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Dynamic Buckling of Bars*

(3rd Report, In the Case of Elastoplastic Bars)

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In this paper, a dynamic buckling problem of elastoplastic columns axially loaded at one end by a moving rigid body is treated, and the experimental results are compared with those calculated from the theory in which a tangent modulus E_t is used both for compression and bending of the column up to the buckling load. When impact velocities of the rigid body are small, the theoretical values agree with the experimental ones. If the kinetic energy of the moving body is kept constant while its mass and velocity are varied, the results obtained from experiments and numerical calculations are as follows: In the case of low impact velocities with large masses, the column buckles more easily and the buckling load is smaller than that of high impact velocities with small masses. This difference appears distinctly when the impact velocity of the body is high but almost disappears when the impact velocity is low.

1. Introduction

In our previous papers⁽¹⁾⁽²⁾, it has always been assumed that the material of a column is perfectly elastic and follows Hooke's law. When the column is subjected to an axial impact by a mass with a high velocity, plastic buckling takes place. In this paper, the dynamic buckling problem on plastic columns is investigated. Chawla⁽³⁾ has treated an elastoplastic problem similar to that previously studied on elastic columns by Hoff⁽⁴⁾ (that is, the problem of a simply supported column initially bent in the shape of a half sine wave and loaded by displacing one end axially at a constant rate). He assumed that the column had an idealized H-section which consisted of two equal, concentrated elastoplastic flanges separated by a web of negligible area but of infinite shear rigidity. Also, Abrahamson and Goodier⁽⁵⁾ have studied the dynamic buckling problem under a uniform axial force. The former⁽³⁾ is an application of the Shanley model to the dynamic problem, and the deflection is presented by a polynomial expression. The maximum load (the dynamic buckling load) of the column obtained from this numerical analysis becomes smaller than the corresponding static buckling load.

He has inferred that this discrepancy between them is caused partly by the arbitrary choice of the value of order n of the polynomial expression. The latter⁽⁵⁾ is an extension of Lindberg's linearized problem⁽⁶⁾ which treats the column material as rigid and strain-hardening with a constant strain-hardening modulus E_t , so that the nonlinear effects representing axial components of the strain due to bending of the column are not retained in it.

This paper is concerned with a problem of slightly curved elastoplastic columns with a uniform rectangular cross section and fixed ends and having one end axially struck by a rigid body. If we assume that transverse sections of the column remain plane during deformations, a bending moment can be related to a curvature through the tangent modulus E_t . Furthermore, using this tangent modulus E_t , we derive a basic equation to determine the axial force in the plastic region, in which, however, the axial-inertia effects are assumed to be negligibly small. In general, strain-reversal takes place on the convex side before the column actually buckles, so that the theory on dynamic plastic buckling is very complicated. To simplify the theoretical treatment, however, the tangent modulus E_t is used both for compression and bending up to the buckling load. The numerical solutions under this assumption are compared favorably with the experimental results when the impact velocities of the rigid body are low.

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2. On dynamic plastic buckling

A column is made of a work hardening material having a stress-strain curve such as shown in Fig. 1. Now, consider a case when the column is compressed beyond the proportional limit. If a linear distribution is assumed for the bending strain in the cross sections of the column, the corresponding distribution of total strain $\epsilon = \epsilon_n + \epsilon_b$ is represented by a line $I_a II_b$ shown in Fig. 2, where ϵ_n is axial compressive strain and ϵ_b bending compressive strain in the cross sections. The condition to allow no decrease in the compressive total strain is represented by

$$\Delta \epsilon_n + \Delta \epsilon_{b,i}^* \geq 0 \quad (i=1, 2) \quad (1)$$

where $\epsilon_{b,i}^*$ is bending strain on the inner or outer fiber, and $\Delta \epsilon_n$ and $\Delta \epsilon_{b,i}^*$ are respectively defined as follows.

$$\Delta \epsilon_n = \epsilon_n(x, t + \Delta t) - \epsilon_n(x, t)$$

$$\Delta \epsilon_{b,i}^* = \epsilon_{b,i}^*(x, t + \Delta t) - \epsilon_{b,i}^*(x, t)$$

If the condition (1) is satisfied, it can be seen from Fig. 2 that at time $t + \Delta t$ the distribution of the total strain, that is, each longitudinal strain in the cross section must be in the region enclosed by two lines I and II.

