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Balancing of a Flexible Rotor*

(3rd Report, Universal State of Balance of a Flexible Rotor Irrespective of Its Bearing Stiffness)

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In balancing a flexible rotor, two alternative procedures have been proposed; one of them is n -plane-balancing and the other $(n+2)$ -plane-balancing, named after the required numbers of correction planes. In this paper, both procedures are discussed theoretically and the latter is found to be superior for the purpose. Next, analysing how the form of characteristic function of shaft deflection of a rotor is influenced by its bearing stiffness, the conditional equation and practical method of balancing are presented here for the universal state of balance of a flexible rotor which will no more be affected by its bearing stiffness. The $(n+2)$ -plane-balancing guarantees this state of balance of the rotor. Further, the theory suggests that the bearing stiffness used in high-speed-balancing need not coincide with what is used in operating condition at site.

1. Introduction

The problems of balancing flexible rotors have increasing importance in connection with the increasing service speeds of rotating machines. Here, balancing of a flexible rotor includes the elimination of the dynamic shaft deflection due to the centrifugal effect of unbalance on the rotor. When a rotor/bearing system possesses a set of orthogonal functions (characteristic functions) of the shaft deflection curve, the deflection becomes null if each modal component of it is balanced. Since as many as n correction planes are necessary and sufficient to balance the first n -th modal component of it, Bishop⁽¹⁾ and Koder⁽²⁾ proposed n -plane-balancing. On the other hand, there is an alternative method called $(n+2)$ -plane-balancing proposed by Federn⁽³⁾ and the author⁽⁴⁾, which requires 'dynamic balance' besides n -plane-method. In this paper both are discussed in order to decide which is more appropriate. The influence of the bearing stiffness on the state of balance of a flexible rotor⁽⁵⁾ is also investigated in this paper to find the possibility of obtaining the universal state of balance which will no more be affected by the variation of the bearing stiffness.

2. Conditional equation of balance of a flexible rotor

A flexible rotor/bearing system considered is

schematically illustrated in Fig. 1. Assumptions are the same as in the previous paper⁽⁴⁾, i.e. (i) the bending stiffness of the rotor and the bearing stiffness are isotropic, but the latter limitation will be deleted later; (ii) internal or external damping, and influence of gravity are neglected; (iii) the system is linear and the gyroscopic effect of rotor section is out of consideration—the latter assumption may be reasonable, because the balancing is a process of straightening the shaft axis; (iv) bearing stiffness is restrained only in radial direction and twisting stiffness about an axis perpendicular to the shaft axis does not come into question, on which Kikuchi reported that the latter is of little importance to the lateral vibration of a flexible rotor except in the closest vicinities of its critical speeds⁽⁶⁾.

Nomenclature are shown as follows (see Fig. 1).

$EI(z)$: bending elasticity of the rotor per unit shaft length

$\mu(z)$: rotor mass per unit shaft length

Fig. 1 Unbalanced flexible rotor/bearing system

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- $u(z)$: initial unbalance (distributed) per unit shaft length
- U_i : correction unbalance (concentrated) in a correction plane at $z_i (i=1, 2, 3, \dots)$
- $\bar{u}(z) \equiv u(z) + \sum_i U_i \delta(z - z_i)$: generalized unbalance at z , analytical representation of co-existence of $u(z)$ and U_i , where $\delta(z - z_i)$ is a delta-function such that
- $$\delta(z - z_i) = 0 \quad \text{at } z \neq z_i$$
- $$= \infty \quad \text{at } z = z_i, \text{ and}$$
- $$\int_0^l \delta(z - z_i) dz = 1 \quad (l: \text{total length of the rotor})$$
- $r(z)$: amount of dynamic shaft deflection $r(z, t) = r(z)e^{j\omega t}$ due to centrifugal effect of unbalances
- $\phi_n(z)$: the n -th characteristic function of shaft deflection of the rotor on two flexible bearings
- $\psi_n(z)$: the n -th characteristic function of the system with two rigid bearings (pinned-pinned ends)
- $\chi_n(z)$: the n -th characteristic function of the system with two completely soft bearings (free-free ends)
- ω_n : the n -th critical speed for the mode ϕ_n
- σ_n : the n -th critical speed for the mode ψ_n
- τ_n : the n -th critical speed for the mode χ_n

For a system shown in Fig. 1, dynamic forces P_A and P_B will appear at bearings A and B due to the centrifugal effect of generalized unbalance $\bar{u}(z)$ at a speed ω such that

$$\left. \begin{aligned} P_A + P_B &= \omega^2 e^{j\omega t} \int_0^l \{ \bar{u}(z) + \mu r(z) \} dz \\ lP_B &= \omega^2 e^{j\omega t} \int_0^l \{ \bar{u}(z) + \mu r(z) \} z dz \end{aligned} \right\} \dots\dots\dots (1)$$

P_A and P_B will be nullified at any ω , if

$$\left. \begin{aligned} (i) \quad \int_0^l \bar{u}(z) dz &= 0, \quad \int_0^l \bar{u}(z) z dz = 0 \\ (ii) \quad r(z) &= 0 \end{aligned} \right\} \dots\dots\dots (2)$$

