621-25 : 539.374

Elastic, Plastic Behavior of Rotor*

(Part 1, Analysis of Axi-Symmetric Rotor Whose Radius Varies in Axial Direction)

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Elastic stress and strain are analyzed in an axi-symmetric rotor whose radius allows arbitrary variation in the axial direction. Means of theoretical approach using the finite element method is taken also to follow the changes of its shape, stress and strain as well as appearance and growth of plastic zones in it due to successive increase of rotational speed. Yamada's $[D^p]$ matrix is adopted as stress-strain relations after yielding of material, and not only material nonlinearity but also geometric nonlinearity due to finite deformation is taken into account in this analysis. As an example, non-uniform deformation of solid and hollow cylinders of various heights is analyzed through elastic, partially plastic and totally plastic ranges. As the result, it becomes clear that non-uniform deformation is the largest when $h_0 = 0.5$ and that the bulge of side surface is single when h_0 is small but double when h_0 is large, etc. The theoretical result of this analysis agrees qualitatively well with the experimental result by Sato et al.

1. Introduction

Many kinds of rotary machines such as turbo-generators, centrifuges and gas turbines have been working recently under severe conditions with their size and working speed increased. Therefore for the design of these machines, exact data are necessary concerning the stress and strain inside the working rotors not only under elastic but also under plastic deformation.

Recently the finite element method is widely used in various fields. Its application was tried on analysis of elastic deformation of rotors⁽¹⁾. In the present report, first, this method is used to obtain the distribution of elastic stress and strain inside the rotors whose radii change not in the circumferential but only in the axial direction. Next, the means are introduced to follow the appearance and development of plastic regions with an increase of rotating speed. Further, this method is applied to analyze elastic-plastic deformation of solid and hollow cylinders in rotation. In the present report, the experimental result by Sato et al.⁽³⁾ is quoted to compare it with

the authors' theoretical result.

2. Method of analysis

The relation of the increment of stress $\{d\sigma\}$ with that of strain $\{d\varepsilon\}$ inside the rotor is written as follows when its rotating speed increases:

$$\{d\sigma\} = [D]\{d\varepsilon\} \dots \dots \dots (1)$$

After the material yields, the matrix $[D]$ becomes $[D^p]$ as obtained by Yamada⁽²⁾ by changing the plastic stress-strain relation of Prandtl-Reuss reversely. The rotor is divided into many ring elements. The stiffness matrix which shows the relation between the increment of nodal displacement $\{d\delta\}$ and that of nodal force $\{dF\}$ is described as $[k]$, so the following formula is valid about each element:

$$\{dF\} = [k]\{d\delta\} - \{dF_0\} \dots \dots \dots (2)$$

where $\{dF_0\}$ is the increment of the centrifugal force working at the nodes within each element. Using the principle of virtual work, $[k]$ and $\{dF_0\}$ are obtained as follows:

$$[k] = 2\pi \int [B]^T [D] [B] r dA \dots \dots \dots (3)$$

$$\{dF_0\} = 2\pi \int [N]^T \left\{ \begin{smallmatrix} \rho d\omega^2 r \\ 0 \end{smallmatrix} \right\} dA \dots \dots \dots (4)$$

where ρ is the density of the material, $d\omega$ is the increment of angular velocity and their integrations are carried out all over the section of each element. $[B]$ and $[N]$ are matrices which show the relations of the increment of nodal displacement with the

* Received 5th January, 1971.

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increment of strain and with the displacement inside the element respectively⁽⁴⁾. As the integration of Eq. (3) was obtained already by Zienkiewicz⁽⁴⁾, its description is omitted here. The integration of Eq. (4) is described in the present report only about the ring element of the triangular section such as in Fig. 1, but the integration about sections of other shapes can be obtained by the same method as this.