In such cases the stresses at points A, C and B in Fig. 1, which are respectively at the inner and outer fiber and on the central axis of the column, are continuously increasing. Therefore, the incremental stress-strain relationship in the axial direction is given by

$$\Delta \sigma_n(x, t) = E_t(\epsilon_n) \Delta \epsilon_n(x, t) \quad (2)$$

Replacing $\Delta \sigma_n$ in the above equation by Δp yields

Fig. 1 Static stress-strain curve

Fig. 2 Condition to allow no decrease in compressive total strain

$$\Delta p(x, t) = A E_t(\epsilon_n) \Delta \epsilon_n(x, t) \quad (3)$$

If the linear distribution is assumed for the bending stress in the cross sections of the column, the bending moment is represented by

$$M_b(x, t) = E_t(\epsilon_n) I \frac{\partial^2 (y - y_0)}{\partial x^2} \quad (4)$$

In the above equations, A is the cross-sectional area of the column, I the moment of inertia of the area of the column, $p(x, t)$ the axial compressive force, $y(x, t)$ the total deflection of the column, $y_0(x)$ the initial deflection of the column and $E_t(\epsilon_n)$ the tangent modulus corresponding to the axial strain ϵ_n at the point B in Fig. 1. In this paper, the effects of axial inertia are neglected, so that ϵ_n , p and therefore E_t are the functions depending on time alone.

Even though the axial force is still increasing the problem becomes much more complicated if the stress at the outer or inner fiber begins to decrease owing to the effects of bending. However, since the deflection is small up to the buckling load, it can be assumed that the condition (1) is approximately satisfied by this time.

3. Basic equations

When Eqs. (3) and (4) are used up to the buckling load, the following equations given by the nondimensional form, subject to suitable initial and boundary conditions, represent a complete system of equations by this time for determining the nondimensional lateral deflection $\bar{y}$ and the nondimensional axial force $\bar{p}$ of the elastoplastic column, and the nondimensional velocity $\bar{v}$ of the striking body:

$$\frac{\partial^2 \bar{y}}{\partial \bar{t}^2} + \bar{p}(\bar{t}) \frac{\partial^2 \bar{y}}{\partial \bar{x}^2} + \frac{1}{\lambda^2} \frac{E_b}{E} \frac{\partial^4 (\bar{y} - \bar{y}_0)}{\partial \bar{x}^4} = 0 \quad (5.a)$$

$$\bar{p}(\bar{t}) = \int_0^1 \frac{E_b}{E} \left\{ \bar{v} - \frac{1}{2} \frac{\partial}{\partial \bar{t}} \int_0^1 \left(\frac{\partial \bar{y}}{\partial \bar{x}} \right)^2 d\bar{x} \right\} d\bar{t} \quad (5.b)$$

$$\bar{p}(\bar{t}) = -\bar{m} \frac{d\bar{v}}{d\bar{t}} \quad (5.c)$$

where $E_b(\bar{t})$ is Young's modulus E in the elastic region and the tangent modulus $E_t(\bar{t})$ corresponding to the axial strain ϵ_n in the plastic region. Also, λ is the slenderness ratio of the column and $\bar{m}$ the nondimensional mass of the striking body. As is well-known, Eqs. (5.a) and (5.c) express equilibriums, respectively, of the column in the lateral direction and of the striking body (assumed as rigid), and Eq. (5.b) is the relation between the axial force and the axial strain caused by it. The nondimensional variables and quantities are defined by

$$\bar{p} = \frac{P}{AE}, \quad \bar{y} = \frac{y}{l}, \quad \bar{y}_0 = \frac{y_0}{l}, \quad \bar{m} = \frac{m}{\rho A l}$$

$$\bar{v} = \frac{v}{c}, \quad \bar{v}_0 = \frac{v_0}{c}, \quad \bar{x} = \frac{x}{l}, \quad \bar{t} = \frac{ct}{l}$$

where l is the length of the column, ρ the mass per unit volume of the column material and c the propagation velocity of wave in the column material.

