Numerical Methods in Transient Heat Conduction*
(3rd Report, One-Dimensional Problems in Cylindrical Co-Ordinate System)

By Kozo Katayama** and Akio Saito***

In this paper, the following problems about one-dimensional transient radial heat conduction in cylindrical co-ordinate system are discussed:

1. Difference equations of explicit type for boundary conditions of no flux across the surface, prescribed heat flux across the surface, linear heat transfer at the surface, non-linear heat transfer, contact with a well-stirred fluid or perfect conductor, and the surface of separation of two media of different conductivities are given.

2. The truncation errors by those difference equations are shown in series expansion.

3. The stability conditions of those difference equations are shown.

4. The effective domain covered with those difference equations is shown, by comparing them with difference equations for one-dimensional rectangular co-ordinate system which are given in our former report.

5. \(at_{min}/|\Delta r|^2 \geq 0.4 \sim 2.6, \ t_{min}/\Delta t \geq 6 \) and \(R/\Delta r \geq 6\) are shown to be of sufficient conditions for the above difference equations, to obtain numerical solutions of accuracy within 2\( \sim \)3\% of the maximum temperature change.

1. Introduction

In the first report\(^1\), we proposed a method to apply an analog computer to many boundary-value problems concerning transient heat conduction in a one-dimensional orthogonal co-ordinate system. We showed ordinary differential equations corresponding to various kinds of boundary conditions, and its analog circuits, and revealed the conditions of ordinarity to obtain the numerical solutions of required accuracy.

In the second report\(^2\), we discussed difference equations which approximate various kinds of boundary conditions, and showed the method to obtain numerical solutions of required accuracy, concerning transient heat conduction problems in an orthogonal one-dimensional co-ordinate system with constant thermal properties.

Difference equations are applied naturally to multidimensional transient heat conduction problems\(^3\). However in former methods, the multidimensional temperature field is usually divided using only a kind of meshes such as meshes of rectangular geometry, as shown in Fig. 1(a).

The method of division, using only the mesh of rectangular geometry, has a merit to make the computing program simple. But on the other hand, the accuracy of the numerical solutions is not so high, because the temperature field cannot be divided along

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\* Received 27th March, 1970.
** Professor, Tokyo Institute of Technology.
*** Associate Professor, Faculty of Mechanical Engineering, University of Yamanashi, Kofu.

(a) An example of dividing extended surface
(b)
(c) An example of dividing turbine blade

Fig. 1 Examples of dividing two-dimensional temperature field
the boundary surface, and the heat flow across the boundary is calculated exaggerately.

We discuss a method to obtain a numerical solution of sufficient accuracy, dividing the temperature field along the boundary by the combination of three kinds of meshes, that is: (a) meshes of rectangular geometry, (b) meshes produced by circles and radial lines, and (c) triangular meshes, as shown in Fig. 1 (b) (c).\(^4\)

In this report, we discuss the finite difference method of heat conduction problems in a one-dimensional cylindrical co-ordinate system of constant thermal properties, as a step to extend "the method of division of the temperature field to obtain the solution of required accuracy" to two-dimensional problems, namely:

(1) Explicit type difference equations, which approximate various kinds of boundary conditions of heat conduction theory, are proposed.

(2) A method to expand the error of difference equation is proposed, and the stability conditions of the above difference equations are shown, using the method of error expansion. And the convergence-divergence conditions are discussed, through actual numerical calculations.

(3) The limit of application of the above difference equations is discussed, considering the influence of the radius of the boundary on temperature distribution.

(4) By comparing the numerical solution with the exact solutions in transient heat conduction problems of various boundary conditions, we reveal means to decide finite differences of time and space to obtain numerical solutions of required accuracy, concerning the radius of the boundary.\(^\dagger\)

Nomenclature

\(T\): temperature  
\(T_n^p\): numerical solution of temperature at \(t = p\Delta t\) and \(r = n\Delta r\)  
\(\theta\): exact solution of temperature  
\(E\): absolute error of numerical solution of temperature  
\(q\): heat flux  
\(\lambda\): thermal conductivity  
\(\varepsilon\): specific heat  
\(\rho\): density  
\(a\): thermal diffusivity \((a = \lambda/\varepsilon)\)  
\(h\): heat transfer coefficient  
\(\sigma\): Stefan-Boltzmann's constant  
\(\varepsilon\): emissivity  
\(F\): geometrical factor  
\(t\): time  
\(r\): co-ordinate of location  
\(S\): surface area  
\(V\): volume, of medium or perfect conductor, per unit area of boundary surface  
\(\Delta t\): finite difference in time  
\(\Delta r\): finite difference of length in the direction of \(r\) co-ordinate  
\(R\): boundary radius  
\(\Theta\): \(= a\Delta t/\Delta r^2\)

