middle layer of wall thickness in the center of its axial length. The state of plane stress may be assumed in the x - θ plane for a sufficiently thin shell.

Equilibrium conditions in the infinitesimal deformation of a thin cylindrical shell with mean radius a , wall thickness $2h$ and length $2l$ under internal pressure p are expressed by the equations

$$\frac{dT_x}{dx} = 0 \quad (1)$$

$$\frac{dQ_x}{dx} + \frac{T_\theta}{a} = p \quad (2)$$

$$\frac{dM_x}{dx} - Q_x = 0 \quad (3)$$

where M_x , T_x and T_θ denote the components of bending moment and membrane force, and Q_x expresses the shear force.

According to Kirchhoff's hypothesis, the strain components ε_x and ε_θ may be expressed by means of the components of unit elongation e_x and e_θ as well as the variations of principal curvatures α_x and α_θ of the middle surface. Since the value of α_θ is negligibly small for infinitesimal deformation, the strain components may be expressed as

$$\varepsilon_x = e_x + z\alpha_x, \quad \varepsilon_\theta = e_\theta \quad (4)$$

On the other hand, e_x , e_θ and α_x are related to the radial and axial displacements w and u by the following relations

$$e_x = \frac{du}{dx}, \quad e_\theta = -\frac{w}{a}, \quad \alpha_x = -\frac{d^2w}{dx^2} \quad (5)$$

By eliminating w from (5), the condition of compatibility is found as

$$\frac{d^2e_\theta}{dx^2} = \frac{\alpha_x}{a} \quad (6)$$

In order to reduce the order of derivative in Eq. (6) for convenience of numerical analysis, the equation is divided into the following two equations

$$\frac{de_\theta}{dx} = \frac{s}{a} \quad (7)$$

and

$$\frac{ds}{dx} = \alpha_x \quad (8)$$

by introducing a new parameter s related to the derivative of w

$$s = -\frac{dw}{dx} \quad (9)$$

As is well known, the components of bending moment and membrane force in the thin shell are related to the components of stress σ_x and σ_θ as follows:

$$\left. \begin{aligned} T_x &= \int_{-h}^h \sigma_x dz, & T_\theta &= \int_{-h}^h \sigma_\theta dz, \\ M_x &= \int_{-h}^h \sigma_x z dz, & M_\theta &= \int_{-h}^h \sigma_\theta z dz \end{aligned} \right\} \quad (10)$$

In solving the simultaneous equations consisting of the equilibrium conditions (1), (2), (3) and the compatibility conditions (7), (8), the stress components σ_x and σ_θ contained in (10) should be expressed by the strain components ε_x and ε_θ .

For a material having a discrete yield point, the incompressibility of material in the plastic region is assumed to hold with high accuracy. In the same consideration, the incompressibility has been assumed in the analysis of deformation of a material having non-linear stress-strain relation. However, such an approximation will lead to a serious error in the deformation analysis in the range of small strains.

In order to eliminate such an error, the present authors⁽⁵⁾ introduced a parameter c expressing the compressibility of material and an effective strain for the plane stress state $\bar{\varepsilon}_c$ in which the compressibility of material was taken into account by the following definition with the parameter c

$$\frac{3}{4}\bar{\varepsilon}_c^2 = \frac{1+c+c^2}{(1+2c)^2}(\varepsilon_x + \varepsilon_\theta)^2 - \varepsilon_x \varepsilon_\theta \quad (11)$$

The parameter c is defined with the effective stress and strain in the ordinary forms, $\bar{\sigma}$ and $\bar{\varepsilon}$, and the coefficient of volumetric strain $\alpha (=e/\sigma, e: \text{mean strain}, \sigma: \text{mean stress})$ by the next relation

$$c = \frac{2}{3} \frac{\bar{\sigma}}{\bar{\varepsilon}} \alpha \quad (12)$$

and it may be obtained in relation to $\bar{\varepsilon}$ from the calibration test in the uniaxial tension.