The initial conditions are

$$\left. \begin{aligned} \bar{t}=0: \quad \bar{y}(\bar{x}, 0) &= \bar{y}_0(\bar{x}), \quad \frac{\partial \bar{y}}{\partial \bar{t}} = 0 \\ \bar{v}(0) &= \bar{v}_0, \quad \frac{d\bar{v}}{d\bar{t}} = 0 \end{aligned} \right\} \dots\dots\dots (6)$$

and the boundary conditions corresponding to the column with the fixed ends are

$$\bar{x}=0, 1: \quad \bar{y} = 0, \quad \frac{\partial \bar{y}}{\partial \bar{x}} = \frac{d\bar{y}_0}{d\bar{x}} \dots\dots\dots (7)$$

Equations (5) through (7) are solved by replacing the derivatives with their corresponding differences. In this paper as in our previous papers⁽¹⁾⁽²⁾, the accuracy of the method is shown by calculating the following relative error $\delta(\bar{t})$ of the total energy in the system in every step of time $\bar{t}$.

$$\begin{aligned} \delta(\bar{t}) &= \frac{\bar{W} - \bar{W}_0}{\bar{W}_0} \\ \bar{W} &= \frac{1}{2} \int_0^1 \left(\frac{\partial \bar{y}}{\partial \bar{t}} \right)^2 d\bar{x} + \int_0^{\epsilon_n} \bar{p} d\epsilon_n \\ &\quad + \frac{E_b}{2\lambda^2 E} \int_0^1 \left\{ \frac{\partial^2 (\bar{y} - \bar{y}_0)}{\partial \bar{x}^2} \right\}^2 d\bar{x} + \frac{1}{2} \bar{m} \bar{v}^2 \\ \bar{W}_0 &= \frac{1}{2} \bar{m} \bar{v}_0^2 \end{aligned}$$

As the numerical values of $\delta(\bar{t})$ do not exceed 0.5 % for $\Delta \bar{x} = 1/20$ and $\Delta \bar{t} = \lambda(\Delta \bar{x})^2 / 20 \sqrt{E_b/E}$, solutions of these difference equations do not diverge and seem to be accurate enough.

4. Experiments

4.1 Experimental apparatus and method

Specimens are an annealed aluminum having 150 mm length and a uniform rectangular cross section of 10 mm width and 5 mm depth. The maximum value of the initial deflection of the specimens is about 0.5 mm. Static compression test of the material used was made with a specimen having the same cross section and length of

Fig. 3 Nominal stress-strain relation of static compression test

15 mm. Strains up to 2 % were measured by resistance strain gauge and strains over 2 % were measured by micrometer after each unloading. A nominal stress-strain curve obtained is shown in Fig. 3.

The axial force at the supported end of the specimen was measured by the axial force detector attached at this end and a synchroscope. This detector is composed of two semiconductor strain gauges and a bar of steel SK2 having the same cross section as the specimen and length of 20 mm. The measured values of the axial force were determined by using the load-scale relation of static compression test of this detector and the synchroscope.

The detector above-mentioned is put into an apparatus to support the specimen A as shown in Fig. 4. Thus, as the specimen is supported by means of eight rollers R in a cap C attached to a rigid wall D, the specimen slides only in the axial direction and therefore we can measure the axial force at the supported end by means of the detector F as mentioned above. Also, plates of steel SK2 with thickness of 3 mm and length of 50 mm attached near the supported end to every side of the specimen prevent this part from causing the deflection. The same devices are used here again as in the experiment of our previous paper⁽²⁾ except that a shock tester is employed as impact loader.

4.2 Experimental results

Figure 5 shows the axial force at the supported end in the case when the mass of the striking body is $m = 0.0437 \text{ kgsec}^2/\text{m}$ ($\bar{m} = 21$) and its impact velocity is $v_0 = 4.9 \text{ m/sec}$. A fine lacerated wave is found just after impact but the curve is comparatively smooth after that. By comparing Fig. 5 with Fig. 6, which corresponds to the case of $v_0 = 20.5 \text{ m/sec}$, it can be seen that when the impact velocity is large, the axial force increases rapidly with time, soon reaching a large buckling load and then rapidly decreases.