In other words,

- (i) no dynamic unbalance exists in the rotor,
- (ii) no shaft deflection due to unbalance forces.
- For the system considered there exists a complete system of characteristic functions $\phi_n(z) (n=1, 2, \dots)$, so the shaft deflection $r(z)$ can be expanded to a serial form of

$$r(z) = r_1 \phi_1(z) + r_2 \phi_2(z) + \dots \dots\dots (3)$$

where r_n is a function of rotational speed ω together with the unbalances on the rotor. The previous paper⁽⁴⁾ revealed that

$$r_n = \frac{(\omega/\omega_n)^2}{1 - (\omega/\omega_n)^2} \bar{e}_n \quad (n=1, 2, \dots) \dots\dots\dots (4)$$

where

$$\left. \begin{aligned} \bar{e}(z) &\equiv \bar{u}(z)/\mu(z) = \sum_{n=1}^{\infty} \bar{e}_n \phi_n(z) \\ \bar{e}_n &= \int_0^l \bar{u}(z) \phi_n(z) dz / \int_0^l \mu(z) \phi_n^2(z) dz \end{aligned} \right\} \dots\dots (5)$$

$$(n=1, 2, \dots)$$

is the local mass eccentricity due to $\bar{u}(z)$.

If unbalances are so arranged that they make $\bar{e}_n$ zero for $n=1, 2, \dots$, then r_n will vanish for any ω . Hence, Eq. (2) can be re-arranged relating to $u(z)$ as

$$\left. \begin{aligned} (i) \quad \int_0^l \bar{u}(z) dz &= 0, \quad \int_0^l \bar{u}(z) z dz = 0 \\ (ii) \quad \int_0^l \bar{u}(z) \phi_n(z) dz &= 0, \quad (n=1, 2, \dots) \end{aligned} \right\} \dots\dots (6)$$

The condition (i) is automatically satisfied if we take n of (ii) up to infinity, because $\bar{e}(z)$ and $\bar{u}(z)$ will be zero at any axial position z as seen in Eq. (5). However, it is impossible to take n up to infinity, and we can only balance practically the first h -th modal components of shaft deflection. The maximum order of the nullified mode, h , may be the order of critical speed ω_h lying close to the maximum operational speed Ω_m ⁽⁷⁾. But if any modal components of shaft deflection higher than $(h+1)$ -th remain unbalanced, difference will appear between n - and $(n+2)$ -plane-methods as for residual bearing forces.

(a) n -plane-method

This procedure requires that $\bar{e}_n$ is zero for $n=1, 2, \dots, h$. Residual bearing forces existing after balancing are readily estimated by Eqs. (3) (4) and (5) as

$$\left. \begin{aligned} P_A' + P_B' &= \omega^2 e^{j\omega t} \int_0^l \left[\mu \bar{e}(z) + \mu \sum_{n=h+1}^{\infty} r_n \phi_n(z) \right] dz \\ &= \omega^2 e^{j\omega t} \int_0^l \mu \left[\sum_{n=h+1}^{\infty} \bar{e}_n \phi_n(z) \right. \\ &\quad \left. + \sum_{n=h+1}^{\infty} \frac{(\omega/\omega_n)^2}{1 - (\omega/\omega_n)^2} \bar{e}_n \phi_n(z) \right] dz \\ &= \omega^2 e^{j\omega t} \int_0^l \sum_{n=h+1}^{\infty} \frac{1}{1 - (\omega/\omega_n)^2} \bar{e}_n \int_0^l \mu \phi_n dz \\ lP_B' &= \omega^2 e^{j\omega t} \sum_{n=h+1}^{\infty} \frac{1}{1 - (\omega/\omega_n)^2} \bar{e}_n \int_0^l \mu \phi_n z dz \end{aligned} \right\} \dots\dots\dots (7)$$

The magnification factor $1/[1 - (\omega/\omega_n)^2]$ varies with the rotational speed and is plotted in Fig. 2. The maximum operating speed Ω_m being lower than any m -th critical speed ω_m of $m > h$, the magnification factor is always larger than unity over the range of ω lower than Ω_m . Hence, residual bearing forces shall be

$$\left. \begin{aligned} P_A' + P_B' &> \omega^2 e^{j\omega t} \sum_{n=h+1}^{\infty} \bar{e}_n \int_0^l \mu \phi_n dz \\ lP_B' &> \omega^2 e^{j\omega t} \sum_{n=h+1}^{\infty} \bar{e}_n \int_0^l \mu \phi_n z dz \end{aligned} \right\} \dots\dots\dots (7a)$$

(b) $(n+2)$ -plane-method

Application of conditions (i) and (ii) of Eq.

