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A Transient Coupled Thermoelastic Problem in the Semi-Infinite Medium*

By Masashi DAIMARUYA** and Masachika NAITO***

This paper deals with the analysis of a transient thermoelastic problem of a semi-infinite medium which is subjected to a thermal shock on its plane boundary. For the cases where the boundary temperatures are given by the "step-type" and the "ramp-type", the dynamic thermoelastic problems were previously studied in the absence of the thermoelastic coupling effect by Danilovskaya, Sternberg and Chakravorty, respectively.

In the present investigation the analysis is concerned with the coupled thermoelastic problem of a case in which the given boundary temperature is of the time-dependent "exponential-type". In the special case the exponential-type becomes the step-type or the ramp-type. Solving the problem by means of the Laplace transform, we obtained approximate solutions by a method of series expansion and numerical exact solutions by a numerical inversion technique.

1. Introduction

The thermoelastic problems are conventionally solved from Fourier's heat-conduction equation and equilibrium equations, including the temperature terms in the stress-strain relations. This procedure, however, omits two effects, namely, the thermoelastic coupling effect on the temperature distribution as a result of straining of the body, and the dynamic effects resulting from the inertia forces. Although the neglect of these effects is generally permissible in the usual problems of the thermoelastic stress analysis, in dynamic thermoelastic problems where the thermoelastic damping is of primary interest it is necessary that both these effects should be included. Hence such problems must be treated as coupled thermoelastic problems⁽¹⁾. The uniqueness of solution for the coupled thermoelastic theory has been proved by Weiner⁽²⁾.

The earliest study of inertia effects in the uncoupled thermoelastic problem is apparently attributed to Danilovskaya⁽³⁾. The problem considered there concerns an elastic half space free from traction being subjected to the "step-type" temperature change on its entire plane boundary, and being constrained against transverse displacements. A similar analysis was also treated independently by Mura⁽⁴⁾. Later

on, Danilovskaya⁽⁵⁾ extended her results to accommodate convective boundary conditions. Sternberg and Chakravorty⁽⁶⁾ studied a modified Danilovskaya's problem in which the half space was subjected to the "ramp-type" heating of the boundary plane. The preceding works clarified the inertia effects, but the thermoelastic coupling term in an equation of motion was disregarded.

Investigations on the effect of thermoelastic coupling, as well as inertia, have been made by many authors, such as Hetnarski⁽⁷⁾, Muki and Breuer⁽⁸⁾, Dillon⁽⁹⁾, Nickell and Sachman⁽¹⁰⁾, and others.

This paper is concerned with the coupled thermoelastic problem which is related to Danilovskaya's problem and Sternberg & Chakravorty's one. In this problem considered here, the boundary condition on temperature will be given in a form of the "exponential-type" heating, which seems to be physically realizable. In the special case, the exponential-type will become the step-type or the ramp-type.

The integral transformations are generally advantageous mathematical methods for an analysis of these problems. So we adopt the Laplace transform. To analyse easily we have obtained approximate solutions by means of series expansion as well as numerical exact solutions by a numerical inversion technique. The numerical results have shown that the approximate solutions are in good agreement with the numerical solutions.

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** Assistant, Faculty of Engineering, Hokkaido University, Kita-ku, Sapporo.

*** Professor, Muroran Institute of Technology.

2. Governing equations

The equations of motion for a linear isotropic solid are expressed as follows:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (i, j=1, 2, 3) \quad (1)$$

where x_i and t denote the Cartesian coordinates and the time, respectively, while u_i , σ_{ij} and ρ denote the Cartesian displacement components, the components of stress, and the density of material. In the infinitesimal displacement theory the strain components ε_{ij} are connected with the displacements by the relations

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (i, j=1, 2, 3) \quad (2)$$

while the displacement-stress relations, that is, the Duhamel-Neumann relations are given by

$$\sigma_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + 2\mu \varepsilon_{ij} - (3\lambda + 2\mu)\alpha T \delta_{ij} \quad (i, j, k=1, 2, 3) \quad (3)$$

Here λ and μ are Lamé's constants, T is the temperature, α is the coefficient of linear thermal expansion, and δ_{ij} is Kronecker's delta. Substituting Eqs. (2) and (3) into Eqs. (1), and representing them in the vector form, one can obtain a generalized Navier's equation.

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \text{grad div } \mathbf{u} = (3\lambda + 2\mu)\alpha \text{grad } T + \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \quad (4)$$

where $\mathbf{u}$ denotes the displacement vector.

On the other hand, the generalized equation of heat conduction, in which the coupling of the temperature and strain fields is considered, was deduced by Biot⁽¹¹⁾, Lessen⁽¹²⁾, and Boley⁽¹³⁾. The corrected heat equation is represented as follows:

$$\kappa \nabla^2 T = \frac{\partial T}{\partial t} + \frac{(3\lambda + 2\mu)\alpha T^*}{\rho c_v} \text{div } \frac{\partial \mathbf{u}}{\partial t} \quad (5)$$

where κ is the thermal diffusivity of the material, c_v is the specific heat at constant volume, T^* is an absolute temperature for which the material is stress-free, and T is the excess of temperature over the reference temperature T^* .

3. Formulation of problem

Departing from the indicative notation, let the Cartesian coordinates henceforth be (x, y, z) and we consider an elastic medium occupying the half-space $x \geq 0$. Assuming that the body is subjected to a time-dependent temperature change at its entire boundary plane and is constrained to the uniaxial motion, the components of displacement are

$$u_x = u(x, t), \quad u_y = u_z = 0 \quad (6)$$

Accordingly, the stress-displacement relations of Eqs. (3) reduce to

$$\left. \begin{aligned} \sigma_x &= (\lambda + 2\mu) \frac{\partial u}{\partial x} - (3\lambda + 2\mu)\alpha T \\ \sigma_y &= \sigma_z = \lambda \frac{\partial u}{\partial x} - (3\lambda + 2\mu)\alpha T \end{aligned} \right\} \quad (7)$$

while Eqs. (4) and (5) become

$$\left. \begin{aligned} (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} &= \rho \frac{\partial^2 u}{\partial t^2} + (3\lambda + 2\mu)\alpha \frac{\partial T}{\partial x} \\ \kappa \frac{\partial^2 T}{\partial x^2} &= \frac{\partial T}{\partial t} + \frac{(3\lambda + 2\mu)\alpha T^*}{\rho c_v} \frac{\partial^2 u}{\partial x \partial t} \end{aligned} \right\} \quad (8)$$

It is evident that the coupling of temperature and strain fields is due to the last term in the right hand sides of the second equation of Eqs. (8). If the coupling term in Eqs. (8) is neglected, the governing equations become uncoupled ones, which can be easily solved separately, first for the temperature and then for the stress and displacement fields.