A: Specimen, C: Cap, D: Rigid wall, F: Axial force detector, R: Roller
 l_a : length of deflected part (100 mm)
 l_b : length of grip (50 mm)

Fig. 4 Axial force measurement

Fig. 5 Synchroscope record for axial force at supported end ($v_0=4.9$ m/sec)

Fig. 6 Synchroscope record for axial force at supported end ($v_0=20.5$ m/sec)

Fig. 7 Buckling load-impact velocity relation (Experiment)

Fig. 8 Photograph of final deflection of specimen

In Fig. 7 experimental results are shown for three kinds of the striking masses $m=0.0437$, 0.0087 and 0.0055 kgsec²/m. Each point shown represents a separate test. When the striking mass is constant, the buckling load becomes large as the impact velocity increases and for the bodies having the same impact velocities it does as the mass increases. The signs $\times$ and $\triangle$ in this figure, respectively, represent points corresponding to initial kinetic energies $W_0=2.480$ kgm ($\bar{W}_0=W_0/AE=4.723 \times 10^{-5}$) and $W_0=1.745$ kgm ($\bar{W}_0=3.324 \times 10^{-5}$) of the striking body. Consequently, the solid line and the longer dotted line give the buckling load-impact velocity relations for the case when the bodies with the same amount of initial kinetic energies strike against the specimens. These curves show that in the case of low impact velocities with large masses, the buckling loads are smaller and the specimens buckle more easily than in the case of high impact velocities with small masses. Also, the difference between them becomes small as the impact velocity becomes small and at the limit where the impact velocity vanishes they both will converge to the static buckling load $p_E=50$ kg.

Figure 8 shows a photograph of the final deflection of the specimens under three experimental conditions ($m=0.0437$ kgsec²/m, $v_0=6.9$ m/sec), ($m=0.0087$ kgsec²/m, $v_0=20.2$ m/sec) and ($m=0.0055$ kgsec²/m, $v_0=24.5$ m/sec) which correspond close to $W_0=1.745$ kgm shown in Fig. 7. For the case when initial kinetic energy of the body is constant, it can be also seen from this figure that the larger the mass of the body, the larger the final deflection is and the more likely the specimen is to buckle. In the final deflected shape there is a point of inflection near the impact end, so that this end is considered to be a fixed end. This is probably due to the fact that no rotation takes place on the impact end while the body is in contact with this end.

5. Comparison of experimental and numerical results

The axial forces shown in Figs. 5 and 6 are compared with the corresponding numerical results in Figs. 9 and 10 respectively. In experiments the grip of the specimens has length $l_b=50$ mm. Since this portion receives no deflection, calculations are carried out, also, assuming that only the compressive strain due to the axial force is applied in this portion. The values of E_b/E are obtained by interpolating between given values in Table 1 prepared from the static stress-strain curve given in Fig. 3. The symbol $\bullet$ represents the limit for

Eqs. (3) and (4) to be effective, after which the stress begins to decrease on the convex side at the center of the column, but since this decrease is small enough, calculations are continued with Eqs. (3) and (4) being assumed to be available up to the buckling load. Both curves in Fig. 9 are smooth and agree well with each other up to the buckling load. In Fig. 10, however, the axial force in the experiment lags behind the numerical value at the outset and overshoots it later. Thus, there is a large difference between both curves from the outset. This may be because the effects of axial inertia become large when the impact velocity is high and therefore it can no longer be assumed that the axial force is uniform along the axial line. That is, in this paper an infinite propagation velo-

Table 1 Values of E_b/E used in calculation

$p \times 10^4$	0~1.14	1.14~1.78	1.78~2.19	2.19~3.04
E_b/E	1	0.2054	0.1151	0.0521
3.04~3.87	3.87~5.12	5.12~6.91	6.91~8.71	8.71~10.79
0.0343	0.0257	0.0161	0.0100	0.0061

Table 2 Calculated values of $\bar{p}_c$ for various amounts of $\bar{W}_0$

$\bar{W}_0$	1×10^{-5}	2×10^{-5}	3×10^{-5}	6×10^{-5}
$\bar{p}_c$	6.8×10^{-4}	8.3×10^{-4}	9.2×10^{-4}	1.11×10^{-3}

Fig. 9 Comparison of numerical and experimental results for axial force ($v_0=4.9$ m/sec) Fig. 10 Comparison of numerical and experimental results for axial force ($v_0=20.5$ m/sec)

city of wave is assumed, while actually it propagates as a plastic wave with the finite velocity. Consequently, the experimental value lags behind the numerical value for the case when the axial force is measured at the supported end as in our experiments. Also, it will be owing to reflection of the plastic wave that the measured value of the axial force is considerably higher than the calculated value.