(6) reduces the residual bearing forces such that

$$\left. \begin{aligned} P_A'' + P_B'' &= \omega^2 e^{j\omega t} \int_0^l \left[\mu \sum_{n=n+1}^{\infty} r_n \phi_n(z) \right] dz \\ &= \omega^2 e^{j\omega t} \sum_{n=n+1}^{\infty} \frac{(\omega/\omega_n)^2}{1-(\omega/\omega_n)^2} \bar{e}_n \int_0^l \mu \phi_n dz \\ lP_B'' &= \omega^2 e^{j\omega t} \sum_{n=n+1}^{\infty} \frac{(\omega/\omega_n)^2}{1-(\omega/\omega_n)^2} \bar{e}_n \int_0^l \mu \phi_n z dz \end{aligned} \right\} \dots\dots\dots (8)$$

The magnification factor of this case being $(\omega/\omega_n)^2 / \{1-(\omega/\omega_n)^2\}$ as plotted in Fig. 3, it is evident for any ω of $\omega \leq \Omega_m \doteq \omega_h$ that

$$\frac{(\omega/\omega_n)^2}{1-(\omega/\omega_n)^2} \ll \frac{(\omega/\omega_{h+1})^2}{1-(\omega/\omega_{h+1})^2} \quad (n=h+2, h+3, \dots\dots)$$

Therefore,

$$\left. \begin{aligned} P_A'' + P_B'' &\doteq \omega^2 e^{j\omega t} \frac{(\omega/\omega_{h+1})^2}{1-(\omega/\omega_{h+1})^2} \bar{e}_{h+1} \int_0^l \mu \phi_{h+1} dz \\ lP_B'' &\doteq \omega^2 e^{j\omega t} \frac{(\omega/\omega_{h+1})^2}{1-(\omega/\omega_{h+1})^2} \bar{e}_{h+1} \int_0^l \mu \phi_{h+1} z dz \end{aligned} \right\} \dots\dots\dots (8 \cdot a)$$

As obviously seen, the only $(h+1)$ -th modal component of mass eccentricity dominates in the residual bearing forces. Comparing Eqs. (7·a) and (8·a), it is observed that the latter is considerably smaller than the former over the speed range of $\omega \leq \Omega_m$. This

Fig. 2 Magnification factor $1/[1-(\omega/\omega_n)^2]$ with varying ω

Fig. 3 Magnification factor $(\omega/\omega_n)^2/[1-(\omega/\omega_n)^2]$ with varying ω

difference has come from the application of 'dynamic balancing' in $(n+2)$ -plane-method, so we conclude that $(n+2)$ -plane-method is more appropriate than n -plane-method as for the residual bearing forces after balancing.

Conditional equation of balance of a flexible rotor is thus presented by taking the upper limit of n in Eq. (6) equal to h as

$$\left. \begin{aligned} \int_0^l \bar{u}(z) dz &= 0, \quad \int_0^l \bar{u}(z) z dz = 0 \\ \int_0^l \bar{u}(z) \phi_n(z) dz &= 0 \quad (n=1, 2, \dots\dots, h) \end{aligned} \right\} \dots\dots\dots (9)$$

This equation is further expressed in terms of initial unbalance $u(z)$ and correction unbalance U_i as

$$\left. \begin{aligned} \int_0^l u(z) dz + \sum_i U_i &= 0, \quad \int_0^l u(z) z dz + \sum_i U_i z_i = 0 \\ \int_0^l u(z) \phi_n(z) dz + \sum_i U_i \phi_n(z_i) &= 0 \quad (n=1, 2, \dots\dots, h) \end{aligned} \right\} \dots\dots\dots (9 \cdot a)$$

which is well-known as Kellenberger-Miwa's equation of balance.

3. Characteristic function of the shaft deflection in connection with the variation of the bearing stiffness

Even when the rotor of same design is dealt with, the algebraic representation of characteristic function $\phi_n(z)$ of the shaft deflection appearing in Eq. (9) will generally change its form depending upon the bearing stiffness, as schematically shown in Fig. 4 for a uniform and symmetrical rotor/bearing system. This implies that the state of balance of a flexible rotor may be affected by the bearing stiffness. Hence, the following is to be investigated:

- (i) How $\phi_n(z)$ changes its form depending on the magnitude of bearing stiffness?
- (ii) Is it possible to obtain the universal state of balance which is unaffected by the bearing stiffness?

The shaft deflection $r(z)$ of the rotor in Fig. 1 at

Fig. 4 Mode shapes of the uniform and symmetrical rotor/bearing system (As a function of the rotor flexibility k and bearing stiffness K)

its steady state of rotation is given as a solution of the following equation

$$EI(r''(z))'' - \mu\omega^2 r(z) = 0 \dots\dots\dots(10)$$

under the end condition of flexible support given in Table 1. In general, $r(z)$ is a vector in a radial plane at axial position z and can be resolved in this plane into two components r_x and r_y perpendicular to each other. Since there is no coupling motion in the system between r_x and r_y , it is analytically sufficient to consider simply one of them, r_x or r_y . Hence, we consider $r(z)$ in Eq. (10) to be representative of such a component and treat it as a scalar.

