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A Numerical Method for Solving Elastic-Plastic Problems of Rotationally Symmetric Shells*

(2nd Report, Comparison of Analytical Results with Experimental Ones)

By Minoru HAMADA** and Masataka TANAKA***

The authors have previously proposed a numerical method of solution for the elastic-plastic problems of axisymmetric shells by the finite difference method, using the incremental theory of plasticity with the Prandtl-Reuss stress-strain relations. Few numerical examples, however, have been presented for comparison with the experimental results. In this paper, numerical computations corresponding to the elastic-plastic problems of torispherical vessels and of bellows with various shapes are carried out in order to confirm the validity of the method, and the results obtained are compared with the experimental ones by other researchers and those by the authors themselves.

1. Introduction

On the basis of the incremental theory of plasticity, a numerical method of solution has been established in a previous report⁽¹⁾ for the large deflection problems of elastic-plastic shells of revolution, and some problems of bellows have been analyzed with the use of the Prandtl-Reuss equations based on the isotropic hardening theory. This solution procedure was extended to the case in which the stress-strain relations of Ziegler's kinematic hardening theory were used, and the elastic-plastic behaviors of circular plates, which are examples of axisymmetric shells, were analyzed when the plates were subjected to a cyclically varying uniform pressure⁽²⁾. Moreover, the stress-strain relations which include the isotropic and the kinematic hardening in an arbitrary ratio were treated⁽³⁾, and they were applied to numerical analysis of the elastic-plastic behaviors of circular plates subjected to completely reversed cycling of uniform pressure⁽⁴⁾.

From comparison of the results obtained using the above three kinds of constitutive equation, it has been made clear⁽²⁾⁽⁴⁾ that sufficiently accurate results can be furnished by the analysis based on the isotropic hardening theory. Thus in this report, a numerical method of solution for the elastic-plastic problems of axisymmetric shells using the Prandtl-Reuss equations

is also discussed. Namely in this paper, the analytical results are compared with the measured ones by other researchers and those of bellows by the authors.

2. Outline of solution procedure

Though the basic equations of the large deflection problem of axisymmetric shells based on the incremental theory of plasticity have been described in detail in the previous report⁽¹⁾, they are outlined in the following using the matrix notation.

Using the same notation and coordinate system as in reference (1), we introduce the following matrices:

$$\left. \begin{aligned} \{\bar{\epsilon}\}^T &= [\bar{\epsilon}_\xi \bar{\epsilon}_\theta], & \{\dot{\epsilon}\}^T &= [\dot{\epsilon}_\xi \dot{\epsilon}_\theta] \\ \{\bar{\epsilon}_m\}^T &= [\bar{\epsilon}_{\xi m} \bar{\epsilon}_{\theta m}], & \{\dot{\epsilon}_m\}^T &= [\dot{\epsilon}_{\xi m} \dot{\epsilon}_{\theta m}], \\ \{\bar{r}\dot{\epsilon}_m\}^T &= [\bar{r}\dot{\epsilon}_{\xi m} \bar{r}\dot{\epsilon}_{\theta m}] \\ \{Z\}^T &= [\beta \bar{\psi}_H], & \{\dot{Z}\}^T &= [\dot{\beta} \dot{\bar{\psi}}_H] \\ \{\bar{\kappa}\}^T &= [\bar{\kappa}_\xi \bar{\kappa}_\theta], & \{\dot{\kappa}\}^T &= [\dot{\kappa}_\xi \dot{\kappa}_\theta] \\ \{\bar{M}\}^T &= [\bar{M}_\xi \bar{M}_\theta], & \{\dot{M}\}^T &= [\dot{M}_\xi \dot{M}_\theta], \\ \{\bar{r}\dot{M}\}^T &= [\bar{r}\dot{M}_\xi \bar{r}\dot{M}_\theta] \\ \{\bar{N}\}^T &= [\bar{N}_\xi \bar{N}_\theta], & \{\dot{N}\}^T &= [\dot{N}_\xi \dot{N}_\theta], \\ \{\bar{r}\dot{N}\}^T &= [\bar{r}\dot{N}_\xi \bar{r}\dot{N}_\theta] \end{aligned} \right\} \dots\dots (1)$$

where suffix T , $\{ \}$ and $[\]$ denote the transpose matrix, column vector and row vector, respectively. With the use of the above matrix notation, equilibrium equations (17)₁* and (20)₁ of the previous paper⁽¹⁾ are rewritten as in Eq. (2), Eqs. (18)₁ and (19)₁ as in Eq. (3), and compatibility equations (21)₁ and (22)₁ are also rewritten as in Eqs. (4) and (5). Equilibrium equations:

* Received 18th December, 1971.