According to Hencky's stress-strain relation, the stress components σ_x and σ_θ may be expressed by the strain components ε_x and ε_θ as follows⁽⁵⁾:

$$\left. \begin{aligned} \sigma_x &= \frac{2}{3} \frac{\bar{\sigma}}{\bar{\varepsilon}_c} \left(\frac{2+c}{1+2c} \varepsilon_x + \frac{1-c}{1+2c} \varepsilon_\theta \right) \\ \sigma_\theta &= \frac{2}{3} \frac{\bar{\sigma}}{\bar{\varepsilon}_c} \left(\frac{1-c}{1+2c} \varepsilon_x + \frac{2+c}{1+2c} \varepsilon_\theta \right) \end{aligned} \right\} \quad (13)$$

The stress-strain relation is expressed by a polynomial form as

$$\bar{\sigma} = a_0 \bar{\varepsilon} + a_1 \bar{\varepsilon}^2 + a_2 \bar{\varepsilon}^3 + a_3 \bar{\varepsilon}^5 \quad (14)$$

for convenience of integrating the relation (10) analytically. The components of the unit elongation e_x , e_θ and the variations of principal curvatures α_x and α_θ of the middle surface are transformed into the following expressions by introducing new parameters E_1 , β , ω_1 , and ω_0

$$\left. \begin{aligned} e_x &= \frac{2E_1}{\sqrt{3}} \left[\cos \left(\omega_1 \mp \frac{\pi}{3} \right) + c \cos \omega_1 \right] \\ e_\theta &= \frac{2E_1}{\sqrt{3}} \left[\cos \left(\omega_1 \mp \frac{\pi}{3} \right) + c \cos \omega_1 \right] \\ \alpha_x &= \frac{2E_1 \beta}{\sqrt{3} h} \left[\cos \left(\omega_0 \mp \frac{\pi}{3} \right) + c \cos \omega_0 \right] \\ \alpha_\theta &= \frac{2E_1 \beta}{\sqrt{3} h} \left[\cos \left(\omega_0 \mp \frac{\pi}{3} \right) + c \cos \omega_0 \right] \end{aligned} \right\} \quad (15)$$

As mentioned above, since α_x is not equal to zero identically while $\alpha_\theta = 0$, ω_0 should be a constant, the components of bending moment and membrane force may be found as functions of E_1 , β and ω_1 by integrating the stress components and their moments through thickness by taking into account (11) and

(15). The concrete forms of these expressions are omitted here for saving space; they may be found in the previous paper⁽¹⁾⁽³⁾.

By substituting these expressions and (15) into equations (1), (2) and (8), a system of non-linear differential equations of first order is obtained. The system may be expressed symbolically in the following form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} E_1' \\ \beta' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \dots\dots\dots (16)$$

where the prime means the differentiation with respect to x , for example, $E_1' = dE_1/dx$. Elements of the coefficient matrix (a_{ij}) and the column vector (b_i) are of very complicated expressions in terms of the coefficients $a_0 \sim a_3$ and the parameters E_1 , β and ω_1 . These expressions also have been given in the previous papers⁽¹⁾⁽³⁾.

On the other hand, a condition

$$T_x = \text{const} \dots\dots\dots (17)$$

is obtained by integrating Eq. (1), where the constant may be determined by the boundary condition. Accordingly, one of the parameters contained in Eq. (16) cannot vary independently. When ω_1 is considered as a function of independent variables E_1 and β , the first derivatives of E_1 and β may be obtained by solving the simultaneous algebraic equations (16) with respect to them.

These derivatives and the first derivatives of Q_x and s expressed with E_1 , β and ω_1 by Eqs. (2) and (9) are integrated simultaneously by a numerical method.

3. Numerical integration

Since the simultaneous non-linear differential equations are difficult to solve analytically, a numerical integration is performed by using digital computer with the Runge-Kutta-Gill method. The method is excellent in that the gradient of variable to be found can be made as small and uniform as possible easily throughout the whole range of calculation by adjusting the increment of each step of calculation automatically, which enables us to minimize the error of calculation in a simple manner. Such a merit is restricted in the other methods using the previously fixed integration step such as the finite difference method or matrix method.

Numerical integration is started at $x=0$. The conditions $Q_x=s=0$ at $x=0$ are easily found from the symmetry of deformation. However, the values of E_1 and β at $x=0$ cannot be determined in advance. The boundary conditions at $x=l$ which should be used for determining them remain intractably at first because the calculation in the method is performed

from $x=0$ to $x=l$. Therefore, the values of E_1 and β at $x=0$ should be found by the trial method so as to satisfy the conditions at $x=l$. The conditions at $x=l$ may be expressed as follows:

$$F_1(E_1, \beta) = 0, \quad F_2(E_1, \beta) = 0 \dots\dots\dots (18)$$

where F_1 and F_2 denote the functions of the values of their arguments E_1 and β at $x=0$.

