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Numerical Method of Transient Heat Conduction with Temperature Dependent Thermal Properties*

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Finite difference methods were developed to solve transient problems of nonlinear heat conduction with temperature dependent thermal properties, and the following steps were presented.

(1) Finite difference equations of heat conduction and of boundary conditions were presented and the stability conditions of them were discussed.

(2) The types of boundary conditions which could be treated by these finite difference equations were clarified.

(3) Difference equations of the explicit type were shown to be applicable to most nonlinear problems of heat conduction, and the sufficient conditions to obtain numerical solutions of required accuracies were discussed and presented.

(4) Numerical results of temperature response in plastic plates were compared with experimental results and these were shown to agree with sufficient accuracies.

1. Introductions

Analysis of heat conduction problems applicable to a variety of materials coming into use in the modern industrial technology over a wide range of temperatures, has been extended and it is very important. Examples of variations of thermal conductivities and heat capacities of typical industrial materials are shown in Figs. 1 and 7, and it is reasonable to consider that for analysis of conduction heat transfer these thermal properties should be treated as functions of temperature.

Efforts have been made to obtain analytical solutions of nonlinear heat conduction problems with temperature dependent thermal properties. Most problems except simple cases, however, have been left unsolved because of the mathematical difficulties.

Recent development in high speed computers allow numerical method to be used as effective means to solve these problems. Especially be the

finite difference method, and by selecting a time difference scheme, complicated nonlinear heat conduction problems have been successfully handled. Research findings on stability and accuracy of the finite difference equations of linear heat conduction with constant thermal properties were issued by R. Courant⁽²⁾, G. G. O'Brien,⁽³⁾ authors⁽⁴⁾ and others, but little research has been done on nonlinear heat conduction with temperature dependent thermal properties. Furthermore, some kinds of unsolvable boundary value problems exist using the finite difference method, but so far no research findings have been published to clarify the limit of applicability of the finite difference method.

In this paper, the following listed problems concerning the analysis of transient heat conduction with temperature dependent thermal properties using the finite difference method are clarified:

(1) The finite difference equations of forward time difference approximating the partial differential equation and the boundary conditions are presented, and stability conditions of them are shown.

(2) The boundary value problems of heat conduction in which heat flux across the surface increases with temperature, e. g. some kinds of chemical reactions, are discussed and it is shown that they are unsolvable by the finite difference method, and the limit of applicability of the finite

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difference method is clarified.

(3) The variation of thermal properties is not so great except in the vicinity of the melting point. The explicit method is shown to be applicable to the problems regarding the temperatures except near the melting point and the conditions imposed to obtain the numerical solutions having desired accuracies are examined.

Analyzing examples by the finite difference method combined with measurement of thermal properties and their temperature dependencies are shown and the results are compared with experimental results of temperature response.

In summation, the finite difference method presented in this paper is shown to be available for analyzing transient heat conduction with temperature dependent thermal properties.

Table 1 Nomenclatures

α : thermal diffusivity	m ² /hr	δ : error
c : specific heat	kcal/kg°C	ϕ : $\lambda \Delta t / c\rho(\Delta x)^2$
$c\rho$: heat capacity	kcal/m ³ °C	θ : dimensionless temperature
F : heat flux	kcal/m ² hr	θ : temperature °C
T : temperature	°C	λ : thermal conductivity kcal/mhr°C
t : time	hr	ρ : specific weight kg/m ³
Δt : increment of time t	hr	Suffixes
x : spatial coordinate	m	n (lower): at $x=n\Delta x$
Δx : increment of x	m	p (upper): at $t=p\Delta t$