** Professor, Faculty of Engineering, Osaka University.

*** Research Assistant, Faculty of Engineering, Osaka University, Suita.

* Equation number in ()₁ denotes that of reference (1).

$$\begin{aligned} \{\bar{r}\dot{N}\} &= \frac{h}{a} \begin{bmatrix} 0 & 0 \\ 0 & \bar{r} \end{bmatrix} \{\dot{Z}\}' \\ &+ \frac{h}{a} \begin{bmatrix} \bar{\psi}_V \cos \Phi - \bar{\psi}_H \sin \Phi & \cos \Phi \\ 0 & 0 \end{bmatrix} \{\dot{Z}\} \\ &+ \frac{h}{a} \begin{bmatrix} \bar{\psi}_V \sin \Phi \\ \bar{r}^2 \bar{p}_H \end{bmatrix} \dots\dots\dots (2) \\ [1 \ 0] \{\bar{r}\dot{M}\}' + [0 \ -\cos \Phi / \bar{r}] \{\bar{r}\dot{M}\} \\ &- [(\bar{\psi}_V - \bar{M}_\theta) \sin \Phi + \bar{\psi}_H \cos \Phi \\ &\sin \Phi] \{\dot{Z}\} + \bar{\psi}_V \cos \Phi = 0 \dots\dots\dots (3) \end{aligned}$$

Compatibility equations:

$$\begin{aligned} [0 \ 1] \{\bar{r}\dot{\epsilon}_m\}' + [-\cos \Phi / \bar{r} \ 0] \{\bar{r}\dot{\epsilon}_m\} \\ + [(a^2/h^2 + \bar{\epsilon}_{\xi m}) \sin \Phi \ 0] \{\dot{Z}\} = 0 \dots\dots\dots (4)^* \\ \{\dot{\kappa}\} = \begin{bmatrix} a/h & 0 \\ 0 & 0 \end{bmatrix} \{\dot{Z}\}' + \begin{bmatrix} 0 & 0 \\ (a/h) \cos \Phi / \bar{r} & 0 \end{bmatrix} \{\dot{Z}\} \dots\dots\dots (5) \end{aligned}$$

Equation (23)₁ representing the Kirchhoff-Love hypothesis is written as follows:

$$\{\dot{\epsilon}_i\} = \{\dot{\epsilon}_m\} + \zeta \{\dot{\kappa}\} \dots\dots\dots (6)$$

The stress-strain relations (36)₁ are expressed as

$$\{\dot{\sigma}\} = [\bar{D}^p] \{\dot{\epsilon}\} \dots\dots\dots (7)$$

where $[\bar{D}^p]$ is a nondimensional expression of the plastic stress-strain matrix $[D^p]$, given by

$$[\bar{D}^p] = [D^p]/E \dots\dots\dots (8)$$

The matrix $[D^p]$ is described in detail in Eq. (25)₁.