The first approximations of E_1 and β given at $x=0$ are modified by the increments ΔE_1 and $\Delta \beta$ obtained by the Newton-Raphson method with respect to the two variables according to the deviations between the conditions (18) and the corresponding results of numerical calculation at $x=l$. Moreover, the result of analysis in elastic range is applicable as the first approximation of E_1 and β .

The parameter ω_1 should be found in each step of x by solving the transcendental equation (17). Though the equation cannot be solved analytically, the Newton-Raphson method may be used in the numerical calculation. As the first approximation of ω_1 at every step, the corresponding value of ω_1 just before one step may be used. As the error of ω_1 affects seriously the result of calculation, the allowable error is restricted to less than 10^{-5} of its maximum value throughout the whole range. Since such a procedure cannot be applied at $x=0$, the first approximation of ω_1 at $x=0$ is referred to the result of elastic analysis.

The value of c cannot be found in advance because it depends on the deformation state as well as the mechanical property of material. Accordingly, the material is supposed to be incompressible ($c=0$) at first. Then, the first approximation of the deformation is obtained for the assigned internal pressure after solving the fundamental equations with the condition $c=0$, from which the distribution of effective strain $\bar{\epsilon}$ is obtained.

Assuming that $\bar{\epsilon}_{eq}$ (volumetric mean of $\bar{\epsilon}$ throughout the body) may express a measure of deformation state as a mean, the value of c corresponding to $\bar{\epsilon}_{eq}$ is obtained from the relation between c and $\bar{\epsilon}$ shown in Fig. 3. The modified state of deformation may be determined by solving again the fundamental equations by using the value of c supposed to be constant for each assigned value of internal pressure.

4. Result of calculation and discussion

Material of the long cylindrical shell is the same aluminum alloy as one considered in the previous papers⁽¹⁾⁽³⁾. Figure 2 shows the stress-strain relations obtained from a uniaxial tension test using two calibration specimens each cut out from a raw cylinder in the directions parallel and perpendicular to its generator. As shown in the figure, since the results obtained from both directions are very close to each

other, the mechanical property of the material can be considered to be approximated by the mean curve of them illustrated with a thick one. The values of coefficients in the relation (14) obtained by approximating the mean curve are shown in (19). Figure 3 shows the relation between c and $\bar{\epsilon}$ obtained from the same calibration test.

$$\left. \begin{aligned} a_0 &= 1.139 \times 10^4, & a_1 &= -2.877 \times 10^6 \\ a_2 &= 2.784 \times 10^8, & a_3 &= -9.541 \times 10^{11} \text{ kg/mm}^2 \end{aligned} \right\} \dots\dots\dots (19)$$

In the previous papers⁽¹⁾⁽³⁾, the deformation without axial force under internal pressure of a short cylindrical shell ($l/\sqrt{ha} = \sqrt{10}$) with supported ends free to move in the axial direction was considered as a numerical example. Therefore, a long cylinder ($l/\sqrt{ha} = 2\sqrt{10}$) having the same size as that except its axial length is considered in the same manner for comparing the difference between their deformation modes in the non-linear analysis.

With the absence of axial force, the condition (17) becomes $T_x = 0$. Therefore, with the condition $T_x = 0$ throughout the cylinder, the following conditions may be used as the boundary conditions (18) at the supported ends.

$$w = M_x = 0 \dots\dots\dots (20)$$

As the result of numerical evaluation, Figs. 4(a) and 5(a) show the distributions of radial and axial displacements. The distributions of the components of bending moment are shown in Figs. 6(a) and 7(a), and the distribution of circumferential membrane force is shown in Fig. 8(a).

As shown in Fig. 4(a), the radial deflection w attains to its maximum at a location lying between the center and the supported ends and decreases to zero

Fig. 2 Stress-strain relation of aluminum alloy

Fig. 3 Relation between parameter c and effective strain

monotonously towards the ends while it slightly decreases with a point of inflection towards the center. Such a mode closely resembles that of elastic deformation. However, the point of maximum deflection approaches the center with an increase of internal pressure. Such a trend does not appear in the linear elastic deformation because the deflection is proportional to internal pressure. That is, this trend depends on the non-linear effect appearing in the developed plastic deformation. Moreover, such a variation of deformation mode with the increase in internal pressure characterizes the plastic deformation of a long cylinder. The deflection of a short cylinder obtained in the previous paper⁽¹⁾ is reproduced in Fig. 4(b) for comparison, where the deflection decreases monotonously always from the maximum at the center to zero at the supported ends. Comparing Figs. 4(a) with 4(b), it is seen that the maximum deflections in both cases are nearly equal to each other in the cases when p is less than 60 kg/cm² while the maximum deflection of the long cylinder becomes less than that of the short one at $p = 70$ kg/cm² and the former becomes almost 80 percent of the latter at $p = 80$ kg/cm².