Subscripts

0,1,2,\ldots, \(p\) (upper): the value at the time \(t = 0\), \(\Delta t, 2\Delta t, \ldots, p\Delta t\)  
0,1,2,\ldots, \(n\) (lower): the value at the place \(r = 0\), \(\Delta r, 2\Delta r, \ldots, n\Delta r\)  
\(n^*\): the value at the boundary \((r = R)\) (\(n^* = R/\Delta r\))  
\(n, n+1\): mean value between the place \(r = n\Delta r\) and the place \(r = (n+1)\Delta r\)  
\(\Omega\): the value of media  
\(\Omega^*\): the value of radiational source  
\(M\): the value of well-stirred fluid or perfect conductor  
\(A, B\): the values of solids \(A, B\)

2. Difference equations of heat conduction problems

2.1 Difference equation of heat conduction equation

For calculations of transient heat conduction problems, it is convenient to use explicit type difference equations. A well-known difference equation of regular division in a one-dimensional cylindrical co-ordinate system is,

\[
T_n^{p+1} = \theta T_n^p + \theta (1 - \frac{1}{2n}) T_{n+1}^p + \theta (1 - \frac{1}{2n}) T_{n-1}^p + (1 - 2\theta) T_n^p 
\]

(1)

\[
T_n^{p+1} = 4\theta T_n^p + (1 - 4\theta) T_n^{p+1} 
\]

(2)

Where, \(T_n^p\) is the temperature in the place \(r = n\Delta r\), and at the time \(t = p\Delta t\), and non-dimensional parameter \(\theta = a\Delta t/\Delta r^2\).

When the boundary condition is given by prescribed boundary temperature, a numerical solution is obtained from Eq. (1) and Eq. (2); however, when the boundary conditions of the other type are given, as shown in Table 1, difference equations which correspond to the boundary conditions must be used simultaneously.
2.2 Difference equations for various kinds of boundary conditions, where the temperature field is inside of the boundary

2.2.1 The difference equations for boundary conditions of (1) no flux across the surface, (2) prescribed heat flux across the surface, (3) linear heat transfer at the surface, and (4) non-linear heat transfer.

As is shown in Fig. 2(a), dividing the temperature field into cylinders of thickness 1/2 Δr, Δr, Δr, ... and considering the balance of heat about the cylinder at the boundary, we get

\[
\left( \pi R^2 - \pi \left( R - \frac{\Delta r}{2} \right)^2 \right) \rho \left( \frac{\partial T}{\partial n} \right)_r = 2\pi R q^{n*} + 2\pi \left( R - \frac{\Delta r}{2} \right) q^{n*-1, n*} \Delta t,
\]

considering \( R = n^* \Delta r \),

\[
T^{n*=1, n*} = T^{n*} + \left( \frac{8n^n}{4n^n-1} \right) \frac{\Delta t}{\Delta r} q^{n*} + \frac{8n^n-4}{4n^n-1} \frac{\Delta t}{\Delta r} q^{n*-1, n*} + \frac{\Delta t}{\Delta r} q^{n*-1, n*}.
\]

Where,

\[
q^{n*-1, n*} = \lambda \left( T^{n*-1} - T^{n*} \right) / \Delta t
\]

and for the boundary conditions (3), (4), (5), in Table 1, \( q^{n*} \) is shown in Table 2. Therefore, substituting \( q^{n*} \) and \( q^{n*-1, n*} \) into the above equation, the difference equations corresponding to the various kinds of the boundary conditions are obtained. The results are Eq. (3) ~ Eq. (6) in Table 2.

2.2.2 The difference equations for the boundary conditions of (6) contact with well-stirred fluid or perfect conductor, (7) the surface of separation of two media A, B of different conductivities.

Considering the balance of heat in Fig. 2(a), (c), as paragraph 2.2.1, the difference equations corresponding to the boundary conditions are

\[
T^{n=1} \frac{\Delta r h}{\lambda} T^{n=*-1} = \frac{8n^n}{4n^n-1} \frac{\Delta t}{\Delta r} q^{n*} + \frac{8n^n-4}{4n^n-1} \frac{\Delta t}{\Delta r} q^{n*-1, n*} + \frac{\Delta t}{\Delta r} q^{n*-1, n*}.
\]

where,

\[
T^{n=1} \frac{\Delta r h}{\lambda} T^{n=*-1} = \frac{8n^n}{4n^n-1} \frac{\Delta t}{\Delta r} q^{n*} + \frac{8n^n-4}{4n^n-1} \frac{\Delta t}{\Delta r} q^{n*-1, n*} + \frac{\Delta t}{\Delta r} q^{n*-1, n*}.
\]

Table 1 Thermal boundary conditions and their equations\(^{59}\)