$$\begin{aligned} & \left\{ [X_1] \begin{bmatrix} 0 & 0 \\ 0 & \bar{r} \end{bmatrix} + \left(\frac{a}{h}\right)^2 [X_3] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \{\dot{Z}\}'' + \left\{ [X_1] \begin{bmatrix} \bar{\psi}_V \cos \Phi - \bar{\psi}_H \sin \Phi & \cos \Phi \\ 0 & \bar{r}' \end{bmatrix} \right. \\ & + [X_2] \begin{bmatrix} 0 & 0 \\ 0 & \bar{r} \end{bmatrix} + \left(\frac{a}{h}\right)^2 [X_3] \begin{bmatrix} 0 & 0 \\ \cos \Phi / \bar{r} & 0 \end{bmatrix} + \left(\frac{a}{h}\right)^2 [X_4] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left. \right\} \{\dot{Z}\}' \\ & + \left\{ [X_1] \begin{bmatrix} (\bar{\psi}_V \cos \Phi - \bar{\psi}_H \sin \Phi)' & (\cos \Phi)' \\ 0 & 0 \end{bmatrix} + [X_2] \begin{bmatrix} \bar{\psi}_V \cos \Phi - \bar{\psi}_H \sin \Phi & \cos \Phi \\ 0 & 0 \end{bmatrix} \right. \\ & + \left(\frac{a}{h}\right)^2 [X_3] \begin{bmatrix} 0 & 0 \\ (\cos \Phi / \bar{r})' & 0 \end{bmatrix} + \left(\frac{a}{h}\right)^2 [X_4] \begin{bmatrix} 0 & 0 \\ \cos \Phi / \bar{r} & 0 \end{bmatrix} + \frac{a}{h} [X_5] \left. \right\} \{\dot{Z}\} \\ & = -[X_1] \begin{bmatrix} \bar{\psi}_V \sin \Phi \\ \bar{r}^2 \bar{p}_H \end{bmatrix}' - [X_2] \begin{bmatrix} \bar{\psi}_V \sin \Phi \\ \bar{r}^2 \bar{p}_H \end{bmatrix} - \frac{a}{h} \begin{bmatrix} \bar{\psi}_V \cos \Phi \\ 0 \end{bmatrix} \dots\dots\dots (12) \end{aligned}$$

where

$$\left. \begin{aligned} [X_1] &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} [\bar{D}_{12}] [\bar{D}_{11}]^{-1}, \quad [X_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ([\bar{D}_{12}] [\bar{D}_{11}]^{-1})' + [0 \ -\cos \Phi / \bar{r}] [\bar{D}_{12}] [\bar{D}_{11}]^{-1} \\ [X_3] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \bar{r} ([\bar{D}_{22}] - [\bar{D}_{12}] [\bar{D}_{11}]^{-1} [\bar{D}_{12}]) \\ [X_4] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} (\bar{r} ([\bar{D}_{22}] - [\bar{D}_{12}] [\bar{D}_{11}]^{-1} [\bar{D}_{12}]))' + [0 \ -\cos \Phi / \bar{r}] \bar{r} ([\bar{D}_{22}] - [\bar{D}_{12}] [\bar{D}_{11}]^{-1} [\bar{D}_{12}]) \\ [X_5] &= \begin{bmatrix} (\bar{M}_\theta - \bar{\psi}_V) \sin \Phi - \bar{\psi}_H \cos \Phi & -\sin \Phi \\ (a^2/h^2 + \bar{\epsilon}_{\xi m}) \sin \Phi & 0 \end{bmatrix} \end{aligned} \right\} \dots\dots\dots (13)$$

If the differential equation (12) is solved under certain boundary conditions, the increment $\{\dot{Z}\}$ can be obtained. Since the elimination procedure used in references (2) and (4) is more convenient for the numerical solution of this type of differential equation than the iterative method in reference (1), the former method is applied in this paper.

3. Comparison of analytical results with experimental ones and its considerations

3.1 Torispherical vessels subjected to internal pressure

Let us consider the problem of a pressure vessel subjected to internal pressure p as shown in Fig. 1. In this case, we may put in Eq. (12)

which is identical with that derived by Yamada et al.⁽⁵⁾ in the analysis of elastic-plastic tension of a notched plate.

On the other hand, Eq. (31)₁ which prescribes the relations between the stress increment $\{\dot{\sigma}\}$ and the membrane force increment $\{\dot{N}\}$ or the bending moment increment $\{\dot{M}\}$ reduces to

$$\{\dot{N}\} = \int \{\dot{\sigma}\} d\bar{\zeta}, \quad \{\dot{M}\} = \int \{\dot{\sigma}\} \bar{\zeta} d\bar{\zeta} \dots\dots\dots (9)**$$

The use of Eqs. (6) through (9) leads to

$$\begin{Bmatrix} \dot{N} \\ \dot{M} \end{Bmatrix} = [\bar{B}] \begin{Bmatrix} \dot{\epsilon}_m \\ \dot{\kappa} \end{Bmatrix} \dots\dots\dots (10)$$

which corresponds to Eq. (35)₁, where

$$\begin{aligned} [\bar{B}] &= \begin{bmatrix} \int [\bar{D}^p] d\bar{\zeta} & \int [\bar{D}^p] \bar{\zeta} d\bar{\zeta} \\ \int [\bar{D}^p] \bar{\zeta} d\bar{\zeta} & \int [\bar{D}^p] \bar{\zeta}^2 d\bar{\zeta} \end{bmatrix} \\ &= \begin{bmatrix} [\bar{D}_{11}] & [\bar{D}_{12}] \\ [\bar{D}_{12}] & [\bar{D}_{22}] \end{bmatrix} \dots\dots\dots (11) \end{aligned}$$