Figure 5(a) shows the distributions of axial displacement u of the long cylinder. Figure 5(b) for the short one is reproduced from the previous papers⁽¹⁾⁽³⁾ for comparison. As shown in these figures, the modes of distribution in both cases are almost the same. The axial displacement appears as a shrinkage towards

(a) ($l/\sqrt{ha} = 2\sqrt{10}$) (b) ($l/\sqrt{ha} = \sqrt{10}$)

Fig. 4 Distributions of radial displacement

the center and increases monotonously towards the ends. The maximum value of u for the long cylinder attains almost two times as that for the short one. From these figures, it may be said that the increase of diameter is remarkable for the deformation of short cylinder while the axial shrinkage is remarkable for that of long one and the trend grows from elastic deformation to plastic.

The distribution of bending moment M_x shown in Fig. 6(a) is different from that shown in Fig. 6(b) for the short one. That is, M_x has small negative values in the central part with a slightly wavy distribution. For the short cylinder, the ratio of the value of M_x at $x=0$ to its maximum value is 0.11 for $p=25 \text{ kg/cm}^2$ and increases with an increase of internal pressure and attains to 0.62 for $p=80 \text{ kg/cm}^2$, which shows the remarkable end effect throughout the short cylinder. However, for the long cylinder, the ratio is 0.016 for $p=25 \text{ kg/cm}^2$ and attains barely to 0.095 even for $p=80 \text{ kg/cm}^2$. In other words, the end effect is restricted to within a certain range near the ends for the long cylinder.

It is interesting to note that a decrease of maximum value appears with an internal pressure larger than $p=70 \text{ kg/cm}^2$ for the long cylinder or $p=80$

Fig. 5 Distributions of axial displacement

kg/cm^2 for the short one. Such a feature does not appear in the linear elastic case. The distribution curves are somewhat flattened with an increase of internal pressure by the growing non-linear effect.

Figure 7(a) shows the distribution of bending

Fig. 6 Distributions of axial bending moment

Fig. 7 Distributions of circumferential bending moment

moment M_θ whose feature almost resembles that of M_x shown in Fig. 6(a). However, in this case, in contrast to the case of M_x , the decrease of maximum value with an increase of internal pressure does not appear. Each value of M_θ attains almost 50 percent of M_x . Figure 7(b) showing the distribution of M_θ for the short cylinder has almost the same feature as in the long one in comparison with the distribution of M_x shown in Fig. 6(b).

Figures 8(a) and 8(b) show the distributions of membrane force T_θ for the long and short cylinders, respectively. For the short cylinder, the value of T_θ decreases from the center to the ends monotonously. However, for the long one, the value of T_θ has its maximum at a mid point, decreasing to zero towards the ends and slightly decreasing towards the center. The point of maximum value of T_θ for the long cylinder moves towards the center with an increase of internal pressure, which shows the growing non-linear effect.

In the foregoing discussion, it may be noted that the rate of increase of displacement is accelerated while that of bending moment or membrane force is retarded gradually with an increase of internal pressure. Such a trend depends on the characteristics of stress-strain curve and shows a feature obtainable in the non-linear analysis only.

As shown in these figures, the ratio of maximum

Fig. 8 Distributions of circumferential membrane force

bending moment to maximum membrane force can reach a significant value. For the long cylinder, $|M_x/h^2|_{\max}/|T_\theta/h|_{\max}$ is nearly equal to 0.17 for $p=60$ kg/cm² for example. This shows that the membrane theory in which the effect of bending moment is neglected cannot be applied in the precise analysis of the deformation of cylindrical shell.