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>The equations for boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) No heat flux across the surface</td>
<td>( \frac{\partial T}{\partial r} \mid_{r=R} = 0 )</td>
</tr>
<tr>
<td>(2) Prescribed surface temperature</td>
<td>( T_{r=R} = T_{m(t)} )</td>
</tr>
<tr>
<td>(3) Prescribed heat flux across the surface</td>
<td>( -\lambda \left( \frac{\partial T}{\partial r} \right) \mid_{r=R} = q(t) )</td>
</tr>
<tr>
<td>(4) Linear heat transfer at the surface (Newton's cooling)</td>
<td>( \pm \lambda \left( \frac{\partial T}{\partial r} \right) \mid_{r=R} = h(T_{m} - T) )</td>
</tr>
<tr>
<td>(5) Non-linear heat transfer (Radiation)</td>
<td>( \pm \lambda \left( \frac{\partial T}{\partial r} \right) \mid_{r=R} = \delta F(T^4 - T^4) )</td>
</tr>
<tr>
<td>(6) Contact with a well-stirred fluid or perfect conductor</td>
<td>( \pm \lambda \left( \frac{\partial T}{\partial r} \right) \mid_{r=R} = \delta q_{m} \frac{dT}{dt} \mid_{r=R} - q(t)S = 0 )</td>
</tr>
<tr>
<td>(7) The surface of separation of two media A, B of different conductivities</td>
<td>( -\lambda_{A} \left( \frac{\partial T}{\partial r} \right) \mid_{r=R} = -\lambda_{B} \left( \frac{\partial T}{\partial r} \right) \mid_{r=R} , T_{A r=R} = T_{B r=R} )</td>
</tr>
</tbody>
</table>

\(^{59}\) According to this division, the nodal point for the boundary temperature comes at the boundary, and the distance between the adjacent nodal points becomes \( \Delta r \) everywhere.

Fig. 2 The temperature field of cylindrical co-ordinate system divided into cylinders.
\[ T_{n^*+1} = \frac{(8n^* + 4)k\Theta_A}{(4n^* + 1)l + (4n^* - 1)} T_{n^*+1}^0 + \frac{(8n^* - 4)\Theta_A}{(4n^* + 1)l + (4n^* - 1)} T_{n^*-1}^0 + \left[ \frac{1 - (8n^* + 4)k + (8n^* - 4)}{(4n^* + 1)l + (4n^* - 1)} \Theta_A \right] T_{n^*} \] 

respectively. Where, Eq. (8) is the difference equation for the well-stirred fluid or perfect conductor, and \( T_n^* \) in Eq. (9) is the boundary temperature when the temperature field \( A \) exists within the boundary. And \( k = \lambda_B/\lambda_A, l = (\varphi\rho)B/(\varphi\rho)_A, \Theta = \Theta_B f/l(A)^2 \).

3. Difference equations for various kinds of boundary conditions, where the temperature field is outside of the boundary

For the case of Fig. 2 (b), the difference equations of Table 3 is obtained, in the same way as paragraph 2.2, where, the difference equation for the boundary condition of \( \Omega \) the surface of separation of two media \( A, B \) of different conductivities, is the same as Eq. (9).

As shown above, the difference equations of Eq. (1) \~ Eq. (15) for radial heat conduction problems in one-dimensional cylindrical co-ordinate system are obtained. These equations agree with the difference equations for a one-dimensional orthogonal system, which are shown in the former report, when \( n^* = \infty \). (The influence of the boundary radius disappears.)

3. A proposal of a method for expanding the error accompanied by finite difference calculations

In calculating the temperature distributions numerically by explicit type difference equations, Eq. (1) \~ Eq. (15), the truncation error varies oscillatorily with time and space, on account of the mutual interference among the oscillatory waves of the errors at all nodal points.

We propose a method for expanding the truncation error after sufficient iteration processes, based on the following assumptions about the fundamental wave.

(1) When the error at the nodal point on the boundary surface is only a round-off error—for example, when the boundary condition is the condition of \( \Omega \) prescribed surface temperature in Table 1, a node of the fundamental wave of the error appears on the boundary surface [Fig. 3 (a), (c)].

(2) When the error at the nodal point on the boundary surface varies with the oscillation of the error within the temperature field—when the boundary condition is one of the conditions \( \Omega, \Omega, \Omega, \Omega, \Omega \) in Table 1, or the condition for the center of the circle; the side of the fundamental wave of the error exists on the boundary surface [Fig. 3 (b), (c)].

For the case of Fig. 3 (a), we define \( \epsilon_{n^*} (\Omega, \Omega, \Omega) \) and \( (\epsilon_{n^*})_n \) as the error, the fundamental wave, and the \( n \)-th wave of the \((k+1)\)-th nodal point at time \( t = pAf \), respectively, where, \( p \) is sufficiently large.