In the case of the ring element of a triangular section, matrix $[N]$ is shown as follows⁽⁴⁾:

$$[N] = [N_i, N_j, N_m] \dots\dots\dots (5)$$

$$[N_i] = \begin{bmatrix} a_i + b_i r + c_i z, & 0 \\ 0 & a_i + b_i r + c_i z \end{bmatrix}$$

where

$$a_i = r_j z_m - r_m z_j, \quad b_i = z_j - z_m, \quad c_i = r_m - r_j$$

Substituting Eq. (5) into Eq. (4) and integrating,

$$\{dF_0\}^T = \{dF_{0i}, 0, dF_{0j}, 0, dF_{0m}, 0\} \dots\dots\dots (6)$$

$$dF_{0i} = \pi \rho d\omega^2 (a_i J_1 + b_i J_2 + c_i J_3) / \Delta$$

we obtain

$$J_1 = \frac{1}{3} (a_i c_i d_i + a_j c_j d_j + a_m c_m d_m)$$

$$+ \frac{1}{4} (b_i c_i e_i + b_j c_j e_j + b_m c_m e_m)$$

$$J_2 = \frac{1}{4} (a_i c_i e_i + a_j c_j e_j + a_m c_m e_m)$$

$$+ \frac{1}{5} (b_i c_i f_i + b_j c_j f_j + b_m c_m f_m)$$

$$J_3 = -\frac{1}{6} (a_i^2 d_i + a_j^2 d_j + a_m^2 d_m)$$

$$- \frac{1}{4} (a_i b_i e_i + a_j b_j e_j + a_m b_m e_m)$$

$$- \frac{1}{10} (b_i^2 f_i + b_j^2 f_j + b_m^2 f_m)$$

where

$$d_i = (r_m^3 - r_j^3) / c_i^2, \quad e_i = (r_m^4 - r_j^4) / c_i^2$$

$$f_i = (r_m^5 - r_j^5) / c_i^2$$

Fig. 1 Ring element with triangular section

for $r_m \neq r_j$, and

$$d_i = e_i = f_i = 0$$

for $r_m = r_j$

and Δ is the area of the triangular section of the element.

The incremental method of rotation used in the present analysis is shown in Table 1.

3. Example of analysis

The above-mentioned method can be applied to a rotor whose radius varies arbitrarily in the axial direction. In the present report, the analyses are carried out about the rotation of solid and hollow cylinders as fundamental cases, and their non-uniform, elastic-plastic deformation is made clear, changing the height to width ratio of the cylinder h_0 (h_0 = the height/the diameter), that of the hollow cylinder h_0 (h_0 = the height/the outer diameter) and the diameter ratio a_r (a_r = the inner diameter/the outer diameter).

From the result of the material test of pure aluminium, the stress-strain relation is decided as follows:

$$\sigma = 6800 \varepsilon \text{ kg/mm}^2 \quad (\sigma \leq 4.27 \text{ kg/mm}^2 = \text{yield}$$

stress of the material)

$$= 18.7 e^{0.2 \varepsilon} \text{ kg/mm}^2 \quad (\sigma > 4.27 \text{ kg/mm}^2)$$

Poisson's ratio, the density, and the diameter of the cylinder, or the outer diameter of the hollow cylinder,

Table 1 Flow chart of calculation

are assumed to be $\nu=0.33$, $\rho=2.72 \text{ kg/mm}^2$ and 50 mm respectively.

3.1 Rotation of the solid cylinder

A half of the cylinder is divided into 288 elements (169 nodal points) as in Fig. 2. To shorten the computing time, four elements are yielded in each step.

Figure 3 shows the relation between the rotational speed and the height to width ratio (h_0) of the cylinder, and a, b, c and d in this figure show rotational speeds when the first plastic region appears, when one third of the body is yielded, when two thirds of the body is yielded and when the total body is yielded, respectively. When h_0 is extremely small, non-uniformity of deformation is as small as the rotation of the disc in condition of plane stress. Rotational speed necessary for yielding decreases with an increase of h_0 , and is the smallest when $h_0=0.5$, because non-

uniform deformation due to influence of the end surface becomes maximum at this time. This shows that the existence of the end surface gives an influence on deformation of material within the distance of about one fourth of the diameter from the end surface, and that it makes deformation non-uniform. As h_0 increases more, rotational speed necessary for yielding (the curves a and d) increases first rapidly and then gradually till h_0 becomes equal to about 1.5. This shows that rotation of the cylinder of $h_0>1.5$ may be regarded as that of the cylinder of infinite height in the plane strain condition as a whole.

Figure 4 shows movement of the end surface. The distance between both end surfaces decreases most when $h_0=0.5$, and as h_0 increases more, its decrease becomes small. As a whole, movement of the end surface decreases from the center toward the circumference, but when h_0 is very small, the point where a decrease of the distance becomes the largest exists between the central and the circumference (namely $R \approx 0.3R_e$). This is because the cylinder is yielded only partially at 30 000 rpm, and the plastic zone is exposed on the end surface at this point ($R \approx 0.3R_e$). Both when all parts are in elastic stage and when all parts are yielded, displacement of the end surface becomes the largest at the center regardless of h_0 .