Unknown increments in Eqs. (2) to (5) and (10) are

$$\{\dot{N}\}, \{\dot{M}\}, \{\dot{\epsilon}_m\}, \{\dot{\kappa}\} \text{ and } \{\dot{Z}\}$$

Taking note of $\{\dot{Z}\}$ among them, the following second-order ordinary differential equation corresponding to Eqs. (43)₁ and (44)₁ can be derived through the same procedure as in reference (1).

* Since the compatibility equation (21)₁ cannot apply to a cylindrical shell, it is reformed as this expression.
** Integration is carried out along the shell thickness.

$$\bar{p}_V = -\bar{p} \cos \phi, \quad \bar{p}_H = \bar{p} \sin \phi \quad \dots\dots\dots(14)$$

Let the diameter D of the cylindrical shell be the reference length a of the vessel, then the configuration parameters of this case are as follows:

$$\bar{R}_1 = R_1/a, \quad \bar{R}_2 = R_2/a, \quad \lambda = h/a \quad \dots\dots\dots(15)$$

Using these parameters, angles ϕ_1 and ϕ_2 are expressed as

$$\left. \begin{aligned} \phi_1 &= \sin^{-1}\{(1/2 - \bar{R}_2)/(\bar{R}_1 - \bar{R}_2)\} \\ \phi_2 &= \pi/2 - \phi_1 \end{aligned} \right\} \quad \dots\dots\dots(16)$$

The meridional length ξ_t between O and C of Fig. 1 is given as

$$\xi_t \equiv \xi_t/a = \bar{R}_1\phi_1 + \bar{R}_2\phi_2 + \bar{\xi}_{BC} \quad \dots\dots\dots(17)$$

The following four factors which influence the accuracies of solution can be considered in this numerical example.

- 1) The length of cylindrical shell: $\bar{\xi}_{BC}$
- 2) The division number along the meridional length $\widehat{OC}$: N
- 3) The division number along shell thickness: $2M$
- 4) The value of internal pressure increment: $\bar{p}$

These values were determined as follows. Referring to the linear elastic solution⁽⁷⁾ of a semi-infinitely long cylindrical shell subjected to bending moment and shearing force at its end, $\bar{\xi}_{BC}$ was selected as

$$\bar{\xi}_{BC} = 6 \sqrt{(h/a)^2 / [12(1-\nu^2)]} \doteq 3.5 \sqrt{h/a} \quad \dots\dots\dots(18)$$

The boundary conditions at points O and C of Fig. 1 were assumed as

$$\{\dot{Z}\} = 0 \quad \dots\dots\dots(19)$$

Since in this type of pressure vessel the dependent variables undergo a remarkable change in the portion $\widehat{AB}$ of a toroidal shell, the mesh number for the finite difference approximation must be large if the interval $\widehat{OC}$ is equally divided. Therefore, points D and E were so determined that $\bar{\xi}_{DA} = \bar{\xi}_{BE} = \bar{\xi}_{AB}/2$, and the interval $\widehat{DE}$ was divided into finer segments. The interval $\widehat{OD}$ was equally divided into 10, $\widehat{DE}$ into 40 and $\widehat{EC}$ into 10, and sufficiently accurate results were

Fig. 1 Torispherical drum head vessel subjected to internal pressure

obtained in the following numerical examples. The shell thickness being laminated with 16 layers, Eq. (11) etc. were numerically integrated by the Simpson rule. The value of internal pressure increment $\bar{p}$ was so determined that the maximum strain increment $\dot{\epsilon}_{\max}$ was equal to about 1/10 of the yield strain $\dot{\epsilon}_0$ in uniaxial tension. Numerical calculations were carried out under various computation conditions and the accuracies of solution were examined. An example of such results is shown in Fig. 2. This figure shows examined results on the relation between internal pressure and the meridional strain produced at $\phi = 45^\circ$ on the internal surface of the vessel used by Stoddart⁽⁶⁾ in his experiment.