5. Experimental examination

The corresponding experiment was performed by using the same apparatus as in the previous papers⁽²⁾⁽³⁾. Figure 9 shows the scheme of experimental apparatus, and its photograph is shown in Fig. 10. Internal pressure is applied by an oil pump. The components of strain ϵ_x and ϵ_θ on the inner and outer surfaces at $x/l=0, 0.2, 0.4, 0.6$ and 0.8 are measured with electric resistance strain gauges.

As the result of the experiment, the axial strains on the inner surface $\epsilon_x(z=h)$ and the outer surface $\epsilon_x(z=-h)$ and the mean value of circumferential strain ϵ_θ on both surfaces are shown in Figs. 11, 12 and 13 with the symbols showing the measured positions. On the other hand, Figs. 14 and 15 show the distributions of axial strain on the inner and outer surfaces obtained by the analytical evaluation, respect-

Fig. 9 Schematic diagram of experimental arrangement

Fig. 10 Photograph of experimental arrangement

ively. As the strain component to be measured by the strain gauge corresponds to the mean value of its distribution within the gauge length (axial direction) or the gauge width (circumferential direction) around the measuring position, the solid curves in Figs. 11 and 12 showing the analytical values corresponding to the measured values are plotted by using the mean values within the range shown with thick line segments on the abscissas in Figs. 14 and 15 corresponding to gauge length. Though the distributions of circum-

ferential strain ε_θ obtained by the analysis are omitted because they are analogous to Fig. 4 by the relation $w/h = (a/h)\varepsilon_\theta$, the solid curves in Fig. 13 are plotted also in the same manner.

As shown in these figures, the experimental results agree quite well with the corresponding analytical ones. A slight difference between both results on the curve $x/l=0.8$ in Fig. 12 may be attributed to the effect of small deflection at the supported end due to a clearance inevitable even with extreme care taken in the experiment. Therefore, it may be concluded that the validity of the proposed method is verified quite well by the experiment also in the case of long cylinder.

Comparison of both analytical results obtained with and without the consideration of compressibility of material is made for the circumferential strain ε_θ and the axial strain on the inner surface $\varepsilon_x(z=h)$ at $x=0$ in Fig. 16, for example. As shown in the figure, the experimental results agree very well with the solid curves obtained analytically with the consideration of compressibility. Consequently, it is clear that the assumption of incompressibility will lead to a certain amount of error.

6. Conclusions

Deformation under the internal pressure of a supported long cylinder ($l/\sqrt{ha}=2\sqrt{10}$) of aluminum alloy was analysed by the fundamental equations

Fig. 11 Analytical and experimental values of axial strain on the inner surface

Fig. 12 Analytical and experimental values of axial strain on the outer surface

Fig. 13 Analytical and experimental values of circumferential strain

Fig. 14 Distributions of axial strain on the inner surface

Fig. 15 Distributions of axial strain on the outer surface

Fig. 16 Relations between strain components $[\epsilon_\theta, \epsilon_x(z=h)]$ at $x=0$ and internal pressure p with ($c \neq 0$) and without ($c=0$) the consideration of compressibility of the material

proposed previously by the present authors, and the analytical result was compared with the corresponding experiment. These results are summarized in the following in comparison with those for a supported short cylinder ($l/\sqrt{ha} = \sqrt{10}$) of the same kind obtained in the previous paper.

1) Experimental results agreed quite well with the corresponding analytical ones using the fundamental equations according to the deformation theory. Therefore, it may be said that the validity and accuracy of the analytical method are verified quite well by the experiment and the deformation theory is sufficiently useful for estimating precisely the deformation of this extent.

2) Experimental results agreed very well with the analytical ones considering the effect of compressibility of material while a considerable discrepancy is found between the experimental value and the analytical result neglecting it. Therefore, it is clear that the compressibility of material should be considered in analysing precisely the small deformation of a material having non-linear stress-strain relation.

3) The deformation mode of a long cylinder under internal pressure varies with an increase of

internal pressure under the nonlinear effect due to the mechanical property of material. For example, though the maximum deflection appears at first at a point near the supported end as in the linear elastic deformation, the point of maximum deflection moves towards the center of axial length with an increase of internal pressure. Such a trend appears also in the distribution of circumferential membrane force. On the other hand, the maximum deflection appears always at the center and thus the variation of mode is not seen in a short cylinder. For a sufficiently large value of internal pressure, the deformation mode of a long cylinder seems to attain that of a short one.

4) Increase of diameter is remarkable in the deformation of a short cylinder while the axial shrinkage is relatively remarkable for a long one and the trend grows with the plastic deformation.