① Chrome ② Aluminum alloy (2024-T4)
③ Inconel ④ Stainless steel 301

Fig. 1 Thermal conductivity and heat capacity of metals⁽¹⁾

2. Finite difference equations of heat conduction

2.1 Finite difference equations of one dimensional heat conduction

The differential equation of heat conduction with temperature dependent thermal properties,

$$c\rho \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right) \quad \dots\dots\dots (1)$$

may be converted into a finite difference equation. Let increments of x and time t be Δx and Δt respectively and selecting forward time difference, and the finite difference equation may be written as

$$(T_{n,p+1} - T_{n,p})c\rho_n = \frac{\Delta t}{(\Delta x)^2} \{ \lambda_{n,n-1}(T_{n-1,p} - T_{n,p}) + \lambda_{n,n+1}(T_{n+1,p} - T_{n,p}) \} \quad \dots\dots\dots (2)$$

where $\lambda_{n,n}$ and $c\rho_n$ are the spatially mean values of thermal conductivity and average heat capacity in each time increment Δt , given by,

$$\lambda_{n,m} = \frac{1}{T_{n,p} - T_{n,p+1}} \int_{T_{n,p}}^{T_{n,p+1}} \lambda dT \quad \dots\dots\dots (3)$$

$$c\rho_n = \frac{1}{T_{n,p+1} - T_{n,p}} \int_{T_{n,p}}^{T_{n,p+1}} c\rho dT \quad \dots\dots\dots (4)$$

respectively. They are allowed to be arbitrary functions of temperature. Equation (2) may be rewritten in the form

$$\int_{T_{n,p}}^{T_{n,p+1}} c\rho dT = \frac{\Delta t}{(\Delta x)^2} \left(\int_{T_{n,p}}^{T_{n-1,p}} \lambda dT + \int_{T_{n,p}}^{T_{n+1,p}} \lambda dT \right) \quad \dots\dots\dots (5)$$

by using Eqs. (3) and (4). The variation of heat capacity is not so great except near the melting point. Therefore, the value of heat capacity corresponding to temperature $T_{n,p}$, given by

$$c\rho_n = c\rho(T_{n,p}) \quad \dots\dots\dots (6)$$

may be used instead of Eq. (4).

Then the finite difference equation is reduced to an explicit equation,

$$T_{n,p+1} = T_{n,p} + \phi_{n,n-1}(T_{n-1,p} - T_{n,p}) + \phi_{n,n+1}(T_{n+1,p} - T_{n,p}) \quad \dots\dots\dots (7)$$

where,

$$\phi_{n,m} = \frac{\lambda_{n,m}}{c\rho_n} \frac{\Delta t}{(\Delta x)^2} \quad \dots\dots\dots (8)$$

Using Eq. (5), the average value of heat capacity during time increment Δt must be implicitly determined, so that the complicated calculation procedures such as a trial and error method may be required. On the other hand, Eq. (7) is an explicit equation so that calculation can easily be carried out.

2.2 Boundary conditions

① When the surface temperature is prescribed as a function of time, the boundary condition of the finite difference equation at $x=0$ is given

by

$$T_0^p = T(0, p\Delta t) \dots \dots \dots (9)$$

② When the boundary condition of differential equation is given by

$$-\lambda \frac{\partial T}{\partial x} = F \dots \dots \dots (10)$$

the difference equation approximating the boundary condition is given by

$$\int_{T_0^p}^{T_0^{p+1}} c\rho dT = 2 \frac{\Delta t}{(\Delta x)^2} \left(F\Delta x + \int_{T_0^p}^{T_1^p} \lambda dT \right) \dots \dots (11)$$

Using Eq. (6), it is reduced to the explicit equation

$$T_0^{p+1} = T_0^p + \phi_{0,1} \left(\frac{F}{\lambda_{0,1}} \Delta x + T_1^p - T_0^p \right) \dots \dots (12)$$

In Eqs. (10), (11) and (12), F is heat flux across the surface which is given as a function of surface temperature or prescribed function of time.

2.3 Applicable limit of the finite difference equations

The difference equations (5) and (7) can be solved with boundary condition ① and/or boundary condition ② when heat flux F is the prescribed function of time or decrease function of surface temperature. When heat flux F increases as surface temperature increases, heat conduction problems can not be solved by the finite difference method.