Let us compare in the following the analytical results obtained under the above computation conditions with the experimental results on Stoddart's vessel⁽⁶⁾ made of mild steel and on Turner's vessel⁽⁸⁾ of stainless steel.

The configuration parameters of Stoddart's vessel⁽⁶⁾ are

$$\bar{R}_1 = 1, \quad \bar{R}_2 = 0.0833, \quad \lambda = 0.0104 \quad \dots\dots\dots(20)$$

and its material constants are

$$E/Y = 755.1, \quad \nu = 0.31 \quad \dots\dots\dots(21)$$

In the analysis, the elastic-perfectly-plastic material in which $H' = 0$ is assumed.

In Figs. 3 and 4, the analytical results are compared with the measured ones⁽⁶⁾ within elastic range. Figure 3 shows the stress distributions on the internal surface of the vessel at $pa^4/Eh^4 = 279.8$, and Fig. 4 does those on the external surface at the same pressure level. Fairly good agreement is recognized between the analytical and experimental results on both surfaces. This fact would indicate not only the validity of the present solution procedure in the elastic range, but also the vessel possessing nearly perfect configurations in its shell geometry.

Figure 5 shows the relation between the internal pressure and the meridional strain produced at $\phi = 45^\circ$ on the internal surface of the vessel. The results obtained by the present method are closer to the measured

Fig. 2 Influence of magnitude of pressure increment on numerical results

data than the numerical results⁽⁹⁾ by the finite element method based on a small deflection shell theory, denoted in the figure by "F.E.M. (Small)". The analytical results⁽¹⁰⁾ by the finite element method based on a large deflection shell theory, denoted by "F.E.M. (Large)", are also shown in the figure, but they do hardly differ from the results by the present

Fig. 3 Stress distributions on external surface of Stoddart's vessel at $pa^4/Eh^4=279.8$

Fig. 4 Stress distributions on external surface of Stoddart's vessel at $pa^4/Eh^4=279.8$

method. Nonlinear elastic solution shown in the figure is what is obtained using always Hooke's law as the stress-strain relations in this method.

Next let us compare the analytical results with the experimental ones on a similar vessel tested by Turner et al.⁽⁸⁾

The configuration parameters of Turner's vessel⁽⁹⁾ are

$$\bar{R}_1=0.851, \quad \bar{R}_2=0.162, \quad \lambda=0.0132 \dots \dots \dots (22)$$

This vessel is made of austenitic stainless steel and heat-treated after its forming. The stress-strain relation of this material in uniaxial tension, given in reference (8), is redrawn in Fig. 6. The mark $\circ$ in the figure indicates a mean value of the measured results obtained from three test pieces. The analytical results of Turner et al.⁽⁸⁾ by the finite element method are those obtained from a small deflection shell theory under a considerably rough approximation of the uniaxial property as shown in the figure by the dotted curve. By applying the Holmquist-Nadai equation⁽¹¹⁾

$$\epsilon_{11} = \sigma_{11}/E + K_p(\sigma_{11} - Y)^n \dots \dots \dots (23)$$

where $K_p=0$ if $\sigma_{11} \leq Y$, to the uniaxial stress-strain relation, we take account of the hardening property. The coefficients of Eq. (23) determined as best-fitted to the measured data are as follows:

Fig. 5 Internal pressure—maximum strain (Stoddart's vessel)

Fig. 6 Uniaxial stress-strain diagram (Turner's vessel)

$$\left. \begin{aligned} E &= 2.1 \times 10^4 \text{ kg/mm}^2 \\ Y &= 18.9 \text{ kg/mm}^2, \quad \nu = 0.3 \\ n &= 3.475, \quad K_p = 0.906 \times 10^{-5} (\text{kg/mm}^2)^{-n} \end{aligned} \right\} \dots\dots\dots (24)$$

The stress-strain relation is shown in Fig. 6 by a solid curve, when these coefficients are used. The proportional limit σ_{pl} is chosen as the yield stress Y in Eq. (24). In the analysis of a multi-axial stress field, σ_{11} in Eq. (23) is to be replaced by the equivalent stress σ_e . When Eq. (23) is applied, a non-dimensional expression of the plastic tangent modulus H' is as follows:

$$\bar{H}' \equiv H'/E = 1/[n\bar{K}_p(\bar{\sigma}_e - \bar{Y})^{n-1}] \dots\dots\dots (25)$$

where

$$\left. \begin{aligned} \bar{K}_p &= K_p E^n (h/a)^{2(n-1)}, \quad \bar{Y} = Y a^2 / E h^2 \\ \bar{\sigma}_e &= \sigma_e a^2 / E h^2 \end{aligned} \right\} \dots\dots\dots (26)$$