The series of characteristic functions $\phi_j (j=1, 2, \dots)$ of the rotor on rigid bearings (pinned-pinned ends) conforms to "the complete system of functions", so $r(z)$ can be expressed in a serial form of

$$r(z) = a + b(z/l) + \sum_{j=1}^{\infty} c_j \phi_j(z) \dots\dots\dots(11)$$

Here, the first and second terms of right hand side represent the effect of flexible support, as treated later. Next problem is to evaluate unknown coefficients a , b and c_j for each of the eigen-value $\omega_n (n=1, 2, \dots)$ of Eq. (10). To apply Ritz's method, take m as the maximum order of j in Eq. (11) as*

$$r(z) = a + b(z/l) + \sum_{j=1}^m c_j \phi_j(z) \dots\dots\dots(12)$$

It is evident that $r(z)$ in Eq. (12) satisfies end condition of $r''=0$ for flexible support in Table 1. Another end condition of the shearing force leads us to the following relations:

$$\text{at } z=0: \sum_{j=1}^m c_j (EI\phi_j'')_0' + K_A a = 0 \dots\dots\dots(13)$$

$$\text{at } z=l: \sum_{j=1}^m c_j (EI\phi_j'')_l' - K_B(a+b) = 0 \dots\dots\dots(14)$$

As clearly seen, a and b are expressed in linear com-

bination of $c_j (j=1, 2, \dots, m)$.

To evaluate c_j , calculus of variation is applied. Let Eq. (12) be the solution of Eq. (10) and the shaft deflection at both ends $r(0)=a$ and $r(l)=a+b$ be constrained by the boundary conditions, Eqs. (13) and (14). Then c_j shall be of such magnitude that makes Eq. (12) a stationary function of Eq. (10). Since ϕ_j is zero at both ends by definition, Galerkin's method is applicable in this case, and the following relation is readily obtained as the condition of determining c_j as

$$\int_0^l [(EI r'')'' - \mu\omega^2 r] \phi_j dz = 0 \quad (j=1, 2, \dots, m) \dots\dots\dots(15)$$

where r is to be substituted by Eq. (12). It is clear from Eq. (12) that

$$(EI r'')'' = \sum_{j=1}^m c_j (EI \phi_j'')''$$

On the other hand, σ_j being the eigen-value of ϕ_j , we have

$$(EI \phi_j'')'' = \mu \sigma_j^2 \phi_j$$

Therefore

$$\begin{aligned} (EI r'')'' - \mu\omega^2 r &= \mu \sum_{j=1}^m c_j \sigma_j^2 \phi_j \\ &- \mu\omega^2 \left[\left(a + b \frac{z}{l} \right) + \sum_{j=1}^m c_j \phi_j \right] \\ &= \mu \left[\sum_{j=1}^m c_j (\sigma_j^2 - \omega^2) \phi_j - \omega^2 \left(a + b \frac{z}{l} \right) \right] \end{aligned}$$

Multiplying ϕ_j by both hand sides of above equation and integrating it from 0 to l of z , the orthogonality of ϕ_j gives

$$c_j (\sigma_j^2 - \omega^2) \int_0^l \mu \phi_j^2 dz - \omega^2 \int_0^l \mu \left(a + b \frac{z}{l} \right) \phi_j dz = 0 \quad (j=1, 2, \dots, m) \dots\dots\dots(16)$$

If the bearing stiffness is of any value except infinity

Table 1 Characteristic functions and their eigen-values of shaft deflection for various end conditions

Support		Flexible	Simply supported	Free
n -th charact. function		$\phi_n(z)$	$\psi_n(z)$	$\chi_n(z)$
n -th critical speed		ω_n	σ_n	τ_n
Equation of n -th shaft deflection		$(EI\phi_n'')'' - \mu\omega_n^2 \phi_n = 0$	$(EI\psi_n'')'' - \mu\sigma_n^2 \psi_n = 0$	$(EI\chi_n'')'' - \mu\tau_n^2 \chi_n = 0$
Boundary condition	$z=0$	$\phi_n''=0$ $(EI\phi_n'')' + K_A \phi_n = 0$	$\psi_n=0$ $\psi_n''=0$	$\chi_n''=0$ $(EI\chi_n'')'=0$
	$z=l$	$\phi_n''=0$ $(EI\phi_n'')' - K_B \phi_n = 0$	$\psi_n=0$ $\psi_n''=0$	$\chi_n''=0$ $(EI\chi_n'')'=0$
Eigen-value of system of $EI, \mu = \text{const.}$		λ_n (see Fig. 6)	$\mu_n = n\pi (n=1, 2, \dots)$	$\nu_n = 0 (n=1, 2)$ $= [(2n-3)/2]\pi (n=3, 4, \dots)$
		$\omega_n = (\lambda_n/l)^2 \sqrt{EI/\mu}$	$\sigma_n = (\mu_n/l)^2 \sqrt{EI/\mu}$	$\tau_n = (\nu_n/l)^2 \sqrt{EI/\mu}$

* How to determine m will be described later. The general evaluation of possible error due to ignoring the higher term of ϕ_j of $j \geq m+1$ is treated in the literature⁽⁸⁾.