In the present paper, however, we will analyze the following problem as a coupled thermoelastic problem. The initial and boundary conditions are taken as follows: If the medium is initially undisturbed and at zero temperature, then the initial conditions for this problem are

$$\left. \begin{aligned} T(x, 0) &= 0 \quad (0 \leq x < \infty) \\ u(x, 0) &= \left[\frac{\partial u}{\partial t} \right]_{t=0} = 0 \end{aligned} \right\} \quad (9)$$

and if the half space is heated on its surface plane, whose temperature changes exponentially, and the surface plane is free of tractions, the boundary conditions are taken as

$$\left. \begin{aligned} \sigma_x(0, t) &= 0 \quad (0 \leq t < \infty) \\ T(0, t) &= T_0 \left[1 - \exp\left(-\frac{t}{t_0}\right) \right] \end{aligned} \right\} \quad (10)$$

where T_0 and t_0 are constants (refer to Fig. 1). The latter of Eqs. (10) shows the "exponential-type" boundary condition on temperature, so when $t_0 = 0$ it becomes the "step-type" one, i.e. $T(0, t) = TH(t)$, where $H(t)$ denotes a Heaviside step function defined by

$$H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases} \quad (11)$$

On the other hand, expanding the exponential-type in a series and neglecting its higher order, one can obtain the "ramp-type" proposed by Sternberg &

Fig. 1 Boundary conditions

Chakravorty⁽⁶⁾.

Finally, we add the regularity requirements

$$T(x, t), u(x, t), \sigma_x(x, t) \rightarrow 0 \text{ as } x \rightarrow \infty \quad \dots\dots(12)$$

Here, with the introduction of the following dimensionless quantities

$$\left. \begin{aligned} \xi &= \frac{cx}{\kappa}, & \tau &= \frac{c^2 t}{\kappa} \\ U &= \frac{(1-\nu)cu}{(1+\nu)\kappa\alpha T_0}, & \Theta &= \frac{T}{T_0} \end{aligned} \right\} \dots\dots\dots(13)$$

where ν is Poisson's ratio and c is the dilatational wave velocity given by

$$c = \sqrt{\frac{2(1-\nu)\mu}{(1-2\nu)\rho}}$$

Equations (8) can be written as

$$\frac{\partial^2 U}{\partial \xi^2} = \frac{\partial^2 U}{\partial \tau^2} + \frac{\partial \Theta}{\partial \xi}, \quad \frac{\partial^2 \Theta}{\partial \xi^2} = \frac{\partial \Theta}{\partial \tau} + \delta \frac{\partial^2 U}{\partial \xi \partial \tau} \quad \dots\dots\dots(14)$$

where δ denotes the thermoelastic coupling parameter given as

$$\delta = \frac{2(1+\nu)^2 \mu \alpha^2 T_0}{(1-\nu)(1-2\nu)\rho c_v} \quad \dots\dots\dots(15)$$

while the dimensionless components of stress are expressed as

$$\sigma_{x0} = \varepsilon_{x0} - \Theta, \quad \sigma_{y0} = \nu \sigma_{x0} - (1-2\nu)\Theta \quad \dots\dots\dots(16)$$

where

$$\begin{aligned} \sigma_{x0} &= \frac{(1-2\nu)\sigma_x}{2(1+\nu)\mu\alpha T_0}, & \sigma_{y0} &= \frac{(1-2\nu)\sigma_y}{2(1+\nu)\mu\alpha T_0} \\ \varepsilon_{x0} &= \frac{1-\nu}{(1+\nu)\alpha T_0} \frac{\partial u}{\partial x} = \frac{\partial U}{\partial \xi} \end{aligned}$$

In view of Eqs. (13) and (16), initial conditions Eqs. (9), boundary conditions Eqs. (10) and regularity conditions Eqs. (12) are written as follows.

Initial conditions:

$$\Theta(\xi, 0) = 0, \quad U(\xi, 0) = \left[\frac{\partial U}{\partial \tau} \right]_{\tau=0} = 0 \quad \dots\dots\dots(17)$$

Boundary conditions:

$$\sigma_{x0}(0, \tau) = 0, \quad \Theta(0, \tau) = 1 - \exp\left(-\frac{\tau}{\tau_0}\right) \quad \dots\dots\dots(18)$$

Regularity conditions:

$$\Theta(\xi, \tau), \quad U(\xi, \tau), \quad \sigma_{x0}(\xi, \tau) \rightarrow 0 \text{ as } \xi \rightarrow \infty \quad \dots\dots(19)$$

where $\tau_0 = c^2 t_0 / \kappa$.

Figure 2 shows the time-dependence of the

Fig. 2 Impact temperatures for various values of τ_0

dimensionless temperature $\Theta(0, \tau)$ at the boundary $\xi=0$ for $\tau_0=0.25, 1, 2$ and 3 as well as for the limiting case $\tau_0=0$.

4. Solution of the laplace transform domain

The problem represented by Eqs. (17) to (19) is most conveniently treated with the aid of the Laplace transform. We write $\bar{\Theta}(\xi, p)$ and $\bar{u}(\xi, p)$ for the Laplace transforms with respect to τ of $\Theta(\xi, \tau)$ and $U(\xi, \tau)$.

Thus

$$\left. \begin{aligned} \bar{\Theta}(\xi, p) &= \int_0^\infty \Theta(\xi, \tau) \exp(-p\tau) d\tau \\ \bar{u}(\xi, p) &= \int_0^\infty U(\xi, \tau) \exp(-p\tau) d\tau \end{aligned} \right\} \dots\dots\dots(20)$$

Similarly, let the Laplace transforms of $\sigma_{x0}(\xi, \tau)$ and $\varepsilon_{x0}(\xi, \tau)$ be $\bar{\sigma}_{x0}(\xi, p)$ and $\bar{\varepsilon}_{x0}(\xi, p)$ respectively.

A formal application of the Laplace transform to Eqs. (16) and (14), in view of the initial conditions Eqs. (7), yields

$$\left. \begin{aligned} \bar{\sigma}_{x0}(\xi, p) &= \frac{d}{d\xi} \bar{u}(\xi, p) - \bar{\Theta}(\xi, p) \\ \bar{\varepsilon}_{x0}(\xi, p) &= \frac{d}{d\xi} \bar{u}(\xi, p) \end{aligned} \right\} \dots\dots\dots(21)$$

$$\frac{d^2 \bar{u}}{d\xi^2} - p^2 \bar{u} = \frac{d\bar{\Theta}}{d\xi}, \quad \frac{d^2 \bar{\Theta}}{d\xi^2} - p\bar{\Theta} = \delta p \frac{d\bar{u}}{d\xi} \quad \dots\dots\dots(22)$$

Equations (22), by virtue of Eqs. (18) and (19), are subjected to the following boundary and regularity conditions:

$$\left. \begin{aligned} \left[\frac{d\bar{u}}{d\xi} \right]_{\xi=0} &= \bar{\Theta}(0, p) = \frac{1}{p_0} \\ \bar{u}(\xi, p), \quad \bar{\Theta}(\xi, p) & \rightarrow 0 \text{ as } \xi \rightarrow \infty \end{aligned} \right\} \dots\dots\dots(23)$$

where $p_0 = p(\tau_0 p + 1)$.