5) The decrease of maximum value of bending moment M_x appears for the internal pressure larger than a certain value depending on the value of ratio $l/\sqrt{ha}$. This also shows the non-linear effect due to the mechanical property of material.

6) The rate of increase of displacement is accelerated while those of bending moment and membrane force are retarded gradually with an increase of internal pressure for both kinds of cylinders with the non-linear characteristics of mechanical property of material.

References

- (1) Ohashi, Y. and Kamiya, N., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 35, No. 279 (1969-9), p. 2136.
- (2) Ohashi, Y. and Kamiya, N., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 37, No. 293 (1971-1), p. 27.
- (3) Ohashi, Y. and Kamiya, N., *Int. Jour. Solids Structure*, Vol. 6 (1970), p. 975.
- (4) Timoshenko, S. and Krieger, S.W., *Theory of plates and shells* (2nd ed.), (1952), p. 466, McGraw-Hill.
- (5) Ohashi, Y. and Kamiya, N., *Int. Jour. Mech. Sci.*, Vol. 9, No. 4 (1967), p. 183.

Discussion

K. MIZOGUCHI (Kinki University):

(1) The analysis is performed under the condition $T_x=0$. What kind of problem in reality is considered for such an analysis?

(2) The ring in the apparatus shown in Fig. 9 may be supposed to serve as a packing for avoiding the leakage. To avoid the leakage for a pressure as high as 80 kg/cm² in addition to a fairly large amount of axial displacement u , the packing must be pressed against the cylinder with a pretty high pressure.

Doesn't the pressure affect the cylinder as the axial force T_x ?

(3) Isn't it necessary to show the allowable range for applying the result of analysis according to the infinitesimal deformation theory?

Authors' closure

(1) The analysis is performed as a typical example in a series of this kind of deformation analysis and not on a special problem in reality. However, as a special example in reality, the authors are

considering the following problem. In relation to the investigation of general theory of plastic deformation, an experiment in which a hollow cylinder is subjected to tension, torsion and internal pressure is performed generally. The circumferential strain due to internal pressure is found generally only by the measurement of variation of diameter at the center of gauge length in the experiment, but as the deformation mode in the plastic range varies with an increase of internal pressure as mentioned in the present paper, the circumferential strain found at the center of gauge length is not sufficient as the measure of circumferential strain throughout the gauge length comparable with the other kind of strain.

(2) The axial displacement appears as shrinkage towards the center because of the condition $T_x = 0$. Therefore, a slight axial tension due to the friction at the supported ends may appear in relation to the displacement.

When the "O-ring" is pressed against the cylinder by a fairly high pressure, the action of O-ring against the cylinder is preferable in reducing the harmful friction at the supported ends.

On the other hand, if the effect of O-ring is sufficiently significant, some differences between the experimental points and the analytical curves must appear in Fig. 12. However, as the agreement between them is very good for a wide range of internal pressures as shown in the figure, it may be considered that the effect is trivial.

(3) When the validity of analytical result is discussed only theoretically, it is necessary to show the allowable range of applications for the deformation theory. For discussing the allowable range, there are two criterions proposed by Budiansky^(*1) and Zhukov^(*2) at present. However, the authors think that any definite criterion has not yet been established. Therefore, the analytical result should be discussed in comparison with the corresponding experimental result instead. As the agreement between them is very good as shown in the paper, the result is confirmed by the experiment to be within a sufficiently allowable range.

The deformation and flow theories have their allowable ranges of application as special cases of general theory of plasticity. Therefore, it is necessary to make clear applicable ranges in relation to their allowable errors of them for the actual problem.

K. MIZOGUCHI (Kinki University):

(2) Is it sufficient to conclude directly that the effect of T_x is trivial only from the good agreement

between the theoretical and experimental results? Isn't there any possibility that the analytical result would deviate from the experimental one when the condition $T_x = 0$ is realized perfectly by some method? Otherwise, is the effect of experimental error to realize the condition $T_x = 0$ verified to be trivial by establishing the theoretical equation taking account of T_x ?

Authors' closure

(2) In this experiment, each end of 1.5 mm width of the cylindrical specimen is supported from both sides by the supporting edge of two disks having a sufficient rigidity. Though the mechanical finishing of their contact surfaces is performed with the greatest care, a slight clearance is inevitable because of their large size. Therefore, to restrict the radial deflection due to the clearance at the supported end, the clearance is filled up with celluloid, for the filler may be supposed preferable to restrict mainly the radial deflection due to the clearance with its high resistance to volumetric strain while it affects little the axial displacement with its low resistance to shear.