(pinned-pinned ends), then $\omega \neq \sigma_j$ as shown later in Figs. 6 and 8. Hence,

$$c_j = \frac{(\omega/\sigma_j)^2 \int_0^l \mu \left(a + b \frac{z}{l}\right) \phi_j dz}{1 - (\omega/\sigma_j)^2 \int_0^l \mu \phi_j^2 dz} \quad (j=1, 2, \dots, m) \quad (17)$$

Substitution of a and b in Eq. (16) by Eqs. (13) and (14) gives a set of simultaneous linear algebraic equations of c_j of order m , and a set of solutions of c_j other than the trivial one will be obtained, if the determinant having coefficients of c_j as its elements is zero. This gives the frequency equation. The order of frequency equation being m as for ω^2 , we have real and positive values of ω^2 as many as m in number, i.e. $\omega = \omega_n$ ($n=1, 2, \dots, m$). Denoting the coefficients determined for $\omega = \omega_n$ as a_n for a , b_n for b and c_{nj} for c_j , the solution of Eq. (10) is perfectly expressed in the form of Eq. (12) for $\omega = \omega_n$, which corresponds to the n -th characteristic function $\phi_n(z)$ of the general solution of

$r(z) = \sum_{n=1}^{\infty} r_n \phi_n(z)$. Hence, ϕ_n is written as

$$\phi_n(z) = a_n + b_n(z/l) + \sum_{j=1}^m c_{nj} \phi_j(z) \quad (n=1, 2, \dots, \leq m) \quad (18)$$

It is clear that ω_n is the n -th critical speed of the rotor on flexible bearings.

To evaluate the maximum order $j=m$ of ϕ_j in Eq. (18), we compare the magnitude of c_{nj} for $j=m, m+1, m+2, \dots$. Re-writing Eq. (17) for n -th mode gives

$$c_{nj} = \frac{(\omega_n/\sigma_j)^2 \int_0^l \mu \left(a_n + b_n \frac{z}{l}\right) \phi_j dz}{1 - (\omega_n/\sigma_j)^2 \int_0^l \mu \phi_j^2 dz} \quad (n=1, 2, \dots, \leq m; j=1, 2, \dots, m) \quad (17 \cdot a)$$

Considering that $(\omega_n/\sigma_j)^2 / \{1 - (\omega_n/\sigma_j)^2\} \ll 1$ for any j larger than n and that both a_n and b_n are of certain finite magnitude for flexible support (see Fig. 4), it will be clear that

$$c_{nn} \gg c_{n, n+1} \gg c_{n, n+2} \gg \dots$$

where any c_{nj} of magnitude zero is excluded. Thus

Fig. 5 A uniform and symmetrical rotor/bearing system

we conclude that $\phi_n(z)$ can well be approximated by taking the upper limit of $j=m$ to be equal to n , as

$$\phi_n(z) = a_n + b_n(z/l) + \sum_{j=1}^n c_{nj} \phi_j(z) \quad (n=1, 2, \dots) \quad (19)$$

[Numerical example] To verify the above statement, $\phi_n(z)$ thus determined is compared with the exact solution of Eq. (10). Consider, for example, the uniform and symmetrical rotor/bearing system shown in Fig. 5, where $EI = \text{const.}$, $\mu = \text{const.}$, and $K_A = K_B = K$. Of this system $\phi_j(z) = \sin(j\pi z/l)$ and $\sigma_j = (j\pi/l)^2 \sqrt{EI/\mu}$ have been given in Table 1. The system being symmetrical as for mid-span, the modes of the shaft deflection will be separated into two groups of odd-numbered modes and even-numbered ones. They are determined separately.

(a) Odd-numbered modes, $n=1, 3$: In Eq. (12) it is evident that $b=0$, so that

$$r(z) = a + c_1 \phi_1(z) + c_3 \phi_3(z) = a(1 + \gamma_1 \phi_1 + \gamma_3 \phi_3) \quad (20)$$

where

$$\gamma_1 = c_1/a, \quad \gamma_3 = c_3/a$$

γ_1 and γ_3 are to be determined as dependents of bearing stiffness K . Boundary conditions, Eqs. (13) and (14), are of the same form in this case. Substituting Eq. (20) into Eq. (13) gives

$$\gamma_1 + 3\gamma_3 = \frac{1}{\pi^3} \frac{K}{EI/l^3} = \frac{24}{\pi^3} \alpha \quad (21)$$

where $\alpha = 2K/k$ is the ratio of total bearing stiffness to bending stiffness k of the rotor and is the dimensionless parameter giving the measure of relative flexibility of bearings to that of the rotor itself. Here, $k = 48EI/l^3$ is rotor bending stiffness at mid-span for a simply supported system.