In Figs. 7(a) to (d), the analytical results are compared with the measured ones⁽⁸⁾ concerning the relations between internal pressure and strains at various points. The analytical results indicated by the dot-dashed curve in the figures are those of Turner et al.⁽⁸⁾ obtained by the finite element method when the uniaxial property is approximated as shown in Fig. 6 by a dotted curve. The difference cannot be so clearly appreciated from the analytical results by the present method based on a large deflection shell theory, in which the uniaxial property is more closely approximated. Generally in this type of shell structure, the initial imperfections in its shape and measuring errors in strain gauge locations do affect the experimental results, and hence such a discrepancy between the analytical and measured results may be inevitable.

3.2 Bellows subjected to axial loading

Some problems concerning a U-shaped bellows have been treated in the previous paper⁽¹⁾. The bellows is here generalized to an S-shaped one as shown in Fig. 8. Since it is usual that bellows has at least several convolutions, the influences of its end convolutions are regarded as small and the half pitch shown in Fig. 9 is analyzed⁽¹⁾. As the boundary conditions at points A and B of Fig. 9, the following equation is used.

$$\{\dot{Z}\} = 0 \dots\dots\dots (27)$$

When only the axial force acts, the relations:

$$\dot{\bar{P}}_V = \dot{\bar{P}}/(2\pi), \quad \dot{\bar{\Psi}}_V = 0, \quad \dot{\bar{p}}_V = \dot{\bar{p}}_H = 0 \dots\dots\dots (28)$$

can be assumed in Eq. (12). In the generalized S-shaped bellows treated here, the configuration parameters are as follows:

$$\mu = b/a, \quad \lambda_1 = r_1/a, \quad \lambda_2 = r_2/a, \quad \lambda = h/a, \quad \theta \dots\dots\dots (29)$$

Numerical calculations were carried out under the same computation conditions as in reference (1), i.e., the interval AB of Fig. 9 being divided into 25 and shell thickness into 20. The axial force incre-

(a) Meridional strain at $\xi/a=0.485$ on internal surface

(b) Circumferential strain at $\xi/a=0.485$ on external surface

(c) Meridional strain at $\xi/a=0.329$ on external surface

(d) Circumferential strain at $\xi/a=0.498$ on external surface

Fig. 7 Relations between internal pressure and strains at various points (Turner's vessel)

ment $\dot{\bar{P}}$ was so chosen that the axial displacement increment $\dot{\bar{w}} = \dot{w}/h$ was always equal to about 1/10 of the displacement $\bar{w}_1$ at elastic limit. The accuracies were examined by carrying out calculations under various computation conditions for a special case of bellows having the configuration parameters $\mu=0.875$, $\lambda_1=\lambda_2=0.0699$, $\lambda=0.0129$ and $\theta=\pi/2$ and the constants of linear hardening material $E/Y=950$, $\nu=0.3$ and $H'=0.02E$. Figure 10 shows an example of such examined results. That is, the figure indicates that to make $\dot{\bar{w}}$ equal to about 1/10 of $\bar{w}_1$ can be admitted in the relation between the deflection per half pitch w/h and the maximum strain. The influences of the division number N for the finite difference approximation and of $2M$ for the lamination of shell thickness have been already checked in reference (1).

Next let us compare the analytical results with the experimental ones by Marcal et al.⁽⁹⁾

Figures 11 and 12 show the results with respect to a C-shaped bellows of mild steel, which has the configuration parameters

$$\mu=1, \quad \lambda_1=\lambda_2=0.284, \quad \lambda=0.0245, \quad \theta=\pi/2 \quad \dots\dots\dots(30)$$

Fig. 8 Generalized S-shaped bellows subjected to axial force

Fig. 9 Half pitch of generalized S-shaped bellows

and the material constants

$$E/Y=772.4, \quad \nu=0.3 \quad \dots\dots\dots(31)$$

The bellows used by Marcal et al.⁽⁹⁾ in their experiment has three corrugated shapes as shown in Fig. 13. The half pitch is taken out and the elastic-perfectly-plastic material in which $H'=0$ is assumed in the analysis. Figure 11 shows the relation between the axial compressive load and deflection* per half pitch,