Eliminating $\bar{u}(\xi, p)$ from Eqs. (22), we find the fourth-order differential equations for $\bar{\Theta}(\xi, p)$ as

$$\frac{d^4 \bar{\Theta}}{d\xi^4} - p(p+1+\delta) \frac{d^2 \bar{\Theta}}{d\xi^2} + p^3 \bar{\Theta} = 0 \quad \dots\dots\dots(24)$$

and using Eqs. (21), (22) and (23) the associated boundary and regularity conditions are given by

$$\left. \begin{aligned} \bar{\Theta}(0, p) &= \frac{1}{p_0}, \quad \left[\frac{d^2 \bar{\Theta}}{d\xi^2} \right]_{\xi=0} = (1+\delta) \frac{p}{p_0} \\ \bar{\Theta}(\xi, p) & \rightarrow 0 \text{ as } \xi \rightarrow \infty \end{aligned} \right\} \dots\dots\dots(25)$$

On the other hand, eliminating $\bar{\Theta}(\xi, p)$ from Eqs. (22), we find the equation for $\bar{u}(\xi, p)$ as

$$\frac{d^4 \bar{u}}{d\xi^4} - p(p+1+\delta) \frac{d^2 \bar{u}}{d\xi^2} + p^3 \bar{u} = 0 \quad \dots\dots\dots(26)$$

and the associated boundary and regularity conditions become

$$\left. \begin{aligned} \left[\frac{d\bar{u}}{d\xi} \right]_{\xi=0} &= \frac{1}{p_0} \\ \left[\frac{d^3 \bar{u}}{d\xi^3} \right]_{\xi=0} &= (p+1+\delta) \frac{p}{p_0} \\ \bar{u}(\xi, p) & \rightarrow 0 \text{ as } \xi \rightarrow \infty \end{aligned} \right\} \dots\dots\dots(27)$$

Equations (24) and (26) are of the same form, hence both these characteristic equations are given by

$$\lambda^4 - p(p+1+\delta)\lambda^2 + p^3 = 0 \quad \dots\dots\dots(28)$$

Consequently, the general solutions of Eqs. (24) and (26), under considering Eqs. (25) and (27), can be expressed as

$$\left. \begin{aligned} \bar{\theta}(\xi, p) &= A_\theta \exp(-\lambda_1 \xi) + B_\theta \exp(-\lambda_2 \xi) \\ \bar{u}(\xi, p) &= A_u \exp(-\lambda_1 \xi) + B_u \exp(-\lambda_2 \xi) \end{aligned} \right\} \quad \dots\dots\dots(29)$$

in which A_θ , B_θ , A_u and B_u are integral constants to be determined from boundary conditions, while λ_1 and λ_2 are the solutions of Eq. (28), given by

$$\left. \begin{aligned} \lambda_{1,2} &= \left(\frac{p}{2}\right)^{1/2} [p+1+\delta \pm \{p^2 - 2(1-\delta)p + (1+\delta)^2\}^{1/2}]^{1/2} \end{aligned} \right\} \quad \dots\dots\dots(30)$$

Using the boundary conditions Eqs. (25) and (27), the arbitrary constants are determined as follows:

$$\left. \begin{aligned} A_\theta &= \frac{\lambda_1^2 - p^2}{\lambda_1^2 - \lambda_2^2}, & B_\theta &= -\frac{\lambda_2^2 - p^2}{\lambda_1^2 - \lambda_2^2} \\ A_u &= -\frac{\lambda_1}{\lambda_1^2 - \lambda_2^2}, & B_u &= \frac{\lambda_2}{\lambda_1^2 - \lambda_2^2} \end{aligned} \right\} \quad \dots\dots\dots(31)$$

With Eqs. (29), (31) and (21), we obtain the following solutions for the temperature $\bar{\theta}(\xi, p)$, the strain $\bar{\epsilon}_{x0}(\xi, p)$ and the stress $\bar{\sigma}_{x0}(\xi, p)$ in the Laplace transform domain:

$$\left. \begin{aligned} \bar{\theta}(\xi, p) &= \frac{\lambda_1^2 - p^2}{(\lambda_1^2 - \lambda_2^2)p_0} \exp(-\lambda_1 \xi) \\ &\quad - \frac{\lambda_2^2 - p^2}{(\lambda_1^2 - \lambda_2^2)p_0} \exp(-\lambda_2 \xi) \\ \bar{\epsilon}_{x0}(\xi, p) &= \frac{\lambda_1^2}{(\lambda_1^2 - \lambda_2^2)p_0} \exp(-\lambda_1 \xi) \\ &\quad - \frac{\lambda_2^2}{(\lambda_1^2 - \lambda_2^2)p_0} \exp(-\lambda_2 \xi) \\ \bar{\sigma}_{x0}(\xi, p) &= \frac{p^2}{(\lambda_1^2 - \lambda_2^2)p_0} \exp(-\lambda_1 \xi) \\ &\quad - \frac{p^2}{(\lambda_1^2 - \lambda_2^2)p_0} \exp(-\lambda_2 \xi) \end{aligned} \right\} \quad \dots\dots\dots(32)$$

5. Approximate solution by means of a method of series expansion

The Laplace inverse transform of Eqs. (32) can be done with the contour integral by Jordan's lemma, and can be obtained by the integration along these paths shown in Fig. 3. However, there are branch points at $p = 1 - \delta + 2\sqrt{\delta}$, which cause most of the difficulty in the inversion process. In the present paper, we seek approximate solutions by means of series expansion based on the initial-value theorem, and then numerical solutions by means of a numerical inversion technique. It is noted that the initial-value theorem is represented as follows: Provided that $\bar{f}(p)$ is the Laplace transform, with respect to τ , of a function $f(\tau)$, and $\lim_{\tau \rightarrow 0} f(\tau)$ exists, then a

relation $\lim_{\tau \rightarrow 0} f(\tau) = \lim_{p \rightarrow \infty} p\bar{f}(p)$ should hold. Namely, this theorem will signify short-time solutions corresponding to large p solutions in the Laplace transform domain.