Substituting Eq. (20) into Eq. (16) and calculating gives

$$\gamma_j(\sigma_j^2 - \omega^2) - \frac{4}{j\pi} \omega^2 = 0, \quad (j=1, 3) \quad (22)$$

Considering that $\omega \neq \sigma_j$ for any α of finite magnitude gives

$$\gamma_j = \frac{(\omega/\sigma_j)^2}{1 - (\omega/\sigma_j)^2} \frac{4}{j\pi} \quad (j=1, 3) \quad (23)$$

Let $\omega/\sigma_1 = \eta$ and consider that $\sigma_3/\sigma_1 = 3^2 = 9$, and then combine Eq. (23) with Eq. (21), which reduces

$$\frac{\eta^2}{1 - \eta^2} + \frac{9\eta^2}{9^2 - \eta^2} = \frac{6}{\pi^2} \alpha$$

Re-arranging gives the frequency equation as

$$(1 + 0.061\alpha)\eta^4 - (9 + 4.98\alpha)\eta^2 - 4.93\alpha = 0 \quad (24)$$

from which two critical speeds ω_1 and ω_3 are determined depending upon the stiffness parameter α . Substituting these ω_1 and ω_3 separately into Eq. (23), two sets of $(\gamma_{11}, \gamma_{13})$ and $(\gamma_{31}, \gamma_{33})$ are evaluated for the first and third modes, respectively. To evaluate the coefficient a in Eq. (20), $\phi_n(z)$ in Eq. (19) is

normalized, for example, as

$$\int_0^l \mu \phi_n^2 dz = M, \quad M = \int_0^l \mu dz \quad (25)$$

Substituting Eq. (20) into Eq. (25) gives the magnitude of $a=a_1$ and a_3 . In the problem of balancing, however, $\phi_n(z)$ need not be 'normalized' as seen in Eq. (9), so that it is of no use to evaluate a_n .

(b) Even-numbered mode, $n=2$: The shaft deflection will be anti-symmetrical as for the mid-span for even-numbered mode, so that $b=-a/2$ or

$$r(z) = a \left[\left(1 - \frac{2z}{l} \right) + \gamma_2 \phi_2(z) \right], \quad \gamma_2 = c_2/a \quad (26)$$

The boundary condition of $r''=0$ at $z=0$ gives

$$\gamma_2 = \frac{1}{(2\pi)^3} \frac{K}{EI/l^3} = \frac{3}{\pi^3} \alpha \quad (27)$$

On the other hand, condition to give c_j or γ_j , Eq. (17) leads us to the relation of

$$\gamma_2 = \frac{2}{\pi} \frac{(\omega/\sigma_2)^2}{1 - (\omega/\sigma_2)^2} \quad (28)$$

Eliminating γ_2 of Eq. (28) by Eq. (27) gives critical speed as

$$\omega^2 = \frac{0.152\alpha}{1 + 0.152\alpha \sigma_2^2} \quad (29)$$

Thus ω_2 and γ_2 are evaluated depending upon the parameter α . The coefficient a_2 need not be determined as before.

Numerical results of ω_n and $\phi_n(z)$ for various magnitudes of α are illustrated in Figs. 6 and 7, respectively. Solid lines in these figures correspond to the exact solution of Eq. (10) and dashed lines to the result of above calculation.

there is an alternative scale of ω_1/σ_1 which is uniquely determined by α . According to our experience, ω_1/σ_1 of rotating machinery of medium and larger class such as steam turbines and turbo-generator lies between 0.7 to 0.9, so that mode shapes in Fig. 7 are illustrated for these ranges. As seen in Fig. 6, our calculation is satisfactory for ω_1 , but not sufficiently so for ω_2 and ω_3 . Errors have mainly come from the maximum order of the mode revolved in the calculation by Ritz's method, i.e. $n=3$ in this particular case. If we take m of Eq. (12) much higher than $m=3$ as the basis of calculation by Ritz's method, better results are obtained of ω_2 and ω_3 as shown in chained lines in Fig. 6. For modal shapes in Fig. 7, on the other hand, good coincidence is shown in this example. To adopt the order m much higher than 3 gives better approximation of the modal shapes also, but little improvement is obtained in spite of increasing troublesome calculation.

Generally, it should be noted in Eq. (12) that

$$\phi_n(z) = a_n + b_n(z/l) + \sum_{j=1}^m c_{nj} \phi_j(z) \quad (30)$$

($n=1, 2, \dots, \leq m$) a part of infinite series of ϕ_j ($j=1, 2, \dots$ up to infinity) having m as the maximum of j , does not coincide with an alternative expression

$$\varphi_n(z) = a_n' + b_n'(z/l) + \sum_{j=1}^m c_{nj}' \phi_j(z) \quad (31)$$

($n=1, 2, \dots, \leq m$) which is at first assumed only to have the m -th term of ϕ_j as its highest element and all coefficients a_n' , b_n'

Fig. 6 Eigen-value of uniform and symmetrical rotor/bearing system

and c_{nj}' are determined by Ritz's method. Generally, a_n and a_n' , b_n and b_n' , c_{nj} and c_{nj}' are different between Eqs. (30) and (31). To improve the accuracy of Eq. (31), it is advisable to take the term of ϕ_j up to the order higher than m as the basis of Ritz's method, then cut off the higher terms of ϕ_j greater than $(m+1)$ -th after determining all of possible coefficients. In the discussion of the later part of this paper, $\phi_n(z)$ is regarded in the meaning of Eq. (30). Referring to this notice, accuracy of $\phi_n(z)$ will generally sufficient by taking $m=n$ for any n -th mode, as previously shown in Eq. (19).