Fig. 10 Influence of computation conditions on accuracy of solutions (bellows configurations; $\mu=0.875$, $\lambda_1=\lambda_2=0.0699$, $\lambda=0.0129$, $\theta=\pi/2$; material; $E/Y=950$, $\nu=0.3$, $H'=0.02E$; loading; axial compression)

Fig. 11 Relation between compressive load and axial deflection per half pitch

Fig. 12 Relation between axial deflection and maximum strain (C-shaped bellows)

* Marcal et al.⁽⁹⁾ have measured a relative displacement between points A and D of Fig. 13, using dialgauges.

and Fig. 12 does that of the deflection per half pitch versus the maximum strain. The analytical results by Marcal et al.⁽⁹⁾ based on a small deflection shell theory are shown in the figure by the mark $\triangle$. If better agreement of the present solution with the measured data is taken into consideration, the effect due to the large deflection shell theory is clearly recognized in this example.

Figures 14 and 15 show the results regarding the S-shaped bellows of mild steel, which has the configuration parameters

$$\mu=0.781, \quad \lambda_1=\lambda_2=0.170, \quad \lambda=0.0202, \quad \theta=2.27 \quad \dots\dots\dots(32)$$

and the material constants

Fig. 13 C-shaped bellows employed in experiment (9)

Fig. 14 Relation between compressive load and maximum strain (S-shaped bellows)

Fig. 15 Relation between axial deflection per half pitch and maximum strain (S-shaped bellows)

$$E/Y=505.7, \quad \nu=0.3 \quad \dots\dots\dots(33)$$

The bellows used by Marcal et al.⁽⁹⁾ has only one convolution as shown in Fig. 16, and joins its cylindrical shell component at point A. It seems rather appropriate that the interval $\widehat{CB}$ is taken out and the clamped (2) condition, i.e., Eq. (46)₁, given in reference (1), is to be used as the boundary conditions at point C. But the location of point C cannot be determined from the data given in reference (9). Hence, two kinds of the analytical results are shown here, i.e., the one obtained using Eq. (27) as the boundary conditions at point A and the other using Eq. (46)₁, half a pitch being taken out and an elastic-perfectly-plastic material being assumed in the same manner as in the above example. In Figs. 14 and 15, the present solution (1) denotes the former and the present solution (2) the latter. The actual boundary conditions may be intermediate between them. Figure 14 shows the relation between the axial compressive load and maximum strain and Fig. 15 does that of the deflection per half pitch versus the maximum strain. From such comparisons, it may be concluded that the present solution agrees better with the measured data.

Next let us discuss the compressive experiment on bellows by the authors and its comparison with the corresponding analytical results.

The bellows used in the experiment is made of SUS 27 stainless steel and solution-treated at about 1050°C after its forming. Each bellows has seven convolutions. The stress-strain relation of this material in uniaxial tension is shown in Fig. 17. The coefficients of the Holmquist-Nadai equation (23) are as follows:

$$\left. \begin{aligned} E &= 2.1 \times 10^4 \text{ kg/mm}^2, & Y &= \sigma_{pl} = 9.0 \text{ kg/mm}^2 \\ \nu &= 0.3 \\ n &= 3.63, & K_p &= 1.476 \times 10^{-7} (\text{kg/mm}^2)^{-n} \end{aligned} \right\} \quad \dots\dots\dots(34)$$

which are determined as best-fitted to the measured

Fig. 16 S-shaped bellows employed in experiment (9)

data. From the measurement of the bellows dimensions, the following configuration parameters were obtained. That is, for the test bellows No. 1,

$$\mu=1, \quad \lambda_1=\lambda_2=0.0724, \quad \lambda=0.00529, \quad \theta=\pi/2 \quad \dots\dots\dots(35)$$

and for the No. 2,

$$\mu=1.04, \quad \lambda_1=\lambda_2=0.0761, \quad \lambda=0.00526, \quad \theta=1.31 \quad \dots\dots\dots(36)$$

Fig. 17 Uniaxial stress-strain diagram of test bellows material

Fig. 18 Relation between compressive load and axial deflection per half pitch (Test bellows No. 1)

Fig. 19 Strains at point A (Test bellows No. 1)