Hence, we expand λ_1 and λ_2 of Eqs. (30) for large values of p in a series and obtain the following Laurent's series:

$$\left. \begin{aligned} \lambda_1 &= p + \frac{\delta}{2} + \frac{\delta(4-\delta)}{8p} + O(p^{-2}) \\ \lambda_2 &= \sqrt{p} - \frac{\delta}{2\sqrt{p}} + O(p^{-3/2}) \end{aligned} \right\} \quad \dots\dots\dots(33)$$

By neglecting the negative powers in Eqs. (33), $\lambda_1^2 - \lambda_2^2$ can be expressed as follows:

$$\left. \begin{aligned} \lambda_1^2 - \lambda_2^2 &= (p-a)(p-b) \\ a, b &= (1/2)(1-\delta \pm \sqrt{1-2\delta}) \end{aligned} \right\} \quad \dots\dots\dots(34)$$

Now we assume that a and b are real numbers provided $0 \leq \delta \leq 0.5$. In this case there are the following relations among a , b and δ :

$$a + \delta/2 = \sqrt{a}, \quad b + \delta/2 = \sqrt{b} \quad \dots\dots\dots(35)$$

Substituting Eqs. (33) and (34) into Eqs. (32), we find the following equations.

$$\left. \begin{aligned} \bar{\theta}(\xi, p) &= \frac{1}{p_0} \exp(-\xi\sqrt{p}) + \frac{\delta(4p+\delta)}{4p_0(p-a)(p-b)} \\ &\quad \times [\exp\{-\xi(p+\delta/2)\} - \exp(-\xi\sqrt{p})] \\ \bar{\epsilon}_{x0}(\xi, p) &= \frac{1}{p_0} \exp(-\xi(p+\delta/2)) + \frac{p}{p_0(p-a)(p-b)} \\ &\quad \times [\exp\{-\xi(p+\delta/2)\} - \exp(-\xi\sqrt{p})] \\ \bar{\sigma}_{x0}(\xi, p) &= \frac{p^2}{p_0(p-a)(p-b)} \\ &\quad \times [\exp\{-\xi(p+\delta/2)\} - \exp(-\xi\sqrt{p})] \end{aligned} \right\} \quad \dots\dots\dots(36)$$

Using the convolution integral, we can easily perform the inverse Laplace transform of Eqs. (36).

5.1 Solutions for the step-type

By setting $p_0 = p$ in Eqs. (36), we can seek solutions for the step-type. In applying the convolution integral to Eqs. (36), the solutions for the dimensionless temperature, strain and stress can be obtained as follows.

Fig. 3 Paths of the inversion integral

$$\text{Temperature: } \Theta(\xi, \tau) = \frac{1}{a-b} e^{-(\delta/2)\xi} [\{ (a-b) + (\delta+b)e^{a(\tau-\xi)} - (\delta+a)e^{b(\tau-\xi)} \} H(\tau-\xi) - \{ (\delta+b)e^{a(\tau-\xi)} D_a(\varphi) - (\delta+b)e^{b(\tau-\xi)} D_b(\varphi) \}] \dots (37)$$

$$\text{Strain: } \varepsilon_{x0}(\xi, \tau) = \frac{1}{a-b} e^{-(\delta/2)\xi} [\{ (a-b) + e^{a(\tau-\xi)} - e^{b(\tau-\xi)} \} \times H(\tau-\xi) - \{ e^{a(\tau-\xi)} D_a(\varphi) - e^{b(\tau-\xi)} D_b(\varphi) \}] \dots (38)$$

$$\text{Stress: } \sigma_{x0}(\xi, \tau) = \frac{1}{a-b} e^{-(\delta/2)\xi} [\{ (a-b) + ae^{a(\tau-\xi)} - be^{b(\tau-\xi)} \} H(\tau-\xi) - \{ ae^{a(\tau-\xi)} D_a(\varphi) - be^{b(\tau-\xi)} D_b(\varphi) \}] \dots (39)$$

Here $H(\tau-\xi)$ is a Heaviside step function defined by Eq. (11), $D_a(\varphi)$ and $D_b(\varphi)$ denote the following integrations:

$$D_f(\varphi) = \frac{2}{\sqrt{\pi}} \int_{\varphi}^{\infty} \exp \left\{ - \left(\eta - \sqrt{f} \frac{\xi}{2\eta} \right)^2 \right\} d\eta, \quad \varphi = \frac{\xi}{2\sqrt{\tau}} \left\{ \dots (40) \right.$$

$$= \frac{1}{2} e^{4C} \operatorname{erfc} \left(\varphi + \frac{C}{\varphi} \right) + \frac{1}{2} \operatorname{erfc} \left(\varphi - \frac{C}{\varphi} \right), \quad C = \sqrt{f} \frac{\xi}{2}$$

in which $\operatorname{erfc}(\varphi)$ is a complementary error function defined by

$$\operatorname{erfc}(\varphi) = \frac{2}{\sqrt{\pi}} \int_{\varphi}^{\infty} e^{-\eta^2} d\eta \dots (41)$$

When the coupling effect is neglected in Eqs. (37), (38) and (39), in other words, in the case of an uncoupled problem, then $a=1$, $b=0$, $D_{f=0}(\varphi) = \operatorname{erfc}(\varphi)$, the solutions become as follows, which will coincide with Danilovskaya's solutions;

$$\left. \begin{aligned} \Theta(\xi, \tau) &= \operatorname{erfc}(\varphi) \\ \varepsilon_{x0}(\xi, \tau) &= \operatorname{erfc}(\varphi) + \{ H(\tau-\xi) - D_{f=1}(\varphi) \} \exp(\tau-\xi) \\ \sigma_{x0}(\xi, \tau) &= \{ H(\tau-\xi) - D_{f=1}(\varphi) \} \exp(\tau-\xi) \end{aligned} \right\} \dots (42)$$

5.2 Solutions for the exponential-type

By setting $p_0 = p(\tau_0 p + 1)$, in the same way as section 5.1, we can obtain also the solutions for the case of the exponential-type.

$$\text{Temperature: } \Theta(\xi, \tau) = e^{-(\delta/2)\xi} \left\{ 1 + \frac{\delta\tau_0(1-\delta\tau_0/4)}{(1+a\tau_0)(1+b\tau_0)} e^{(\xi-\tau)/\tau_0} \right\} H(\tau-\xi) - \frac{1+\tau_0}{(1+a\tau_0)(1+b\tau_0)} e^{-\tau/\tau_0} N(\varphi) + e^{-(\delta/2)\xi} \left[\frac{(b+\delta)e^{a(\tau-\xi)}}{(a-b)(1+a\tau_0)} \{ H(\tau-\xi) - D_a(\varphi) \} - \frac{(a+\delta)e^{b(\tau-\xi)}}{(a-b)(1+b\tau_0)} \{ H(\tau-\xi) - D_b(\varphi) \} \right] \dots (43)$$

$$\text{Strain: } \varepsilon_{x0}(\xi, \tau) = e^{-(\delta/2)\xi} \left\{ 1 - \frac{(1+\delta\tau_0)/2}{(1+a\tau_0)(1+b\tau_0)} e^{(\xi-\tau)/\tau_0} \right\} H(\tau-\xi) - \frac{\tau_0}{(1+a\tau_0)(1+b\tau_0)} e^{-\tau/\tau_0} N(\varphi) + e^{-(\delta/2)\xi} \left[\frac{e^{a(\tau-\xi)}}{(a-b)(1+a\tau_0)} \{ H(\tau-\xi) - D_a(\varphi) \} - \frac{e^{b(\tau-\xi)}}{(a-b)(1+b\tau_0)} \{ H(\tau-\xi) - D_b(\varphi) \} \right] \dots (44)$$

$$\text{Stress: } \sigma_{x0}(\xi, \tau) = -\frac{1}{(1+a\tau_0)(1+b\tau_0)} \{ e^{-(\delta/2)\xi} e^{(\xi-\tau)/\tau_0} H(\tau-\xi) - e^{-\tau/\tau_0} N(\varphi) \} + e^{-(\delta/2)\xi} \left[\frac{ae^{a(\tau-\xi)}}{(a-b)(1+a\tau_0)} \{ H(\tau-\xi) - D_a(\varphi) \} - \frac{be^{b(\tau-\xi)}}{(a-b)(1+b\tau_0)} \{ H(\tau-\xi) - D_b(\varphi) \} \right] \dots (45)$$