The oscillatory wave, the node of which exists at \( k = 0 \) and \( k = K \), is

![Fig. 3 The kinds of fundamental waves of oscillation](image-url)
\[ (E\alpha)^p = A_1 \sin \left( \frac{\pi k}{K} + \frac{\pi t}{dt} \right) + A_2 \sin \left( \frac{\pi k}{K} - \frac{\pi t}{dt} \right) = 2A_1 \sin \left( \frac{\pi k}{K} \cos \frac{\pi p}{m} \right) \]  

The \( m \)-th wave is

\[ (E\alpha)^p = 2A_2 \sin \left( \frac{\pi k}{K} \cos \frac{\pi p}{m} \right) \]

where, \( A_0 \) is a function of \( p \), including the constant of zero.

\[ E\alpha^p = \sum_{m=1}^{K-1} (E\alpha)^p \]

Therefore, putting \( 2A_2 \cos \left( \frac{\pi p}{m} \right) = \varphi(m, p) \)

\[ E\alpha^p = \sum_{m=1}^{K-1} \varphi(m, p) \sin \left( \frac{\pi k}{m} \right) \]  

In the same way, for the cases of Fig. 3 (b) and (c),

\[ E\alpha^p = \sum_{m=1}^{K-1} \cos \left( \frac{\pi k}{m} \right) \sin \left( \frac{\pi k}{m} \right) \cos \left( \frac{\pi k}{m} \right) \]

\[ E\alpha^p = \sum_{m=1}^{K-1} \cos \left( \frac{\pi k}{m} \right) \cos \left( \frac{\pi k}{m} \right) \]

respectively.

### 4. The stability conditions of difference equations

As difference equations of Eq. (1) ~ Eq. (15) are the equations of explicit type, therefore, they must be used within the stability ranges, to obtain a stable numerical solution, as is shown in the former report.

#### 4.1 The stability conditions

As the method to derive stability conditions of Eq. (1) ~ Eq. (15) is essentially the same as shown fully in the 2nd report, in this report we explain the method briefly about one example, and show the results only.

We consider a case when the boundary condition of \( \delta \) prescribed heat flux across the surface, in Table 1, is given at the boundary \( r = n^\alpha \cdot \Delta r \) to the inner temperature field. Defining \( \theta \alpha^p, T\alpha^p \) and \( E\alpha^p \) as the exact solution, the numerical solution, and the error at the place \( r = n^\alpha \cdot \Delta r \) and the time \( t = p \Delta t \).

\[ E\alpha^p = T\alpha^p - \theta \alpha^p \]  

\[ \theta \alpha^p \] is considered to satisfy the difference equation of \( T\alpha^p \) approximately. Therefore, putting \( r = n^\alpha \cdot \Delta r \) and substituting \( \theta \alpha^p \) and \( T\alpha^p \) into Eq. (4)

\[ E\alpha^p = 2 \left( \frac{8n - 4}{4n^2 - 1} \right)^2 \theta \alpha^p \]  

from Eq. (19).

When the absolute error \( E\alpha^p \) oscillates unstably, the numerical solution \( T\alpha^p \) also becomes unstable; and when it converges, \( T\alpha^p \) is stable. Therefore, the condition for convergence of \( E\alpha^p \) is the stability condition of difference equation, Eq. (4).

Expanding the absolute error by Eq. (17), and assuming as in the former report(3), the case where the amplitude of the \( m \)-th wave grows, with an increasing \( p \),

\[ \varphi(m, p+1) \leq \varphi(m, p) \left( \frac{8n - 4}{4n^2 - 1} \right)^2 \left( \frac{8n - 4}{4n^2 - 1} \right) \]

Using Neumann's condition for convergence(6)(7),

\[ \varphi(m, p+1) \leq \varphi(m, p) \geq -1 \]

and considering \( 1 \leq m \leq K \), the stability condition is,

\[ \theta \leq \frac{4n - 1}{4(2n^2 - 1)} \]  

The stability conditions of the difference equations of Eq. (1) ~ Eq. (15) are obtained in the same way, and the results are shown in Table 4.

For the case \( n \rightarrow \infty \) in Table 4, the conditions agree with the stability conditions of one-dimensional difference equations in the orthogonal system, which are shown in the former report.

#### 4.2 Discussion of stability conditions by an example

We discuss the stability conditions in Table 4, by an example shown in Table 5.

The exact solution of the problem shown in Table 5 is

\[ \theta = \frac{1}{2} \left( \sum_{r=1}^{N} \right) \left( \frac{1}{\alpha \alpha(x)} \right) \]

Where, \( J_1(x) \) is Bessel function of the i-th order, and \( R \alpha(x) \) is a root of the equation