As the effect of fitting is extremely delicate, when the fitting is slightly too tight, the agreement between analytical and experimental results of the axial strain component cannot be obtained even in elastic range, on account of the friction at the supported end. Moreover, as an error in agreement appears corresponding to $T_x > 0$, it is clear that the error arises from the effect of friction at the supported end rather than the effect of O-ring.

Even in such a case, the above mentioned error can be eliminated and a good agreement between the analytical and experimental values of axial strain is obtained by loosening slightly the fitting at either end. Therefore, such a care may be considered to be suitable for realizing the condition $T_x = 0$ in the experiment. Accordingly, the present authors think that the previous reply is sufficiently proper and any fear mentioned by the discussor does not appear at all.

As the disturbance due to the friction at the supported end may be considered as of local nature, it may be replaced by the axial force T_x in the sufficiently distant range from the end, according to St. Venant's principle. The axial force T_x in the theoretical equations may be considered as such a value in this discussion. Then, if the corresponding value of T_x is obtained for each internal pressure in the experiment, the effect of friction on the deformation of cylinder may be discussed by using the value as a boundary condition. However, the value is little reproducible because it is closely related to the slight secondary effect in the experiment.

S. TAKEZONO (Kumamoto University):

(1) For the improper first approximation, it

(*1) Budiansky, B., *Trans. ASME, Ser. E*, Vol. 26, No. 2 (1959-6), p. 259.

(*2) Zhukov, A.M., *Problems in Theory of Plasticity*, Izd. Akad. Nauk, USSR, (1961), p. 30.

cannot be said that a good convergence of solution is obtained without fail by the Newton-Raphson method. Is the good convergence of solution obtained by selecting the elastic solution as the first approximation?

(2) How the deformations in the unloading and reloading processes are analysed by this theory?

(3) It may be considered that a creep deformation appears even at room temperature in some aluminum alloys. Isn't there any fear of such deformation in the material used in this experiment?

(4) In the experimental procedure, the condition $M_x=0$ might be lost by filling the slight clearance between the end of specimen and the supporting edge to realize the condition $w=0$ at the supported end. Doesn't any trouble arise for the reason of small lap at the end?

(5) What kind of device is used to keep the oil pressure for some period in each step?

(6) How was treated the strain gauge on the inner surface after cemented on the surface?

(7) The discussor thinks that the validity of the analytical method proposed in Ref. 1 has been confirmed perfectly by the experiment performed for the short cylinder in Ref. 2. However, the analytical result for the long cylinder is examined by the present experiment again. Is there any special reason for examining the case of a long cylinder?

Authors' closure

(1) For the load corresponding to small deformation near the linear elastic one, the solution can be obtained by using the elastic solution as the first approximation in the Newton-Raphson method. However, as the difference between the elastic solution and the solution to be found in the large deformation of inelastic range is too large to apply directly the former as the first approximation of the latter, the above mentioned procedure cannot be applied directly. Therefore, in the present analysis, the external load is increased from the one corresponding to the linear elastic range to the one corresponding to non-linear deformation by a sufficiently small step. In such a case, the relevant result obtained just before one step may be used as the first approximation of the value to be found.

(2) Those deformations may be analysed by the same methods as in the papers^{(*)3)~(*)6)} published

(*)3) Ohashi, Y., et al., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.34, No. 262 (1968), p. 1031.

(*)4) Ohashi, Y. and Kawashima, K., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 34, No. 262 (1968), p. 1042.

(*)5) Ohashi, Y. and Kawashima, K., *Preprint of Japan Soc. Mech. Engrs.* (in Japanese), No. 212 (1969-10), p. 25.

(*)6) Ohashi, Y. and Kawashima, K., *Preprint of Japan Soc. Mech. Engrs.* (in Japanese), No. 703-1 (1970), p. 109.

previously by one of the authors and others.

(3) Strictly speaking, the state of equilibrium is not reached just after loading because a transient creep appears even at room temperature not only in aluminum but also in mild steel. Therefore, the state in which the deformation rate shown by the dial indicator attains less than about $10^{-3}/3$ mm/min at the location of maximum displacement in the specimen is considered as that of statical equilibrium. In an extreme case, the state appeared after 30 min of loading.