In Eqs. (43), (44) and (45), $N(\varphi)$ denotes the following integration:

$$N(\varphi) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \varphi \exp \left[-\frac{\tau}{\tau_0} \left\{ 1 - \frac{1}{(\eta+1)^2} \right\} - \varphi^2 (\eta+1)^2 \right] d\eta, \quad \varphi = \frac{\xi}{2\sqrt{\tau}} \dots (46)$$

5.3 Discontinuities at an elastic wave front

The discontinuity will be denoted by the symbol S ; thus,

$$SF(x_0) = F(x_0+0) - F(x_0-0) \dots (47)$$

For the case of the step-type, discontinuities at elastic wave fronts ($\xi=\tau$) are obtained as follows, using Eqs. (37), (38) and (39):

$$\left. \begin{aligned} S\Theta(\xi, \tau) &= \Theta(\xi, \tau+0) - \Theta(\xi, \tau-0) = 0 \\ S\varepsilon_{x0}(\xi, \tau) &= \varepsilon_{x0}(\xi, \tau+0) - \varepsilon_{x0}(\xi, \tau-0) = \exp\left(-\frac{\delta}{2}\xi\right) \quad (\xi=\tau) \\ S\sigma_{x0}(\xi, \tau) &= \sigma_{x0}(\xi, \tau+0) - \sigma_{x0}(\xi, \tau-0) = \exp\left(-\frac{\delta}{2}\xi\right) \end{aligned} \right\} \dots (48)$$

Equations (48) show that discontinuities at the wave front are of the magnitude of $\exp\{-(\delta/2)\xi\}$ for strain and stress, but vanish for temperature. These results coincide with ones of Hetnarski⁽⁷⁾ and Boley⁽¹³⁾⁽¹⁴⁾. Further-

more, when δ is equal to zero, the strain and stress have discontinuities of unity magnitude at all times, and this fact satisfies the results of Danilovskaya⁽³⁾ and Mura⁽⁴⁾. In the case of the exponential-type one can find the discontinuities vanish at the wave front, regardless of existence of the thermoelastic coupling parameter;

$$S\theta(\xi, \tau) = S\varepsilon_{x0}(\xi, \tau) = S\sigma_{x0}(\xi, \tau) = 0 \quad (\xi = \tau) \quad (49)$$

6. Numerical solutions by a numerical inversion technique

Next, we shall seek numerical exact solutions of Eqs. (32), by using a numerical inversion technique for the Laplace transform. If $f(\xi, p)$ denotes the Laplace transform of $F(\xi, \tau)$ of time τ , its inversion is represented in the form

$$F(\xi, \tau) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f(\xi, p) e^{p\tau} dp \quad (50)$$

where γ is an arbitrary real constant such that all singularities of $f(\xi, p)$ are in $R_e(p) < \gamma$.

Putting the transform parameter p as

$$p = \gamma + i\eta \quad (\eta \text{ is a real number}) \quad (51)$$

Eq. (50) may be written in the following form, from Schmittroth's paper⁽¹⁵⁾;

$$\left. \begin{aligned} F(\xi, \tau) &= -e^{\gamma\tau} \frac{2}{\pi} \int_0^\infty I_m[f(\xi, \gamma + i\eta)] \sin(\eta\tau) d\eta \\ F(\xi, \tau) &= e^{\gamma\tau} \frac{2}{\pi} \int_0^\infty R_e[f(\xi, \gamma + i\eta)] \cos(\eta\tau) d\eta \end{aligned} \right\} \quad (52)$$

where $I_m\{\}$ and $R_e\{\}$ denote the real and imaginary parts, respectively. The second equation of Eqs. (52) is also written as follows, by making the transform $\eta = \pi u/\tau$:

$$F(\xi, \tau) = e^{\gamma\tau} \frac{2}{\tau} \int_0^\infty R_e\left\{f\left(\xi, \gamma + i\frac{\pi}{\tau}u\right)\right\} \times \cos(\pi u) du \quad (53)$$

The integral interval of Eq. (53) can be divided into the first quarter-cycle of $\cos(\pi u)$ and each succeeding half-cycle;

$$F(\xi, \tau) = e^{\gamma\tau} \frac{2}{\tau} \left\{ \int_0^{1/2} + \sum_{n=1}^\infty \int_{(2n-1)/2}^{(2n+1)/2} \right\} \times R_e\left\{f\left(\xi, \gamma + i\frac{\pi}{\tau}u\right)\right\} \cos(\pi u) du \quad (54)$$

Procedures on the first equation of Eqs. (52) will be done in much the same way as the preceding. Furthermore, in order to accelerate the convergence of the above integration, the following averaging technique can be employed. Assuming that $F_n(\xi, \tau)$ denotes integral values of the n -th cycle and S_m denotes the partial sum up to the m -th cycle the relation between them would be expressed as

$$S_m = \sum_{n=0}^m F_n(\xi, \tau) \quad (55)$$

The limit of the sequence $S_0, S_1, \dots, S_m, \dots$ is then found by the averaging technique⁽¹⁶⁾. That is,

$$\left. \begin{aligned} S_m^{(1)} &= \frac{1}{2}(S_m + S_{m+1}) \\ S_m^{(k)} &= \frac{1}{2}(S_m^{(k-1)} + S_{m+1}^{(k-1)}) \end{aligned} \right\} \dots\dots\dots (56)$$

hence, $F(\xi, \tau)$ is expressed as follows:

$$F(\xi, \tau) = \lim_{m \rightarrow \infty} S_m = \lim_{k \rightarrow \infty} S_m^{(k)} \quad (57)$$

The summation procedure described here will be useful because the equivalent series converges more rapidly than the original series.