4. Conditional equation and balancing procedure for the universal state of balance irrespective of bearing stiffness

The preceding chapter showed that the character-

Fig. 7 Mode shapes of the first three modes of the system of Fig. 5

istic function $\phi_n(z)$ on flexible bearings can be approximated in the serial form of $\phi_j(z)$, as in Eq. (19).

Substituting Eq. (19) into the second expression of conditional equation (9) and considering the condition of dynamic balance—the first expression of Eq. (9)—gives

$$\int_0^l \bar{u}(z) \phi_n dz = a_n \int_0^l \bar{u}(z) dz + \frac{b_n}{l} \int_0^l \bar{u}(z) z dz + \sum_{j=1}^n c_{nj} \int_0^l \bar{u}(z) \phi_j dz = \sum_{j=1}^n c_{nj} \int_0^l \bar{u}(z) \phi_j dz = 0 \quad (n=1, 2, \dots, h) \quad (32)$$

Here h is the maximum order of the shaft deflection to be balanced. Evidently

$$\int_0^l \bar{u}(z) \phi_n(z) dz = 0 \quad (n=1, 2, \dots, h)$$

if

$$\int_0^l \bar{u}(z) \phi_j(z) dz = 0 \quad (j=1, 2, \dots, h)$$

which is satisfied for any magnitude of c_{nj} . Hence, conditional equation (9) reduces to

$$\left. \begin{aligned} \int_0^l \bar{u}(z) dz &= 0, & \int_0^l \bar{u}(z) z dz &= 0 \\ \int_0^l \bar{u}(z) \phi_j(z) dz &= 0 \quad (j=1, 2, \dots, h) \end{aligned} \right\} \quad (33)$$

Let characteristic function of the rotor on bearing of any assigned stiffness other than that of $\phi_n(z)$ be

$$\Phi_n(z) = A_n + B_n(z/l) + \sum_{j=1}^{\infty} C_{nj} \phi_j(z) \quad (n=1, 2, \dots) \quad (34)$$

then the following relation is always satisfied irrespective of the magnitudes of A_n , B_n and C_{nj} , depends of bearing stiffness, so that

$$\int_0^l \bar{u}(z) \Phi_n(z) dz = 0 \quad (n=1, 2, \dots, h) \quad (35)$$

is true so far as Eq. (33) has been satisfied. This has great importance in balancing, i.e. the n -th modal balance on any flexible bearings, Eq. (35), is automatically satisfied by the condition shown in Eq. (33). Thus, the universal balance of a flexible rotor can successfully be obtained irrespective of its bearing stiffness. In case of anisotropic bearing, radial asymmetry of bearing stiffness, the algebraic expression of Φ_n —or magnitude of coefficients A_n , B_n and C_{nj} in Eq. (35)—changes itself in one revolution of the rotor, but the balanced state of the rotor is maintained independently of bearing anisotropy, provided Eq. (33) is satisfied. In other words, the universal state of balance can be attained by

(i) dynamic balance at low speed where significant shaft deflection does not yet appear,

(ii) the n -th modal balance on rigid bearing (pinned-pinned ends) for $n=1, 2, \dots, h$ at high speeds.

As an alternative of the procedure (ii), the n -th modal balance can be performed on any flexible bearings. Φ_n in Eq. (34) being a characteristic func-

tion on those bearings, then

$$\int_0^l \bar{u}(z) \Phi_n dz = A_n \int_0^l \bar{u}(z) dz + B_n \int_0^l \bar{u}(z) z dz + \sum_{j=1}^n C_{nj} \int_0^l \bar{u}(z) \phi_j dz$$

therefore, the condition of

$$\left. \begin{aligned} \int_0^l \bar{u}(z) dz &= 0, \quad \int_0^l \bar{u}(z) z dz = 0, \\ \int_0^l \bar{u}(z) \phi_i(z) dz &= 0 \quad (i=1, 2, \dots, n-1) \end{aligned} \right\} \dots\dots\dots(36)$$

promises

$$\int_0^l \bar{u}(z) \Phi_n(z) dz = C_{nn} \int_0^l \bar{u}(z) \phi_n(z) dz \quad (n=1, 2, \dots, h) \dots\dots\dots(37)$$

This shows the interchangeability of n -th modal balance for modes Φ_n and ϕ_j , provided $C_{nn} \neq 0$ and the condition of Eq. (36) exists. Whether $C_{nn} \neq 0$ holds or not can be assessed by Eq. (17-a). Detailed discussion is omitted here, but at least in the case of uniform and symmetrical rotor system of Fig. 4 $C_{nn} \neq 0$ so far as bearing support is not free-free. On very flexible bearing of almost free-free ends, characteristic function approaches χ_n (see Table 1). Since balancing for modes χ_1 and χ_2 means mere 'dynamic balancing', so will the balancing for χ_n ($n \geq 3$) correspond to that for ϕ_{n-2} as easily traced by similar process described from Eq. (34) to Eq. (37). Hence, the bearing stiffness at high-speed-balancing cannot be set extremely low. After this notice is remembered, Eq. (9) as well as Eq. (33) is regarded here again as the conditional equation for the universal state of balance. It is, of course, unnecessary that ϕ_n in Eq. (9) coincide with what is used in operating condition of the rotor.