For simplicity, taking constant thermal conductivity, λ , and heat capacity, $c\rho$, the finite difference equations (5) and (7) and boundary conditions (11) and (12) are reduced to

$$T_n^{p+1} = T_n^p + \phi(T_{n-1}^p - 2T_n^p + T_{n+1}^p) \dots \dots (13)$$

$$T_0^{p+1} = T_0^p + 2\phi \left(\frac{F}{\lambda} \Delta x + T_1^p - T_0^p \right) \dots \dots (14)$$

Let an exact solution of temperature at $t = p\Delta t$ be Θ_n^p , and the following equations may be given.

$$\Theta_n^{p+1} = \Theta_n^p + \phi(\Theta_{n-1}^p - 2\Theta_n^p + \Theta_{n+1}^p) \dots \dots (15)$$

$$\Theta_0^{p+1} = \Theta_0^p + 2\phi \left(\frac{F}{\lambda} \Delta x + \Theta_1^p - \Theta_0^p \right) \dots \dots (16)$$

Suppose that the solution of the finite difference equation T_n^p contains error δ_n^p ,

$$T_n^p = \Theta_n^p + \delta_n^p, \dots \dots \dots (17)$$

the difference equations concerning the error δ are obtained as follows by neglecting the high order term of δ .

$$\delta_n^{p+1} = \delta_n^p + \phi(\delta_{n-1}^p - 2\delta_n^p + \delta_{n+1}^p) \dots \dots (18)$$

$$\delta_0^{p+1} = \delta_0^p + 2\phi \left(\frac{\delta}{\lambda} \Delta x \frac{dF}{dT} + \delta_1^p - \delta_0^p \right) \dots \dots (19)$$

Equations (18) and (19) are the finite difference equations approximating the differential equation and boundary condition,

$$\frac{\partial \delta}{\partial t} = a \frac{\partial^2 \delta}{\partial x^2} \dots \dots \dots (20)$$

$$-\lambda \frac{\partial \delta}{\partial x} = \delta \frac{dF}{dT} \quad (x=0) \dots \dots \dots (21)$$

From Eqs. (20) and (21) it is clear that the error decreases when $dF/dT \leq 0$ and increases when $dF/dT > 0$. Consequently, the heat conduction problems can not be solved when $dF/dT > 0$ in Eq. (10) by the finite difference equation presented in this chapter.

It is easily seen that the problems with internal heat generation are also unsolvable when the heat generated increases with temperature.

In the case of backward time differencing this limitation is also valid.

2.4 Stability conditions

The stability condition for the finite difference equation (5) is obtained as follows.

Based on the preceding paragraph, the finite difference equation concerning the error δ with temperature dependent thermal properties is obtained.

$$\delta_n^{p+1} = \delta_n^p + \phi_{n,n-1}(\delta_{n-1}^p - \delta_n^p) + \phi_{n,n+1}(\delta_{n+1}^p - \delta_n^p) \dots \dots (22)$$

Equation (22) will be converted into inequality (23):

$$|\delta_n^{p+1}| \leq |1 - \phi_{n,n-1} - \phi_{n,n+1}| |\delta_n^p| + \phi_{n,n-1} |\delta_{n-1}^p| + \phi_{n,n+1} |\delta_{n+1}^p|$$

$$|\delta_n^{p+1}|_{\max(n)} \leq (|1 - \phi_{n,n-1} - \phi_{n,n+1}| + \phi_{n,n-1} + \phi_{n,n+1}) |\delta^p|_{\max(n)} \dots \dots (23)$$

where $|\delta^p|_{\max(n)}$ denotes the maximum error among all nodal temperatures at time $t = p\Delta t$.

If the relation

$$1 - \phi_{n,n-1} - \phi_{n,n+1} \geq 0 \dots \dots \dots (24)$$

is satisfied, Eq. (23) is reduced to

$$|\delta^{p+1}|_{\max(n)} \leq |\delta^p|_{\max(n)} \dots \dots \dots (25)$$

then the relation (24) is regarded as a condition of convergence for error δ . Since the relation (24) is

$$\frac{\lambda_{n,n-1} + \lambda_{n,n+1}}{c\rho_n} \frac{\Delta t}{(\Delta x)^2} \leq 1$$

the stability condition (sufficient condition) is given by

$$\frac{\lambda_{\max}}{c\rho_{\min}} \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \dots \dots \dots (26)$$

where $\lambda_{\max}$ and $c\rho_{\min}$ are respectively the maximum value of thermal conductivity and the minimum value of heat capacity over the entire temperature range of interest.