For example, by applying the preceding method to σ_{x0} in Eqs. (32), $R_e(\bar{\sigma}_{x0})$ can be represented as follows:

$$\left. \begin{aligned} R_e\{\bar{\sigma}_{x0}(\xi, \gamma + i\eta)\} &= R_e\{f_1\} - R_e\{f_2\} \\ R_e\{f_j\} &= \frac{1}{\xi\sqrt{r}} \cos\left(\xi\sqrt{q_j} \sin\frac{\theta_j}{2} + \frac{\omega}{2} + \phi\right) \\ &\times \exp\left(-\xi\sqrt{q_j} \cos\frac{\theta_j}{2}\right) (j=1, 2) \end{aligned} \right\} \quad (58)$$

where, provided $\gamma > 1 - \delta$,

$$\begin{aligned} \tan \omega &= \frac{2(\gamma - 1 + \delta)\eta}{\gamma^2 - \eta^2 - 2(1 - \delta)\gamma + (1 + \delta)^2} \\ r &= [4(\gamma - 1 + \delta)^2\eta^2 + (\gamma^2 - \eta^2 - 2(1 - \delta)\gamma + (1 + \delta)^2)^2]^{1/2} \\ \tan \phi &= \frac{\tau_0\eta}{\tau_0\gamma + 1}, \quad \zeta = [(\tau_0\eta)^2 + (\tau_0\gamma + 1)^2]^{1/2} \\ \tan \theta_{1,2} &= \frac{(2\gamma + 1 + \delta)\eta \pm \sqrt{r} \left(\gamma \sin \frac{\omega}{2} + \eta \cos \frac{\omega}{2} \right)}{\gamma^2 - \eta^2 + (1 + \delta)\gamma \pm \sqrt{r} \left(\gamma \cos \frac{\omega}{2} - \eta \sin \frac{\omega}{2} \right)} \\ q_{1,2} &= \frac{1}{2} \left[\left\{ (2\gamma + 1 + \delta)\eta \pm \sqrt{r} \left(\gamma \sin \frac{\omega}{2} + \eta \cos \frac{\omega}{2} \right) \right\}^2 + \left\{ \left(\gamma^2 - \eta^2 + (1 + \delta)\gamma \right. \right. \right. \\ &\quad \left. \left. \pm \sqrt{r} \left(\gamma \cos \frac{\omega}{2} - \eta \sin \frac{\omega}{2} \right) \right\}^2 \right]^{1/2} \end{aligned}$$

7. Numerical results

The numerical evaluations are illustrated in Figs. 4~14. The numerical integrations in $D_f(\varphi)$ and $N(\varphi)$ were accomplished by the use of Simpson's rule. On the evaluation of numerical solutions, we used also Simpson's rule because the integrand in an individual cycle was smooth. In calculating Eqs. (56), the five averages were used, i.e. $k=5$, and the computation was stopped if

$$\left| \frac{S_m^{(5)} - S_{m+1}^{(5)}}{S_{m+1}^{(5)}} \right| < 10^{-5}$$

The values of the coupling parameter δ are smaller than unity for most of the real materials. In the case of equilibrium temperature $T^* = 293^\circ\text{K}$ the following values for each material are found from

Eq. (15); steel (0.009), copper (0.018), aluminum (0.031), lead (0.079). However, in the numerical calculation the coupling parameter was taken as 0.1 or 0.3. Figures 4, 5 and 6 show the uncoupled case ($\delta=0$) given by Eqs. (42), (43), (44) and (45). The time-dependences of the dimensionless stress σ_{x0} , strain ε_{x0} and temperature Θ are depicted as a function of dimensionless time τ at the dimensionless distance $\xi=2$, for the five different values of dimensionless time constant τ_0 . The stress and the strain for $\tau_0=0$, as predicted by Eqs. (48) and (49), exhibit unity jump-discontinuity at the wave front, but for $\tau_0>0$ they are continuous. The values of stress and strain at the wave front, being entirely due to an inertia effect, decline rapidly with the increase of τ_0 .

Fig. 4 Time-dependence of the stress at $\xi=2$ for various values of τ_0 (in the case of $\delta=0$)

Fig. 5 Time-dependence of the strain at $\xi=2$ for various values of τ_0 (in the case of $\delta=0$)

Fig. 6 Time-dependence of the temperature at $\xi=2$ for various values of τ_0 (in the case of $\delta=0$)

Regardless of values of τ_0 , the stress tends to the steady-state value which is always equal to zero, as $\tau \rightarrow \infty$, while the strain tends to unity.

The time-dependence of σ_{x0} at the wave front, i.e. $\sigma_{x0}(\tau, \tau+0)$ and $\sigma_{x0}(\tau, \tau-0)$, is illustrated in Fig. 7. In this figure the solid curves represent the results computed on the basis of the numerical exact solutions of Eqs. (58). On the other hand, the circular marks ($\circ$) for $\sigma_{x0}(\tau, \tau-0)$ represent the results computed from the approximate solutions in Eqs. (39). It is evident from Fig. 7 that the series solutions agree well with the numerical solutions at the wave front. When time increases, the magnitude of discontinuities for the coupled case departs from the uncoupled one being always equal to unity, and the former approaches zero with the acceleration of $\exp\{-(\delta/2)\xi\}$.

Figures 8, 9 and 10 show the time-dependences of stress, strain and temperature for the case of the exponential-type $\tau_0=0.5$ at the distance $\xi=1$. In the same manner as Fig. 7, the solid or dashed lines denote numerical solutions and the circular marks ($\circ$ and $\bullet$) denote series solutions. In these figures the coupling parameter has been taken as $\delta=0, 0.1$

Fig. 7 Comparison of series solutions with numerical solutions for the stress at wave fronts (for case of the step-type $\tau_0=0$)

Fig. 8 Time-dependence of the stress at $\xi=1$ (for case of the exponential-type $\tau_0=0.5$)

and 0.3. In Figs. 9 and 10, the coupling effects on the strain and the temperature appear with the increase of time. This phenomenon will be easily understood from the fact that the thermoelastic coupling effect arises as a consequence of the coupling between the strain and temperature fields.

Judging from Figs. 7~10, the approximate solutions are reasonable, so the following figures have been obtained from the computation of the series solution.

Figures 11 and 12 show the stress wave and the temperature distributions for the case of the step-type at $\delta=1$, respectively. These figures show the difference between the coupled case ($\delta=0.1$) and the uncoupled case ($\delta=0$).

Similarly, Fig. 13 shows the stress wave and the temperature distribution for the exponential-type $\delta=0.25$ at $\tau=1$. Here coupling parameters are taken as $\delta=0.0$ and 0.3. It will be seen from Figs. 11, 12 and 13 that the thermoelastic coupling effect suppresses the stress wave propagation and also heat conduction, and affects these significantly at the

wave front.

The propagating stress waves for the exponential-type $\tau_0=0.25$ are shown in Fig. 14, which illustrates a comparison between the uncoupled case ($\delta=0$) and the coupled one ($\delta=0.3$). The time of each stress wave corresponds to the distance where it holds

Fig. 11 Stress waves at $\tau=1$ (for case of the step-type $\tau_0=0$)

Fig. 12 Temperature distributions at $\tau=1$ (for case of the step-type $\tau_0=0$)

Fig. 9 Time-dependence of the strain $\xi=1$ (for case of the exponential-type $\tau_0=0.5$)

Fig. 10 Time-dependence of the temperature at $\xi=1$ (for case of the exponential-type $\tau_0=0.5$)

Fig. 13 Stress waves and temperature distributions at $\tau=1$ (for case of the exponential-type $\tau_0=0.25$)

Fig. 14 Comparison of the stress waves for the coupled case $\delta=0.3$ and the uncoupled case $\delta=0$ (for case of the exponential-type $\tau_0=0.25$)

the peak-value in the negative direction, in other words, each of the stress waves corresponds to one, from the left side, at $\tau=0.5, 1, 2, 3$ and 5 , respectively. It will be seen from Fig. 14 that the stress waves for the coupled case, represented by solid lines, decrease exponentially with these propagations.