(4) The condition $M_x=0$ is realized very well because the arm length as well as the corresponding force of the bending moment is really negligible in the very narrow range of support. This may be confirmed also from the experimental result in the elastic range.

(5) Manual adjustment with a precise pressure gauge is sufficient for it.

(6) As in general cases, the gauges are protected with a kind of wax after inspecting the state of insulation.

(7) It might be natural to think like that at a first glance. However, every theory at present in the elasto-plastic analysis is of approximation. There isn't any criterion verified theoretically to estimate how closely the analytical result obtained from the theory may approximate the corresponding actual phenomenon.

Therefore, it is necessary to confirm the accuracy and allowable range of application of analytical result in comparing with the corresponding experiment at least for the typical size or deformation mode. If there is a systematic trend in the difference between both results in the examination, it would be necessary to investigate its cause and to modify the fundamental equation of the theory or theory itself.

S. TAKEZONO (Kumamoto University):

(3) Was such a care taken also in the uniaxial tension test for finding the stress-strain relation of the material of a cylindrical shell?

Authors' closure

(3) Considerable amount of time effect appears not only in aluminum alloy but also in mild steel in the plastic range in the uniaxial tension test at room temperature.

As the plastic deformation is analysed in the state of statical equilibrium, it is necessary to approach to the state also in the corresponding experiment. Therefore, such a consideration has been made in all the cases performed hither-to by one of the authors.

Y. SEGUCHI (Kobe University):

(1) It is expected that the loading path on the outer surface near the supported end is fairly

complicated. If there are some relevant data, please show them graphically.

(2) Though the authors have performed the elasto-plastic analysis according to the deformation theory, inferring from the discussor's experience in the elasto-plastic analysis according to the flow theory, the authors' method is not considered to be advantageous in comparing with the analysis using the flow theory which consists of only a modification and repetition of elastic analysis, because the author's method is complicated in the calculation procedure as described in the latter half of section 3. Please show the significance of applying the deformation theory to the structure which behaves complicatedly as in this problem.

(3) As the reference for extending the author's method to the other problems, please show the range of validity of the analytical result according to the deformation theory.

Authors' closure

(1) Since the same material and conditions, especially $T_x=0$, as in the previous paper are used in this analysis, the state near the supported ends may be considered to be almost the same as that in the short cylinder. This may be inferable in comparing the distributions of membrane force and bending moment near the ends with those in the result of the previous paper. Therefore, the loading paths on both surfaces near the ends may be expected to have almost the same trend as in the previous paper. Then, please refer to the author's closure to the discussion in the previous paper.

(2) The author's method is a mathematical analysis while the discussor's one is a numerical analysis. The complexity in the equations is due not to the deformation theory but to the fact that the mathematical analysis is executed. If the numerical analysis is performed according to the deformation theory, the computation must be far more simple than that mentioned by the discussor. Deformation theory is adopted here for the necessity of executing

the mathematical analysis as far as possible.

(3) As the merit of mathematical analysis in the non-linear range may be considered to be exhibited only with the use of the Runge-Kutta-Gill method, the mathematical analysis by the authors' method may be applied effectively for the axisymmetric deformation of thin plates or shells within the allowable range of the deformation theory.

It was said that the analytical result according to the deformation theory was effective only for the case of simple loading. However, the result is found to be sufficiently effective for the loading path near the simple loading from the comparison with the corresponding experimental result. Zhukov concluded according to the result of his tests on complex loading that the deformation theory was applicable as a first approximation even for the deformation of complex loading remarkably different from that of simple loading, if the directions of principal axes were kept unchanged and the effective stress increased monotonously.

As the reason for this, he mentions that the plastic deformation consists of a slip in the plane of principal shear stress and some planes in the vicinity of it, and so the directions of these planes do not rotate without rotation of the axis of stress tensor. However, this also cannot be considered to reflect every trend entirely. It cannot be said that the sufficient data have been obtained to estimate theoretically the allowable range of analytical result according to the deformation theory in every problem.

On the other hand, the flow theory also holds only in a restricted range. It has been confirmed that the colinear condition between the stress and plastic strain rate vectors is secured not always (the flow theory may hold as an approximation only for the trajectory of small curvature). The necessity to elucidate theoretically the applicable range of these theories is urgently required. The authors also are conducting an experiment with high accuracy in this respect.