In the finite difference equation of nonlinear heat conduction, the thermal conductivity $\lambda_{m,n}$ and heat capacity $c\rho_n$ are not defined as the values corresponding with the same space and same time, so that the value $\lambda_{\max}/c\rho_{\min}$ is not equal to maximum value of thermal diffusivity $a_{\max}$. Defining a^* as

$$a^* = \lambda_{\max}/c\rho_{\min} \dots \dots \dots (27)$$

the stability condition of nonlinear difference equa-

tion may formally be written the same as the stability condition of linear difference equation.

$$\frac{a^* \Delta t}{(\Delta x)^2} \leq \frac{1}{2} \quad (28)$$

The stability condition of Eq. (7) can be obtained and be given by Eq. (28) similarly to Eq. (5) by neglecting $\delta\{d(c\rho)/dT\}$ because the variation of heat capacity is rather small.

The finite difference equation concerning δ of boundary condition, Eq. (11), is given by

$$\delta_0^{p+1} = \delta_0^p + 2\phi_{0,1} \left(\frac{\partial_0^p}{\lambda_{0,1}} \Delta x \frac{dF}{dT} + \delta_1^p - \delta_0^p \right) \quad (29)$$

From Eq. (29) the relations (30), (31) and (32) are obtained:

$$|\delta_0^{p+1}| \leq \left\{ 2\phi_{0,1} + \left| 1 + 2\phi_{0,1} \left(\frac{\Delta x}{\lambda_{0,1}} \frac{dF}{dT} - 1 \right) \right| \right\} \times |\delta^p|_{\max(0,1)} \quad (30)$$

$$1 + 2\phi_{0,1} \left(\frac{\Delta x}{\lambda_{0,1}} \frac{dF}{dT} - 1 \right) \geq 0 \quad (31)$$

$$|\delta_0^{p+1}| \leq \left(1 + 2 \frac{\Delta x}{\lambda_{0,1}} \frac{dF}{dT} \phi_{0,1} \right) |\delta^p|_{\max(0,1)} \quad (32)$$

The relation (33) is necessarily satisfied for $|\delta_0^{p+1}| \leq |\delta^p|_{\max(0,1)}$

$$1 + 2 \frac{\Delta x}{\lambda_{0,1}} \frac{dF}{dT} \phi_{0,1} \leq 1 \quad (33)$$

Relations (31) and (33) must be satisfied for converging error δ . Therefore the stability condition is given by

$$\frac{dF}{dT} \leq 0 \quad (34)$$

and

$$\frac{a^* \Delta t}{(\Delta x)^2} \leq \frac{1}{2 \{ 1 + (\Delta x / \lambda_{\max}) |dF/dT|_{\max} \}} \quad (35)$$

Relation (34) agrees with the condition presented in paragraph 2.3.

3. Accuracies of numerical results

Accuracies of numerical results of the finite difference equation are affected by the degree of temperature dependencies of thermal properties. The variations of thermal properties are not so great as near the melting point, and maximum value is only two times greater than minimum value, as shown in Figs. 1 and 7.

Here the numerical results are examined where thermal conductivity and heat capacity vary in ranges;

$$-\frac{2}{3} < \frac{d\lambda/d\theta}{\lambda_{\max}} < 2, \quad -\frac{2}{3} < \frac{dc\rho/d\theta}{c\rho_{\max}} < 2 \quad (36)$$

where

$$\theta = (T - T_{\min}) / (T_{\max} - T_{\min})$$

and the conditions sufficient to obtain numerical

solutions having accuracies over 94~99% are presented.