8. Conclusions

In the present paper, we have treated an initial boundary value problem on the thermal shock as the coupled thermoelastic problem. The boundary condition on temperature has been given in the form of the exponential-type, which contains the step-type and the ramp-type, hence the former should seem to be more general besides physically more real than the latter.

The following conclusions have been made as the result of the present study.

(1) If the Laplace transform is employed as an analytical method for the coupled thermoelastic problem of this kind, the method of series expansion used in this paper will be effective to obtain an approximate solution. Furthermore, a numerical inversion technique for the Laplace transform will also be advantageous when the inversion is extremely difficult and complicated.

(2) As might be expected from the second Danilovskaya's problem and the ramp-type one, only in cases where the thermal shock is given by the step-type heating, stress and strain waves arising from it should hold discontinuities at the elastic wave front, thus for the case of the exponential-type heating, the stress and strain waves are continuous there.

(3) It will be expected that since in the actual phenomenon of thermal shock a time-lag of heating

is supposed, the practical stress wave should be very small as compared with the stress wave arising from the imaginary step-type heating.

(4) In this one-dimensional problem the thermoelastic coupling effect does not affect the velocity of a dilatational elastic wave, but acts on the stress and strain waves as well as on the temperature distribution as the damping effect.

(5) The influence of the thermoelastic coupling effect on the stress wave in the present problem gradually increases with the increase of both time and distance.

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Discussion

Y. TAKEUCHI (Sizuoka University):

Although the analysis in this paper has been performed as a one-dimensional problem in the Cartesian coordinate, this study is considered to be valuable because of it being the first report on the coupled thermoelastic problem in Japan.

(1) In the present paper, at first the terms of the negative power of p in Eqs. (33) have been neglected, and then λ_i has been squared. Accordingly this procedure means that the large-value constant is neglected while the small-value constant remains; namely, in this paper, provided

$$\lambda_1^2 - \lambda_2^2 = p^2 + p(\delta - 1) + \frac{\delta^2}{4}$$

Eqs. (34) and (35) have been derived. However, if at first λ_i is squared, and then the terms of the negative power of p are neglected,

$$\lambda_1^2 - \lambda_2^2 = p^2 + p(\delta - 1) + 2\delta$$

would be obtained. This procedure should yield

$$a, b = \frac{1}{2}(1 - \delta \pm \sqrt{1 + \delta^2 - 10\delta})$$

hence, in this case Eqs. (34) and (35) have to be altered, consequently, a and b would become imaginary numbers even though $\delta = 0.3$.

(2) The series expansions in Eqs. (33) can be valid only for short time, hence it raises a question whether the result, for instance at $\tau = 20$ in Fig. 7, is correct or not. For this reason, it is unnatural that the result by the method of series expansions should not depart from the result by the numerical inversion technique, with an increasing time.

Consequently, it is doubtful to adopt the conclusion that the thermoelastic coupling effect increases ever so much with time. Are there any valid criteria to adopt the range of τ ?

(3) Does the influence of the coupling effect increases ever so much with an increasing time or approach asymptotically certain values, or reach the peak-value and then decrease? Are there any authors' consequences on it?

Authors' closure

(1) We also consider the questioner is entirely right. However, in order to carry out the analysis easily, we have made the approximation as $\lambda_1 = p + \delta/2$, $\lambda_2 = \sqrt{p}$. If at first λ_i is squared and then the terms of negative power of p are omitted, for the purpose of adjusting an approximate degree, λ_1 and λ_2 should be used in the form as Eqs. (33);

$$\lambda_1 = p + \frac{\delta}{2} + \frac{\delta(4 - \delta)}{8p}, \quad \lambda_2 = \sqrt{p} - \frac{\delta}{2\sqrt{p}}$$

If these approximate relations are used, it would come into question to handle $\exp\{0(1/p)\}$, complex

numbers a, b and so on. Without this complication, this paper has been accomplished.

(2) The solution by the method of series expansions is only correct for short times, hence according as the time increases, the approximate solution should become inapplicable. However, the results of numerical calculations indicated, as shown in Fig. 7, that at the wave front the approximate solutions almost agreed with the numerical solutions within $\tau \leq 20$, at least.

Since solutions by the numerical inversion technique can be regarded as an exact solution, the authors consider that the series solutions will be available up to the time, when those solutions depart far from the numerical solutions.

We also consider, in view of the stress being free at the boundary, stress waves in this paper correspond to, for instance, the discontinuity at the wave front in the reference (13): Let SF denote a magnitude of discontinuities, in the reference (13)

$$SF(x_1, t_1) \rightarrow 0 \text{ as } x_1, t \rightarrow \infty$$

Consequently, the stress waves in Fig. 7 show such a tendency as $\sigma_{x0} \rightarrow 0$ when $\tau (= \xi) \rightarrow \infty$. However, as pointed out, this behavior does not signify in general that the influence of the thermoelastic coupling effect increases ever so much with an increasing time.

(3) Though not clear from the present paper, the influence of the thermoelastic coupling effect should seem to approach asymptotically certain constant values. Authors intend to discuss this point in future.

Y. TAKEUCHI (Sizuoka University):

(4) I consider the use of an approximate method must be based on what an approximate solution has the closest value to an exact solution when the variable values within the available range are substituted into it. The use of approximations is one thing, it is another that, for the adjustment of an approximate degree, the calculation must be carried out just as the first approximation is. Otherwise giving an instance to the uncoupled thermoelastic problem, it would become nonsense to ignore deformation works, that is, the thermal deformation in the heat conduction equation, accordingly, a number of past investigations on the uncoupled problem are nonsense in which in the first place temperature distributions are obtained under the neglect of thermal stresses and then the stresses are calculated from it. In spite of that, the solution in uncoupled cases is of course significant, by the reason that both the usual strain rate and industrial materials almost do not affect the temperature distribution. In accordance with

the authors' terms such as the adjusting of approximate degrees, it will become unnecessary to seek accuracy agreement between the heat conduction equation and the thermoelastic displacement equation. Hence the authors' approximation, in which the terms having a large value are ignored, will not be good.

However, the foregoing does not only mean that the authors' analysis is not significant, but also that when calculations become difficult, in each step of it, the higher-order terms of Taylor's expansion had better be neglected.