<table>
<thead>
<tr>
<th>Table 4</th>
<th>The stability conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>The equation of</td>
<td>( \theta \leq \frac{1}{2} ) ( \left( \frac{8n - 4}{4n^2 - 1} \right)^2 ) ( \left( \frac{8n - 4}{4n^2 - 1} \right) ) ( \left( \frac{8n - 4}{4n^2 - 1} \right) ) ( \left( \frac{8n - 4}{4n^2 - 1} \right) ) ( \left( \frac{8n - 4}{4n^2 - 1} \right) )</td>
</tr>
<tr>
<td>The boundary conditions of ( \theta \alpha )</td>
<td>( \theta \leq \frac{1}{2} ) ( \left( \frac{8n - 4}{4n^2 - 1} \right)^2 ) ( \left( \frac{8n - 4}{4n^2 - 1} \right) ) ( \left( \frac{8n - 4}{4n^2 - 1} \right) ) ( \left( \frac{8n - 4}{4n^2 - 1} \right) )</td>
</tr>
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<td>( \theta \leq \frac{1}{2} ) ( \left( \frac{8n - 4}{4n^2 - 1} \right)^2 ) ( \left( \frac{8n - 4}{4n^2 - 1} \right) ) ( \left( \frac{8n - 4}{4n^2 - 1} \right) ) ( \left( \frac{8n - 4}{4n^2 - 1} \right) )</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5</th>
<th>The basic equation, initial and boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>The basic equation</td>
<td>( \frac{\partial \theta}{\partial t} + \frac{1}{r} \frac{\partial \theta}{\partial r} - \frac{1}{r^2} \theta = 0 ) ( \left( \frac{r \alpha}{R} \right) )</td>
</tr>
<tr>
<td>Initial condition</td>
<td>( \theta = 0 ), ( (r = 0, \theta \geq R) )</td>
</tr>
<tr>
<td>Boundary condition</td>
<td>( \theta = 1 ), ( (r \geq 0, \theta \geq R) )</td>
</tr>
</tbody>
</table>
\[ J_0(\beta) = 0. \]

The difference equations to solve the problem in Table 5 are shown in Table 6. There, \( n^* = n/\Delta r = 1 \) for simplicity. Putting \( m^* = n^* = 1 \) in Eq. (24), the stability condition of the difference equation in Table 6 is

\[ \theta \leq 1/2. \]

For the cases of \( \theta = a \Delta t/(\Delta r)^2 = 0.48, 0.5, 0.52, \) we calculate numerical solutions by the difference equations of Table 6, compare them with the exact solution of Eq. (31), and discuss the stability of the numerical solutions. The absolute errors at the center, \( r = 0 \), are shown in Fig. 4. It is clear from Fig. 4, that when \( a \Delta t/(\Delta r)^2 \) is a slightly smaller value of 0.48 compared with the stability range, the error converges to zero as the iteration time \( p = t/\Delta t \) increases; and when \( a \Delta t/(\Delta r)^2 \) is the stability limit of 0.5, the absolute value of the error does not change; and when \( a \Delta t/(\Delta r)^2 \) is a slightly greater value of 0.52 compared with the stability limit, the error diverges oscillatory, as \( p = t/\Delta t \) increases.

In the above example, the method, shown in paragraph 3., for expanding the error is considered to be applicable, and the stability conditions in Table 4, derived by the method, are considered adequate.

5. Discussion of the applicable range of the difference equations of the cylindrical co-ordinate system

When the non-dimensional parameter \( R/\sqrt{\alpha t} \) is sufficiently large—the case where the boundary radius \( R \) is relatively larger than the time \( t \)—, the temperature distribution near the boundary surface is slightly affected by the boundary radius, therefore in that case, it is more convenient and simpler to use approximately one-dimensional difference equations of the orthogonal system, which are proposed in the former report, instead of the difference equations, Eq. (1) \( \sim \) Eq. (15).

In this paragraph, we define "the applicable range of the difference equations of the cylindrical co-ordinate system" as the range where the error of the temperature calculation arising from the above-mentioned approximation does not exceed 1%. About the boundary shapes of a circular arc and a line, comparing two exact solutions under the same kind of boundary conditions for the same Fourier modulus, we discuss the condition, where the temperature distribution is affected more than 1% by the boundary surface radius, that is, the applicable condition.

We discuss again the problem of Table 5. The exact solution is Eq. (31). Defining \( \theta_{r/R-1} \) as the solution for \( r/R = 1 \) in Eq. (31),

\[ \theta_{r/R-1} = \text{erfc} \frac{R - r}{2 \sqrt{\alpha t}} \]  

(32)

Where,

\[ \text{erfc} X \equiv \frac{2}{\sqrt{\pi}} \int_X^\infty e^{-y^2} dy \]

Defining \( \theta_{\text{max}} \) as the maximum value of \( \theta \), the values of \( (\theta - \theta_{r/R-1})/\theta_{\text{max}} \) are calculated from Eq. (31) and Eq. (32). A diagram, on which \( |\theta - \theta_{r/R-1}|/\theta_{\text{max}} \) satisfies \( |\theta - \theta_{r/R-1}|/\theta_{\text{max}} = 0.01 \), is shown in \( (R-r)/R-R/\sqrt{\alpha t} \) co-ordinate system, as the diagram of (A) in Fig. 5. For the problems within the range under the curve (A), the relation
\[ \theta_{\text{max}} \geq 0.01 \] is satisfied; therefore, it is adequate to apply difference equations of the cylindrical co-ordinate system.

In the same way, when the boundary condition of constant boundary temperature is given to the external temperature field of the boundary, the result is the curve (B) in Fig. 5; and when the boundary condition of constant heat flux across the boundary is given to the internal temperature field, the result is the curve (C) in Fig. 5, respectively. The curves (A) \sim (C) practically agree with each other, and the hatched section in Fig. 5 is the applicable range of Eq. (1) \sim Eq. (15).