In Fig. 11 the experimental result for the case of $m=0.0437$ kgsec²/m in Fig. 7 is compared with the corresponding numerical one. As is observed from this figure, the numerical curve exhibits the same general behavior as the experimental one. They agree with each other up to $v_0=7\sim 8$ m/sec, after which, however, the experimental curve rises more rapidly than the numerical one and over $v_0=10$ m/sec the difference between them becomes considerably wide. $v_0=10$ m/sec in this experiment corresponds to $\bar{v}_0=2 \times 10^{-3}$ for the nondimensional form. It can be seen from this that up to this value of $\bar{v}_0$, the proper values for the buckling load are obtained from our theory.

Figure 12 shows the buckling load-impact velocity relations obtained from the theory for four values of initial kinetic energies $\bar{W}_0=1 \times 10^{-5}$, 2×10^{-5} , 3×10^{-5} and 6×10^{-5} of the striking body.

Fig. 11 Comparison of numerical and experimental results for buckling load

Fig. 12 Buckling load-impact velocity relation for various amount of initial kinetic energy of striking body (Calculated)

The numerical results have the same tendency as the experimental ones in Fig. 7. That is, at a small impact velocity the buckling loads are almost the same, independently of the values of $\bar{W}_0$ and at the limit where $\bar{v}_0$ tends to zero, they converge to the static buckling load $\bar{r}_E = 1.5 \times 10^{-4}$ of this column. On the contrary, as the impact velocity becomes large, it becomes harder for the column to buckle and the initial kinetic energy of the striking body is changed into axial strain energy almost alone, that is, they approach a certain value $\bar{p}_c$ corresponding to the values of $\bar{W}_0$. This value of p_c is determined by equating initial kinetic energy to axial strain energy.

$$W_0 = Al \int_0^{\epsilon_c} \sigma_n d\epsilon_n$$

By using ϵ_c in the above equation

$$p_c = A\sigma_n(\epsilon_c)$$

The above equation is schematically solved for values of $\bar{W}_0 = 1 \times 10^{-5}$, 2×10^{-5} , 3×10^{-5} and 6×10^{-5} by using the static stress-strain curve in Fig. 3, as represented in Table 2.

6. Conclusions

This paper is concerned with the buckling problem of an elastoplastic column with fixed ends and having one end axially struck by a rigid body, and (1) we derive the basic equation to determine the axial force p in the plastic region assuming that the incremental stress-strain relationship in the axial direction is given by

$$\Delta\sigma_n = E_t(\epsilon_n) \Delta\epsilon_n$$

(2) Up to $\bar{v}_0 = 2 \times 10^{-3}$, the axial forces determined theoretically are favorably compared with

the experimental one at the grip end obtained from the aluminum specimens. However, when the impact velocity $\bar{v}_0$ becomes still large, a wide difference is observed between them. This is because of the axial-inertia effects being neglected in our theory. From these facts, we consider that up to $\bar{v}_0 = 2 \times 10^{-3}$, the buckling load can be deduced from our theory.

(3) When the initial kinetic energy W_0 of the striking body is constant, for the case of low impact velocities with large masses the buckling loads are smaller and the columns buckle more easily. This is confirmed both from the theory and the experiment.

(4) When the impact velocity is constant, the buckling load becomes larger as the mass of the striking body is larger. However, this tendency is not so pronounced at a small impact velocity and at the limit of v_0 the buckling load converges to the static one, and for the case of a large impact velocity it approaches a certain value p_c corresponding to the values of W_0 . Such phenomena are also observed in the experiment.

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