Figure 5 shows the shape of swelling of the side surface (bulge). When $h_0=0.1$, the bulge is extremely small, and deformation may be presumed to be uniform. Non-uniformity of bulge becomes the largest when $h_0=0.5$, because mutual interference of influences by both end surfaces becomes maximum. When $h_0>1$, the bulge becomes double and a part of plane strain deformation appears near the central

$h_0 = H_0/R_e$: Height to diameter ratio
Fig. 2 Element section in a half of solid cylinder
288 equal elements, 169 nodes

Fig. 3 Relation of rotating speed with h_0 in solid cylinder

Fig. 4 Displacement of end surface of solid cylinder 30 000 rpm

surface, because influences of both end surfaces on deformation do not exist here.

Figure 6 shows inside deformation when non-uniform deformation is the largest ($h_0=0.5$). Where the radius $R < 0.5R_0$, the height decreases uniformly, but where $R > 0.5R_0$, the longitudinal section deforms into fan shape. At the part of $R \approx 0.5R_0$, shear deformation of the longitudinal section becomes the

Fig. 5 Shape of bulge of side surface solid cylinder 30 000 rpm

Fig. 6 Deformation of solid cylinder

Fig. 7 Equivalent strain inside solid cylinder

$h_0=1$
Gray parts have already yielded in the previous stage
Fig. 8 Appearance and development of plastic region in solid cylinder

largest.

Figure 7 shows distribution of equivalent strains when a cylinder of $h_0=0.75$ is yielded totally. Equivalent strain becomes about four times larger near the center than near the edge. Apart from the end surface by more than about one fourth of the diameter, axial change of equivalent strain decreases rapidly, for axial non-uniform deformation becomes extremely small at this part.

Figure 8 shows the appearance and development of plastic region when $h_0=1$. Plastic regions appear at the two points on the central axis where distances from both end surfaces are about one fourth of the diameter respectively, and they contain the centroid of the cylinder, immediately becoming a unit part. Next, it reaches the part of $R=0.3R_e$ on the end surface.

a_r : ratio of inside diameter to outside diameter
Fig. 9 Relation of rotating speed with diameter ratio a_r of hollow cylinder

$a_r=0.25$, 24 000 rpm, Partially yielded stages
Fig. 10 Displacement of end surface of hollow cylinder (theoretical result)

It is at the edge that an elastic region remains to the end.

3.2 Rotation of the hollow cylinder

Figure 9 shows the relation between the rotating speed and the diameter ratio a_r when $h_0=0.5$ and 2. When $a_r=0$ (the solid cylinder), rotating speeds are extremely large both at the first appearance of plastic region and at total yielding. As a_r becomes large, rotating speed necessary for plastic deformation decreases first rapidly and then gradually. The time interval from initial yielding to total yielding becomes the largest when $a_r=0.1$, for non-uniformity of deformation becomes maximum. This interval decreases as a_r becomes large, for plastic deformation proceeds nearly at the same time at all parts. Rotating speed at yielding is a little larger when $h_0=2$ than when

$N=54\,000$ rpm $a_r=20/75=0.267$
Fig. 11 Displacement of end surface of hollow cylinder (Experimental results by Sato et al.)

$a_r=0.25$, 22 000 rpm, Partially yielded stages
Fig. 12 Shape of bulge of inner surface of hollow cylinder (theoretical result)

$h_0=0.5$ regardless of a_r , which means the hollow cylinder becomes just as difficult to be deformed plastically with an increase of h_0 as the solid cylinder. The rotating speed at yielding of the hollow cylinder of an extremely small inner diameter ($a_r=0.0001$) is almost the same as that of the solid cylinder, which means existence of a small hole influences the deformation little.

Figure 10 shows displacement of the end surface in the case of $a_r=0.25$. During partial yielding, a decrease of the distance between both end surfaces due to rotation becomes largest when $h_0=0.5$, and it decreases when h_0 becomes large just as in the case of the solid cylinder (Fig. 4). When the rotating speed increases considerably after total yielding, however, displacement of the end surface becomes larger when $h_0=0.1$ than when $h_0=0.5$.