Furthermore, for the numerical calculations of heat conduction problems outside of the hatched section, a solution of sufficient accuracy is also obtained by dividing the temperature field into infinite plate elements.

6. The condition to obtain the solution of required accuracy

As is mentioned in the second report\(^{(32)}\), among a few kinds of errors arising in finite difference calculations, the truncation error, which arises by approximating the partial differential equations of heat conduction with difference equations, Eq. (1) \sim Eq. (15), is dominant. And considering the practical accuracy\(^{(39)}\) of the boundary conditions in Table 1, the numerical solutions of temperature distribution may be accompanied with an error of 2\sim3\% of the maximum temperature value.

As is shown in the former report, in a one-dimensional orthogonal system, the parameters deciding the value of the truncation error are \(a_{\text{min}}/\langle \Delta r \rangle^2\)\(^{(33)}\) and \(t_{\text{min}}/\Delta t\); however, in a cylindrical co-ordinate system, the parameters are \(R/\Delta r, a_{\text{min}}/\langle \Delta r \rangle^2\) and \(t_{\text{min}}/\Delta t\).

When the two-dimensional temperature field is divided into several kinds of meshes, as shown in paragraph 1, it is desirable that the conditions for \(a_{\text{min}}/\langle \Delta L \rangle^2\) and \(t_{\text{min}}/\Delta t\) be respectively comparable among all meshes.

Therefore in this report, comparing numerical solutions by finite difference method with the exact solution, we discuss the value of \(R/\Delta r\) which assures "the condition of \(a_{\text{min}}/\langle \Delta r \rangle^2\) and \(t_{\text{min}}/\Delta t\) to obtain the solution of temperature distribution within the required accuracy—2\sim3\% of the maximum temperature—" to agree with the conditions for the orthogonal system shown in the 2nd report\(^{(32)}\):

\[ a_{\text{min}}/\langle \Delta r \rangle^2 \geq 0.4 \sim 2.6 \] \hspace{1cm} (33)

\[ t_{\text{min}}/\Delta t \geq 6 \] \hspace{1cm} (34)

In this paragraph, we discuss first the minimum value of \(R/\Delta r\), by which the range of \(a_{\text{min}}/\langle \Delta r \rangle^2\) agrees with the range of Eq. (33); and next we discuss the condition of \(a_{\text{min}}/\langle \Delta r \rangle^2\) and \(t_{\text{min}}/\Delta t\) to obtain the solution of required accuracy, using the \([t_{\text{min}}/\Delta t, a_{\text{min}}/\langle \Delta r \rangle^2]\) diagram which we proposed in the former report; and then we show that the conditions of \(a_{\text{min}}/\langle \Delta r \rangle^2\) and \(t_{\text{min}}/\Delta t\), which correspond to the above-obtained minimum value of \(R/\Delta r\), agree with Eq. (33) and Eq. (34).

It is clear that, using the values of \(\Delta r\) and \(\Delta t\) thus decided, a numerical solution of required accuracy is obtained.

6.1 The discussion of the adequate value of \(R/\Delta r\) in reference to the value of \(a_{\text{min}}/\langle \Delta r \rangle^2\)

We show a discussion about the example of Table 7.

<table>
<thead>
<tr>
<th>Table 7</th>
<th>The basic equations, initial and boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_0 = \theta_0 + (1 - 4\theta_0)T_1 + \theta_1 + (1 - \theta_1)T_1 + \cdots + (1 - \theta_n)T_n )</td>
<td></td>
</tr>
<tr>
<td>Initial condition</td>
<td>(T_0 = 0)</td>
</tr>
<tr>
<td>Boundary condition</td>
<td>(T_\infty = 1)</td>
</tr>
</tbody>
</table>

The following table shows the numerical solutions by finite difference method, and the error.

<table>
<thead>
<tr>
<th>Table 8</th>
<th>Numerical solutions by finite difference method, and the error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_{\text{min}}/\langle \Delta r \rangle^2)</td>
<td>(r = 0)</td>
</tr>
<tr>
<td>0.2</td>
<td>0.006 8</td>
</tr>
<tr>
<td>0.4</td>
<td>0.037 9</td>
</tr>
<tr>
<td>0.6</td>
<td>0.091 6</td>
</tr>
<tr>
<td>0.8</td>
<td>0.159 1</td>
</tr>
<tr>
<td>1.0</td>
<td>0.232 4</td>
</tr>
<tr>
<td>1.2</td>
<td>0.306 1</td>
</tr>
<tr>
<td>1.5</td>
<td>0.410 3</td>
</tr>
<tr>
<td>2.0</td>
<td>0.557 8</td>
</tr>
<tr>
<td>2.5</td>
<td>0.671 4</td>
</tr>
<tr>
<td>3.0</td>
<td>0.756 7</td>
</tr>
<tr>
<td>4.0</td>
<td>0.866 9</td>
</tr>
<tr>
<td>5.0</td>
<td>0.927 2</td>
</tr>
</tbody>
</table>