Results are obtained for a semi-infinite solid with initial temperature zero and surface temperature unity ($\theta=1$), which is the most striking change in temperature. The thermal conductivity and heat capacity are put as linear functions of temperature which give the greatest variation in the relation (36). The results with the other boundary conditions are experimentally examined in later chapter.

Numerical results by Runge-kutta method are substituted for exact solutions to examine the accuracies of solutions of the finite difference equation^{††}.

3.1 Influence of temperature dependencies of thermal properties on accuracies of numerical results

The numerical errors caused by the solution using Eq. (7) are shown in Figs. 2 and 3 when either thermal conductivity or heat capacity varies with temperature and the other is constant. The error δ is defined as difference between the solution of Eq. (7), θ , and the results by the Runge-Kutta, Θ ; $\delta = \Theta - \theta$. In the figures the maximum errors among all nodal temperatures at each repeated calculation are plotted.

The magnitude of the error is influenced by the degree of temperature dependencies of thermal

Fig. 2 Relations between accuracy and degree of temperature dependency of thermal conductivity $[a^* \Delta t / (\Delta x)^2 = 0.5]$

^{††} The partial differential equation of heat conduction

$$c\rho \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left(\lambda \frac{\partial \theta}{\partial x} \right)$$

is reduced to an ordinary differential equation

$$\frac{d}{d\zeta} \left(\lambda \frac{d\theta}{d\zeta} \right) + 2a^* \zeta c\rho \frac{d\theta}{d\zeta} = 0$$

with variable $\zeta = x / 2\sqrt{a^* t}$ and is solved numerically by the Runge-Kutta method.

A gradient $d\theta/d\zeta$ at $\zeta=0$ is determined by trial and error on the assumption that the condition of semi-infinite solid ($\theta=0$ at $\zeta \rightarrow \infty$) is satisfied when $|d\theta/d\zeta| < 10^{-14}$ and $|\theta| < 10^{-14}$.

[†] This value will be used to define the dimensionless time in later chapters.

properties and is brought close to the error contained in the solution of the linear finite difference equation as temperature dependencies decrease.

The numerical errors of Eq. (7) are shown in Fig. 4 when both thermal conductivity and heat capacity vary with temperature. Examples shown in Fig. 4 are combined with those having the greatest dependency within the relation (36). The errors arising from numerical operations may be bounded by errors of examples ① and ② in these ranges.

3.2 Influence of Δx on accuracy

Numerical errors of nodal temperatures between $x=0$ and $x=4\Delta x$ for the example ① in Fig. 4 are shown in Fig. 5. Since the greater errors may arise at the nodal point near the boundary, the accuracy of solutions can be increased by refining the subdivision.

3.3 Influence of Δt on accuracy

Preceding results were calculated with $a^*\Delta t/(\Delta x)^2=0.5$ (stability criterion). In Fig. 6 the errors contained in numerical results with variable time increments Δt for example ① in Fig. 4 are shown.

Fig. 3 Relations between accuracy and degree of temperature dependency of heat capacity
[$a^*\Delta t/(\Delta x)^2=0.5$]

Fig. 4 Relations between accuracy and degree of temperature dependency of thermal properties
[$a^*\Delta t/(\Delta x)^2=0.5$]

The errors arose to nearly the same degree and was not only dependent on Δt , but also dependent on dimensionless time $a^*t/(\Delta x)^2$. Therefore, the accuracy can hardly be increased by refining only Δt .

3.4 Condition for obtaining solutions with desired accuracy

To sum up the above findings, the accuracy of solution is affected by temperature dependencies of thermal properties and the greatest error arises in example ① in Fig. 4. Therefore the solutions having desired accuracies can be obtained if the numerical operation is carried out by considering $a^*t/(\Delta x)^2$ as being between two thick solid lines in Fig. 4.

That is to say, the solutions at $a^*t/(\Delta x)^2 > 7$ should be adopted if solutions having accuracies of 98% are desired.

These solid lines are smoothly drawn between $a^*t/(\Delta x)^2=2$ and 20 in order to cover all errors arising in numerical solutions within relation (36), and cover the accuracies over 94~99.3%, (at $a^*t/(\Delta x)^2 > 2$, the errors contained in the solution of example ① in Fig. 4 becomes greater than others).