Sato et al.⁽³⁾ carried out an experimental research about plastic deformation of the hollow cylinder by rotation. As the result, Fig. 11 shows plastic displacement of the end surface of an aluminium hollow cylinder of which the inner diameter is 20 mm and the outer one is 75 mm ($a_r=0.267$). Figure 11 coincides well with Fig. 10 qualitatively as a whole.

Figure 12 shows the shape of a bulge on the inner surface when $a_r=0.25$. The inner diameter always increases by rotation. When h_0 is very small, its increase is axially uniform. Bulge becomes the largest, however, when $h_0=0.5\sim 1$. As h_0 increases more, the bulge becomes small again, and when

$h_0=1.5$, its shape changes from single to double. Bulge on the outer surface has a similar shape to the one in Fig. 12. Figure 13 shows the experimental result by Sato et al.⁽³⁾ concerning the shape of bulges on the inner and outer surfaces. Both on the inner and outer surfaces, bulges are the largest when $h_0=0.53$, and they become double when $h_0>1.6$. The results in Fig. 13 coincide well qualitatively with the theoretic-

Fig. 14 Deformation of hollow cylinder

Aluminum cylinder with 20 mm inside diameter and 75 mm outside diameter ($a_r=0.267$). 54 000 rpm
○ Inner surface ● Outer surface

Fig. 13 Shape of bulge of inner and outer surfaces of hollow cylinder (Experimental results by Sato et al.)

cal results in Fig. 12. Figure 14 shows inside deformation when $a_r=0.25$ and $h_0=1.5$.

Figure 15 shows distribution of circumferential stress when $a_r=0.25$ and $h_0=1.5$. Circumferential stress is the largest at the inner surface on the middle section, but on the other lateral sections, it becomes maximum at the point of a little larger radius than at the inner surface. Circumferential stress is the smallest at the outer surface on the middle section, and its distribution is nearly uniform near the end surface.

Figure 16 shows the distribution of axial stress. Near the middle section, a large tensile stress develops at the inner surface, and a large compressive stress develops at the outer surface. Near the end surface, however, the axial stress becomes extremely small.

4. Conclusion

Means is shown to analyze the elastic-plastic deformation of rotors of which the diameter varies axially by using the finite element method. As fundamental examples, analyses concerning rotation of solid

and hollow cylinders are carried out. Non-uniform deformation, distribution of stress and strain, and appearance and development of the plastic zones are calculated. Non-uniform deformation is the largest when the ratio of height to width $h_0=0.5$. Bulge on the side surface is single when h_0 is small, but it becomes double when h_0 is large. It is within one third of the diameter from the end surface that non-uniform deformation occurs conspicuously.

The authors scincerely thank Professor Kazuo Sato and Dr. Fumio Nagai of Tokyo Institute of Technology for kind permission to reprint their precious experimental results.

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Discussion

H. FUKUOKA (Osaka University):

(1) Does the plastic region appear at the centroid of the cylinder, when $h_0<0.75$, though the plastic regions appear at the two places, when $h_0=1$ as in Fig. 8? Does appearance of it determine the

shape of bulge (single or double)?

(2) When the diameter of the cylinder changes arbitrarily in the axial direction, does calculation by the present method require about the same time for computation as in the examples concerning the

straight cylinders in the present report?

Y. SEGUCHI (Kobe University):

(3) What is the meaning of the explanation that the geometric nonlinearity is taken into account in this report, and how does it influence the results?

(4) Can the present problem be solved also by other methods, for instance, the finite difference method, though only the finite element method is used in the present report? Is the error check of the calculation in the report sufficient? How is yielding of the element determined, when the stress varies within it? Does discrepancy of the stress exist between the neighbouring elements?

Authors' closure

(1) When $h_0 < 0.75$, the plastic region appears at the centroid, and the bulge becomes single. It becomes double, however, when the plastic regions

appear at the two inside parts.

(2) About the same computation time as in the present report may be necessary, when the diameter changes.

(3) This explanation means reformation of the matrix [B] in every stage of calculation with use of nodal displacement. The authors explained in detail about influence of reformation of it^(*).

(4) This problem has not been solved yet by any other methods, as far as the authors know. An error check more accurate than in this report seems to be difficult, though it is not sufficient. The element is supposed to have yielded, when the equivalent stress at the centroid of its section reaches the yield stress of the material. Discrepancy of the stress exists there, but this is small enough to be neglected.

(*) Nagamatsu, A., et al., *Bull. JSME*, Vol. 14, No. 70 (1971-4), p. 322.