\(t_{\text{min}}/\Delta t\) is the minimum value among time \(t\) at which transient solutions are wished to be calculated within the required accuracy. Therefore, \(t_{\text{min}}/\Delta t\) is the number of iteration times to obtain the first transient solution.
5 again, among the discussions concerning all boundary conditions in Table 1. The exact solution is Eq. (31). Difference equations to solve the problem are shown in Table 7. The numerical solutions for $R/\Delta r=3$, $\Delta t=0$ in the problem of Table 7 are shown in Table 8. The error is shown in Fig. 6. It is clear from the figure that "the condition to calculate the numerical solutions within the accuracy of $3\%$ of the maximum temperature, over all the temperature field $(r=0, \Delta r, 2\Delta r)$" is $at_{\text{min}}/(\Delta r)^2 \geq 1.16$. In the same way, the condition of $at_{\text{min}}/(\Delta r)^2$, to calculate the temperature distribution within the above accuracy, is shown as the curve (A) in Fig. 7, varying the value of $R/\Delta r$ as $R/\Delta r=1 \sim 10^2$.

<table>
<thead>
<tr>
<th>Table 9</th>
<th>The basic equation, initial and boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>The basic equation</td>
<td>$\frac{\partial \theta}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) = a \frac{\partial \theta}{\partial t} (r \geq R)$</td>
</tr>
<tr>
<td>Initial condition</td>
<td>$\theta = 0$, $(t = 0, r \geq R)$</td>
</tr>
<tr>
<td>Boundary conditions</td>
<td>$\theta = 1$, $(t \geq 0, r = R)$ \nonumber \theta = 0$, $(t \geq 0, r = \infty)$</td>
</tr>
</tbody>
</table>

In the same way, the curve (B) in Fig. 7 is obtained, discussing the condition of $at_{\text{min}}/(\Delta r)^2$ for various values of $R/\Delta r$, concerning the problem where "the boundary condition of constant boundary temperature" is given to the temperature field outside of the boundary (Table 9). In the solutions of the problems of Table 5 and Table 9, the influence of the boundary radius is considered to appear as a contrary factor; therefore in Fig. 7, for the case $R/\Delta r \geq 6$, when the curves (A) and (B) agree, the error is considered to converge as in the case of the difference equations for the orthogonal system, which are shown in the former report. In fact, the condition $at_{\text{min}}/(\Delta r)^2 \geq 0.32$, for this case, agrees practically with the condition for $at_{\text{min}}/(\Delta x)^2$ shown in the 1st and the 2nd reports.

### 6.2 The discussion of the adequate value of $R/\Delta r$ by $t_{\text{min}}/\Delta t - at_{\text{min}}/(\Delta r)^2$ diagram

We show, in this paragraph, that "the condition to obtain the solution of the required accuracy" is given by Eq. (33) and Eq. (34) for $R/\Delta r \geq 6$, discussing the $t_{\text{min}}/\Delta t - at_{\text{min}}/(\Delta r)^2$ diagrams about the case $R/\Delta r = 6$ of paragraph 6.1 for the problems of various kinds of boundary conditions in Table 1.

6.2.1 The problem of the boundary condition (2) in Table 1

We discuss again the problem in Table 5. The difference equations to solve the problem in Table 5 are shown in Table 7. Deciding the number of iteration $t_{\text{min}}/\Delta t$ required to obtain a sufficiently accurate solution by varying the value of the parameter $\Theta = a \Delta t/(\Delta r)^2$, $t_{\text{min}}/\Delta t - at_{\text{min}}/(\Delta r)^2$ diagram in Fig. 8 is given. As is explained precisely in the 2nd report, for all combinations of the values of $t_{\text{min}}/\Delta t$, $at_{\text{min}}/(\Delta r)^2$ which are included in the hatched

† In this report, the parameter $R/\Delta r$ is an integer, and the diagram (A) is represented by discontinuous values, because the temperature field within the boundary is divided equally.
section of the figure, the numerical solutions of the required accuracy are obtained throughout the temperature field. For \( R/\Delta r = 6 \), as the region which satisfies Eq. (33), Eq. (34) and stability condition \( \varepsilon \Delta t (\Delta r)^2 \leq 1/4 \) is completely included in the hatched section of Fig. 8, so the sufficient conditions to keep the accuracy of the solution are Eq. (33) and Eq. (34).

In the same way, \( t_{\text{min}}/\Delta t - t_{\text{min}}/(\Delta r)^2 \) diagram about the problem in Table 9 is shown in Fig. 9, for \( R/\Delta r = 6 \). From the figure, Eq. (33) and Eq. (34) are also the conditions to acquire the solutions of the sufficient accuracy for the problem.