The test bellows No. 1 is called the C-shaped and No. 2 the imperfect C-shaped bellows. In calculation of the above parameters, the mean value of the measured data at the top and bottom of a bellows convolution was used as the shell thickness. In the experiment, a compressive load was applied by a universal testing machine so carefully that an eccentric force was excluded as much as possible. Meridional and circumferential strains at the bottom of a convolution in the central portion of the bellows were measured, respectively, with the use of three strain gauges of 3 mm length attached to the internal and external surfaces with an equal interval along the circumference. The mean value of the three measured

Fig. 20 Strains at point B (Test bellows No. 1)

Fig. 21 Relation between compressive load and axial deflection per half pitch (Test bellows No. 2)

Fig. 22 Strains at point A (Test bellows No. 2)

results is shown in the following*. For the measurement of the axial displacement, not only the total deflection between the upper and lower flanges, but also the relative displacement per one pitch was measured so that the influences of end convolutions near the flanges were excluded. From such a treatment, it was made clear that 6.4 convolutions among total number 7 were effective for displacement.

Figures 18 through 20 show the results regarding the test bellows No. 1. The analytical results obtained by the present method using the coefficients given in Eq. (34) of the Holmquist-Nadai equation agree well with the experimental results.

Figures 21 through 23 show the results regarding the test bellows No. 2. The present analytical solutions agree also well with the measured data.

4. Conclusions

In this paper, numerical calculations have been carried out to compare with the experimental data given by other researchers and by the authors, in order to confirm the validity of the numerical method of solution⁽¹⁾ for the elastic-plastic problems of axisymmetric shells. The analytical results have been compared with the measured ones. Generally, since it is difficult to get perfect geometrical shapes in thin shells, the occurrence of some discrepancy is inevitable

* The fluctuations of the three measured data were within ± 0.5 to $\pm 13\%$ about their mean value when the experiment was stopped.

Fig. 23 Strain at point B (Test bellows No. 2)

between the analytical and experimental results. If the above matter is taken into account, the comparison of the analytical results with the experimental ones shown here indicates that the present method of solution gives appropriate results for the elastic-plastic problem of axisymmetric shells subjected to monotonously varying loading.

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Discussion

Y. YAMAMOTO (University of Tokyo):

The discussor greatly appreciates the solution procedure presented in this paper for structural analysis of bellows and similar shell structures.

The analytical results are compared with the experimental ones of Marcal et al. in Figs. 11 and 14, and it is concluded by the authors that the discrepancy between the analytical results by Marcal et al. and the measured data appears because of the linearity of their theory. Their analytical results, however, differ from the linear solutions even in the elastic range. Thus, it is guessed that an appropriate modification of their analytical results gives better agreement with the measured data. That is, there may be another cause to produce the above discrepancy. The discussor hopes that the application limits of the linear shell theory are clarified after such a modification.

Authors' closure

The analytical results according to a small deflection shell theory and the measured data by Marcal et al., shown in Figs. 11, 14 and 15, are only redrawn from the values given in reference (9). The reason why their solution differs from the linear elastic solution in the elastic range is not clear from reference (9). An appropriate modification is necessary, as Dr. Yamamoto points out, to make clear the effect due to the large deflection shell theory. As one procedure of such a modification, let us introduce the nondimensional load P^* and deflection w^* :

$$P^* \equiv P/P_1 = \bar{P}/\bar{P}_1, \quad w^* \equiv w/w_1 = \bar{w}/\bar{w}_1$$

Append.-Fig. 1 Alternative representation of Fig. 11 using P^* and w^*

which are dimensionless expression of P and w , divided by their values at the elastic limit. Then, the discrepancy between the solution by Marcal et al. and the linear elastic solution in the elastic range can be excluded. Take Fig. 11, for example, then Append.-Fig. 1 can be obtained. From this figure, it can be concluded that the present solution agrees better with the measured data than the solution by Marcal et al. even after the above modification. The solutions of Marcal et al. are made dimensionless by regarding $\bar{P}_1$ and $\bar{w}_1$ as the solutions at the smallest axial load, shown in Fig. 11 by the mark $\triangle$. For nondimensional expression of the other quantities, $\bar{P}_1$ and $\bar{w}_1$ values at the elastic limit in Fig. 11 are used, which are calculated on the basis of the Mises yield condition and the linear elastic solution.