Authors' closure

(4) As pointed out, the authors also consider that the calculation becoming difficult, in this step the series expansion should be performed, and then the terms of its higher-order should be neglected. Hence, the fact seems to be not always good that, as in this paper, the calculation had to be carried out just as the form of the first approximation, for coincidence of the approximate degree. But the approximate solution in this paper was obtained in the first approximate sense for the following reason. If the terms of negative power of p are neglected in a step of $\lambda_{1,2}$,

$$\lambda_1^2 - \lambda_2^2 = p^2 + (\delta - 1)p + \delta^2/4 \dots\dots\dots (i)$$

On the other hand, if in the first instance $\lambda_{1,2}$ is squared, and then the negative power terms of p are neglected, then

$$\lambda_1^2 - \lambda_2^2 = p^2 + (\delta - 1)p + 2\delta \dots\dots\dots (ii)$$

Since in Eq. (i) δ of the constant term is neglected, Eq. (ii) should yield better approximate solutions than Eq. (i). The above relations and Eqs. (33), however, are obtained by the series expansion on the large $p(p \rightarrow \infty)$, hence the difference between Eqs. (i) and (ii) should seem to be very small. Therefore, as the use of Eq. (i) is considered not to have large effects on the results, in the present paper Eq. (i) has been employed. Furthermore, the inversion will be performed with ease by using only convolution integrals, provided Eq. (i) is employed in calculations. On the other hand, if an approximation is carried out in the form of Eq. (ii), the inversion would become difficult and complicated, for instance, on real or complex numbers a, b due to a magnitude of δ , etc. Of course, Eq. (ii) will yield a better approximate solution, but on account of the preceding, the approximate solutions have been obtained as in this paper, and these solutions have agreed well with the numerical solutions. We would like to ask the discussor's consent to this procedure.

N. SEKI (Hokkaido University):

The temperature changes in this paper seem to be obtained by using Eq. (5), in which a generation of heat due to the stress changes in the temperature field has been taken into consideration, under the boundary conditions Eqs. (10).

I wish to ask about the following.

(1) Are there any contradictions between the considering of the generation of heat having an irreversible characteristic and the assuming of an elastic medium?

(2) The propagation velocity of stress waves seems to be greater beyond all comparison than the diffusion velocity of temperature. Accordingly, it will not be practical to apply the results obtained under the boundary conditions Eqs. (10) to the domain where only the stress wave is dominating. How about it?

Authors' closure

(1) Equation (5) is introduced under the assumption of a reversible process. The entropy of a system can be represented as $ds = d_s s + d_i s$ in which ds denotes the increase of entropy of the system, $d_s s$ denotes the part of this increase due to interaction with the surroundings, and $d_i s$ denotes the part of this increase due to changes taking place inside the system, that is, assuming $d_i s = 0$ (reversible processes), Eq. (5) is introduced. Therefore, there are not any contradictions with the elastic theory.

(2) The thermoelastic coupling term means physically that contributes to the volume changes in a medium. Accordingly, even though in the field of the uniform temperature, it ought to be considered when the strain waves exist.

T. IRIE (Hokkaido University):

(1) The authors have mentioned that giving the generalized boundary condition on temperature in the form of the exponential-type and taking the first term of its series expansion, this boundary condition would become the ramp-type. For in the case of the large time-constant τ_0 or very initial time, this assumption will hold, but these are originally different inputs from each other. How do the authors consider about it?

(2) In what range can the maximum impact temperature be applied in the present theory?

(3) In the example of numerical results, the computation has been performed in the case of coupling parameters $\delta = 0.1, 0.3$. Its value of actual materials is smaller than these, consequently, the thermoelastic coupling effect on the stress wave and the temperature distribution seems to be very small. In what cases must the coupling effect be considered as an industrial problem?

(4) A measurement of stress waves and temperature distributions will be very difficult. How about it?

Authors' closure

(1) In the case of the ramp-type, as pointed out, its input form is of course different from that of the exponential-type, hence the boundary condition

of the ramp-type has to be given as follows:

$$\theta(0, \tau) = \begin{cases} 0 & (\tau_0 < 0) \\ \frac{\tau}{\tau_0} & (0 \leq \tau_0 \leq \tau) \\ 1 & (\tau_0 < \tau) \end{cases}$$

(2) Considering both of the ranges where the temperature-dependence in material constants can be neglected and Eq. (5) is applicable, its range seems to be under 200°C.

(3) As described in the reference (1), the thermoelastic coupling effect must be taken into account for the material of the large coupling parameter and the high speed deformation.

(4) The experimental measurement should involve great difficulties, but some experimental investigations on inertia effects have been published. In regard to the thermoelastic coupling effect, an experiment by mechanical impacts would be easier than one by thermal shocks. As far as the authors know, there is only one report^(*1) on it.

T. KOIZUMI (Tokyo Institute of Technology):

(1) The thermal condition on the free boundary has been given in the form of the exponential-type heating, but since the governing Eqs. (4) and (5) for the problems of this kind are linear, the thermal condition used in reference (8) will be more basic than the exponential-type. How about it? Further when an initial period of heating is noticed, the exponential-type seems to correspond to the large value of a heat transfer coefficient in reference (8). How do the authors think of it?

(2) Can the numerical inversion technique of chapter 5 be applied with good convergence to the large-time range corresponding to quasi-static problems?

Authors' closure

(1) The authors also consider the boundary condition of the step-type is fundamental for the case that inertia terms in an equation of motion are neglected as in the questioner's investigation^(*2).

However, if the inertia terms are taken into account, it will come into question that the discontinuity will arise at wave fronts, hence the behavior at and near there will differ from the practical

phenomenon. Furthermore, only in the case of the step-type, the effect depending on the magnitude of heating speed cannot be studied.

Therefore, in this paper, the exponential-type, which involves also the step-type in a special case, has been employed.

The example of numerical results in this paper, as pointed out, will be corresponding to large values of the heat transfer coefficient in reference (5). However, since τ_0 is defined as arbitrary positive numbers $\tau_0 \geq 0$, it will correspond to small values of the heat transfer coefficient when τ_0 is taken large enough.

(2) In the present paper, as a behavior of the elastic wave mainly has been noticed, the extreme large-range of times has not been considered. But, judging from the numerical results, the convergence of a part corresponding to the inertia term is not good, so that the application for that time will be difficult. If it were assumed as a quasi-static problem, the part corresponding to the inertia term would be removed, hence the application would be possible.

S. HANZAWA (Hokkaido University):

It is very interesting that a thermal shock problem has been analyzed as a coupled thermoelastic problem under the exponential-type heating.

(1) I also consider that the step-type, as you call it, will be an ideal case. Therefore, as the authors have pointed out, a unity discontinuity arises at the wave front of the stress wave.

How do the authors think of it?

(2) It has been described that the stress σ_{xx} is continuous at wave fronts when $\tau_0 > 0$. In this connection, I wish to put the authors a question about the meaning of "continuity".

Authors' closure

(1) An appearance of discontinuities at wave fronts is caused by a discontinuity of the step function. That is, the Heaviside step function has a discontinuous point at time $t=0$ known as ordinary points of discontinuity.

(2) In coupled thermoelastic problems of this kind, "the discontinuity" customarily indicates differences between the limit value of the right side and that of the left side, as expressed by Eq. (47). Therefore, in this paper, if $SF(x_0)=0$, the expression "continuous" has been used. Hence, it is different from the continuity of a differential coefficient.

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