Examples ▽ and ▽ plotted with the solid circles in Fig. 4 are results calculated by Eq. (5). Only a slight difference between results of Eq. (5) and Eq. (7) is observed, so that the con-

Fig. 5 Relations between accuracy and Δx
[$a^*\Delta t/(\Delta x)^2=0.5$]

Fig. 6 Relations between accuracy and time increment Δt

venient explicit method, Eq. (7), is advantageous rather than Eq. (5) for problems with heat capacity variation of such a degree.

4. Experimental examination

An exact solution of heat conduction in a slab with boundary conditions of a prescribed surface temperature or heat flux across the surface cannot be precisely obtained, accordingly, the numerical results must be compared with experimental results and the validity of the finite difference equation (7) can then be determined.

Two kinds of plastics—polymethylmethacrylate (PMMA) and polyethylene (PE)—were used as specimens in which thermal properties varied noticeably in a narrow temperature range.

4.1 Measurement of thermal properties

The thermal conductivity and heat capacity, and their temperature dependency were measured by the authors' "transient comparison method (parallel method)"⁽⁵⁾.

Two sets of specimens were made of the same material. Each specimen was composed of ten plates of material having dimensions 60mm × 60mm × 5mm, and a nichrome foil heater (thickness 0.03mm) was sandwiched at the center of the specimen. One of the specimens was set in a constant temperature bath and maintained at a constant temperature during measurement.

The other was set in a vacuum furnace with a controllable temperature, and measurements were carried out in 10°~20°C steps between room temperature and the melting point.

Thermal properties of PMMA were measured between 20°C and 250°C and those of PE between 20°C and 115°C. The measured results are shown

Fig. 7 Measurement results of thermal conductivity and heat capacity of polyethylene (PE) and polymethylmethacrylate (PMMA)

in Fig. 7.

The thermal conductivity of PMMA was nearly constant and that of PE decreased with temperature and could be approximated by a linear function of temperature. The heat capacity of PMMA changed noticeably at 77.5°C which was a glass transition temperature and was approximated by linear functions of temperature at the lower and upper sides of the glass transition temperature as shown in Fig. 7 with solid lines. The glass transition point was determined from measured results of variation of a specific volume with a dilatometer.

The heat capacity of polyethylene was approximated by a quadratic function of temperature. These coefficients of functions were determined by the least square method.

4.2 Comparison between numerical and experimental results

For a comparison between numerical and experimental results, measured surface temperatures and intensity of heat flux were used as boundary conditions for calculation of temperature responses in PMMA and PE respectively, and measured values of thermal properties were used.

4.2.1 Example I

A temperature response test as outlined in Fig. 8 was carried out using PMMA. A diagram of the apparatus used is shown in Fig. 9. The specimen was composed of 24 plates of PMMA

Fig. 8 The scheme of example I

Fig. 9 A diagram of experimental apparatus (Example I)

(105mm×105mm×5mm) and four sheets of graft carbon heater (thickness 0.1mm), which caused no flow of heat across the center surface. This was set in a vacuum vessel to prevent heat losses through the side faces of the specimen.

Temperatures were measured with C-A thermocouples, 0.1mm diameter, which were set in the interior of the specimen stack ($x=0, 5, 10, 15$ mm), and transient temperature responses were recorded with a multiple recorder.

The temperature response of the heating surface ($x=0$) was measured and used as the boundary condition for analysis. The maximum rise in temperature and duration of the experiment were determined in order that the specimen might be regarded as an infinite plate.

The experimental and numerical results are shown in Fig. 10. The temperature at the heating surface ($x=0$) was approximated by a function of time given by

$$T_{x=0} = 22.0 + 515\sqrt{t} \quad (37)$$

and shown in Fig. 10 with solid line, and used as the boundary condition for the finite difference equation.