6.2.2 The problems of the other six boundary conditions in Table 1.

In the same way, the conditions of Eq. (33) and Eq. (34) are acquired by discussing \( t_{\text{min}}/\Delta t = t_{\text{min}}/(\Delta r)^2 \) diagrams for \( R/\Delta r = 6 \). Examples of \( t_{\text{min}}/\Delta t = t_{\text{min}}/(\Delta r)^2 \) diagrams are shown in Fig. 10, for the boundary conditions of "(3) prescribed heat flux across the surface".

From the above discussions, the conditions to obtain the solutions with sufficient accuracy, using the difference equations of Eq. (1) ~ Eq. (15), are

\[
at_{\text{min}}/(\Delta r)^2 \leq 0.4 \sim 2.6 \\
t_{\text{min}}/\Delta t \geq 6 \\
R/\Delta r \geq 6
\]

for the problems of boundary conditions in Table 1.

7. An example

To explain the above-mentioned method precisely, we show the actual procedure to calculate the numerical solutions within the required accuracy, concerning the example of the heating of the cylinder shown in Fig. 11. We calculate the temperature changes at the places \( r = 60 \text{ mm} \) (heating surface),

50 mm and 0 mm (the center), after the times of 18 sec, 27 sec and 30 sec.

In this problem,

\[ R = 60 \text{ mm}, \ t \geq 18 \text{ sec} \]

and then,

\[ R/\sqrt{\alpha t} \leq 3.46 \]

Therefore from Fig. 5, this problem must be solved by difference equations of the cylindrical coordinate system.

Because,

\[ t_{\text{min}} = 18 \text{ sec} \]

from Eq. (33),

\[ \Delta r \leq 10.74 \text{ mm} \]

\[ R/\sqrt{\alpha t} \leq 3.46 \]

Therefore from Fig. 5, this problem must be solved by difference equations of the cylindrical coordinate system.

Because,

\[ t_{\text{min}} = 18 \text{ sec} \]

from Eq. (33),

\[ \Delta r \leq 10.74 \text{ mm} \]

\[ (a) \text{ The case when the temperature field is inside of the boundary} \]

\[ \begin{array}{c}
\text{Fig. 9 The range of non-dimensional factors which give stable solutions of sufficient accuracy} \\
\text{(The case when the boundary condition of constant temperature is given to the temperature field outside of the boundary)}
\end{array} \]

\[ T = 0^\circ, \ t = 0, \ 0 \leq r \leq R, \ \partial T/\partial r = q/1, \ t \geq 0, \ r = R \]

\[ q/\lambda = 500^\circ \text{C/m}, \ R = 60 \text{ mm}, \ a = 0.080 \text{ m}^2/\text{hr} \]

\[ \text{Fig. 11 An example} \]
And from Eq. (35),
\[ \Delta r \leq 10 \text{ mm} \]
So we decide \( \Delta r = 10 \text{ mm} \). From Eq. (34),
\[ \Delta t \leq 3 \text{ sec} \]
And from the stability conditions of Eq. (23), Eq. (24) and Eq. (25),
\[ \Delta t \leq 1.5 \text{ sec} \]
Therefore, we decide \( \Delta t = 1.5 \text{ sec} \). From Eq. (1), Eq. (2) and Eq. (4), the difference equations are shown in Table 10. The numerical solutions by the difference equations in Table 10 are shown in Table 11, compared with the exact solutions*. It is clear that the sufficient accuracy is secured. The calculating time by the computer HITAC 5020 E is within 2 sec.

8. Conclusions

When we solve two-dimensional transient heat conduction problems by finite-difference method, it is desirable, in keeping the required accuracy, to divide the temperature field along the boundary into combinations of a several kinds of meshes. In this report, concerning the difference equation of the cylindrical co-ordinate system, as basic discussions of the dividing elements, we showed the difference equations which approximate various kinds of boundary conditions, proposed a method to expand the truncation error, decided the stability ranges and the applicable range of the equations, and discussed the calculating accuracy, namely:

1) Radial one-dimensional difference equations of the explicit type for the cylindrical co-ordinate system, which approximate various kinds of boundary conditions in Table 1, are shown as Eq. (1)~Eq. (15).

2) The truncation error by the difference equations of Eq. (1)~Eq. (15) are expanded in series as Eq. (16)~Eq. (18), on assumptions about the fundamental wave of oscillation. Furthermore, the stability conditions of Eq. (22)~Eq. (30) for the difference equations, based on the expansion, agree exactly with the actual converging-diverging property of a numerical example.

3) The applicable range of the difference equations of Eq. (1)~Eq. (15) is shown as the hatched region in Fig. 5. For the problems included outside of the region, the difference equations shown in the former report are more convenient than Eq. (1)~Eq. (15).

4) The conditions to solve transient heat conduction problems within the required accuracy, using the difference equations of Eq. (1)~Eq. (15), are given as Eq. (33), Eq. (34) and Eq. (35).

Acknowledgement

The authors wish to express their appreciation to Dr. R. Shimomura, Professor of University of Yamanashi, for his kind support of this research.

They utilized the computer at the Computer Center of University of Tokyo for the numerical calculations in this research.

References