Numerical results agreed well with experimental results and they differed only $2^\circ\sim 3^\circ\text{C}$ at the most. Although the heat capacity of PMMA varies noticeably near the glass transition tempera-

Fig. 10 Numerical results (solid lines) and experimental results (circles). Dashed lines show the results of linear heat conduction. (Example I)

Fig. 11 A scheme of example II

ture, an accurate enough solution can be obtained by the explicit equation (7).

The dashed lines in Fig. 10 show the result of the analytical solution of the linear heat conduction equation,

$$T = 22.5 + 515\sqrt{t}\pi \sum_{n=0}^{\infty} (-1)^n \times \left\{ \text{ierfc} \frac{2nS+x}{2\sqrt{at}} + \text{ierfc} \frac{2(n+1)S-x}{2\sqrt{at}} \right\}$$

which takes the average values of thermal properties between 20° and 250°C ; $\lambda = 0.161 \text{ kcal/mhr } ^\circ\text{C}$ and $c\rho = 525 \text{ kcal/m}^3\text{C}$.

These results clearly show that accurate results can not be obtained by analysis using constant thermal properties.

4.2.2 Example II

The experimental results shown in Fig. 11 of transient temperature response are shown. Each of four plates of PE (100mm×100mm×6mm) was put in layers at the lower and upper sides of a nichrome foil heater (thickness 0.03mm) and it was sandwiched between cooling plates which maintained both ends of the specimen at a constant temperature (Fig. 12).

Temperatures were measured with C-A ther-

Fig. 12 A diagram of the experimental apparatus (Example II)

Fig. 13 Numerical results (solid lines) and experimental results (circles). Dashed lines show results of linear heat conduction. (Example II)

mocouples ($0.1\text{mm}\phi$) buried inside the specimen stack ($x=0, 6, 12, 18, 24\text{ mm}$) and recorded with a multiple recorder. The intensity of the heat flux released from the electric heater was measured and used as the boundary condition for analyzing the temperature response. Measured intensity of heat flux was $F=2312\text{ kcal/m}^2\text{hr}$.

Numerical and experimental results are shown in Fig. 13. The results agreed well, having a difference of only 1°C at the most.

Consequently, it can be concluded that transient heat conduction in materials with temperature dependent thermal properties can be easily and accurately analyzed by the method presented in this paper.

The dashed lines in Fig. 13 show the result of

$$T = 15.5 + \frac{2F}{\sqrt{\lambda c \rho}} \sqrt{t} \sum_{n=0}^{\infty} (-1)^n \times \left\{ \text{ierfc} \frac{2nS+x}{2\sqrt{at}} - \text{ierfc} \frac{2(n+1)S-x}{2\sqrt{at}} \right\}$$

using an average values of thermal properties, $\lambda = 0.331\text{ kcal/mhr } ^\circ\text{C}$ and $c\rho = 607\text{ kcal/m}^3\text{ } ^\circ\text{C}$.

The time required for calculation of both examples was less than 30 seconds using the HITAC 5020E of the Computer Center of Tokyo University.

5. Conclusions

The finite difference method was applied for analysis of transient heat conduction with temperature dependent thermal properties.

The conclusions from the preceding paragraphs are summarized as follows.

(1) The finite difference equations of forward time difference approximating the partial differential equation and boundary conditions of heat conduction are presented and the stability conditions of them are shown.

(2) Applicability of the finite difference method is discussed and clarified so that numerical solutions can be obtained with boundary conditions in which surface temperature of heat flux across the surface is prescribed as a function of time, and heat flux F is a function of surface temperature when $dF/dT \leq 0$, but they can not be obtained when $dF/dT > 0$.

(3) In the temperature range except in the vicinity of the melting point, the accuracies of the numerical results obtained by the explicit method are discussed and the conditions were shown in which the desired accuracies can be obtained.

(4) The analysis method combined with the measurement of thermal properties using the authors' transient comparison method is given as an example, and the results are shown to agree well with experimental results.

In summation, the analysis method presented in this paper is proven to be a convenient and useful method of analyzing transient heat conduction with temperature dependent thermal properties.

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