The balancing procedure proposed above is summarized as;

(i) dynamic balance at sufficiently low speed, and

(ii) a sequence of n -th modal balancing on any flexible bearings (other than extremely flexible ones) for modes $n=1, 2, \dots$, up to the h -th.

In procedure (ii) it should also be noted that high speed balancing for mode Φ_n on any flexible bearing is interchangeable with that for the mode ϕ_n or ϕ_n , so far as Eq. (36) is satisfied. As for the bearing stiffness when a rotor is balanced, some additional precaution should be taken as denoted below.

Critical speed ω_n for any n -th mode lies in the domain of

$$\tau_n < \omega_n < \sigma_n$$

as seen in Figs. 6 or 8. If the bearing of balancing facility is too stiff, test speed for n -th modal balancing ω_{nt} may be beyond the maximum speed Ω_m allowed to the rotor by design. In the example given in

Fig. 8, Ω_m lies between ω_3 and σ_3 . If the bearing of balancing facility is stiffer than that for the parameter α_{3m} (critical magnitude of α where $\omega_{3t} = \Omega_m$), then the third modal balancing becomes impracticable on this bearing, because ω_{3t} falls beyond Ω_m . For this reason the bearing stiffness should be appropriately selected in high speed balancing so as not to be extremely low nor higher than α_{3m} in this particular case.

Lastly, residual unbalance forces acting on bearings of different stiffness is to be investigated. Let characteristic function on bearings used in balancing and in operation be ϕ_n and Φ_n , respectively, and a rotor be balanced by Eq. (9). Within the range of accuracies where Eqs. (19) and (34) are valid, the following equivalency holds for modes $n=1, 2, \dots, h$ that

balancing for $\phi_n \rightarrow$ that for $\phi_n \rightarrow$ that for Φ_n as deduced from Eqs. (32) (36) and (37). This means that balancing for modes ϕ_n up to $n=h$ is equivalent to that for alternative modes Φ_n also up to $n=h$. Hence, the residual bearing force after balancing is affected only by still unbalanced modal components of Φ_n of $n \geq h+1$, and is expressed in a similar form to Eq. (8-a).

[Numerical example] An example of the universal state of balance is shown here for a system shown in Fig. 5. Rotor is uniform and symmetrical as before, and the initial unbalance distributes uniformly along the shaft axis, i.e.

$$u(z) = U/l = \text{const.}, \text{ total unbalance } U = \int_0^l u(z) dz.$$

Let the rotor be balanced to the first mode of the shaft deflection, so correction planes as many as $i=h+2=3$ are necessary and sufficient. They are arranged symmetrically at axial stations of $z_i=0, l/2$ and l . Characteristic function is approximated to

Fig. 8 Limitation of bearing stiffness applied for modal balancing of a flexible rotor

the term of ϕ_1 in Eq. (20), i.e.

$$\phi_1(z) = 1 + \gamma_1 \sin(\pi z/l), \quad \gamma_1 = 24\alpha/\pi^3$$

where γ_1 is a function of stiffness parameter $\alpha = 2K/k$. Substituting these relations into Eq. (9-a) and rearranging gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1/2 & 1 \\ 1 & 1+\gamma_1 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = - \begin{pmatrix} \int_0^l u(z) dz \\ \int_0^l u(z) z dz/l \\ \int_0^l u(z) \phi_1 dz \end{pmatrix} = - \begin{pmatrix} 1 \\ 1/2 \\ 1 + \frac{2}{\pi} \gamma_1 \end{pmatrix} U$$

Unbalance correction $U_i (i=1, 2, 3)$ is immediately determined as

$$U_1/U = U_3/U = -0.18, \quad U_2/U = -0.64$$

Since numerical values of them are all independent of γ_1 . These three corrections are valid for the rotor whatever magnitude the bearing stiffness may have.

5. Conclusions

(i) The $(n+2)$ -plane-balancing of a flexible rotor reduces to n -plane-one provided the modal components of shaft deflection are balanced to the order of infinity, where dynamic bearing force as well as dynamic shaft deflection due to unbalance will vanish.

(ii) Practically the shaft deflection can be balanced to the h -th mode where h is a certain finite integer determined by machine design. In this circumstance, $(n+2)$ -plane-balancing is preferable to n -plane-one, and the former gives considerably less bearing forces to the rotor after balancing, while residual shaft deflection is of the same degree for both procedures.

(iii) The universal state of balance unaffected by the bearing stiffness can be achieved of a flexible

rotor by $(n+2)$ -plane-balancing, conditional equation of which is given by Eq. (9) or Eq. (33).

(iv) To obtain such universal state of balance, a rotor is balanced 'dynamically' at first, and then modal balancing is made for the first, second, ... to the h -th mode of shaft deflection. Correction planes as many as $h+2$ are necessary and sufficient.

(v) The stiffness of bearings when a rotor is balanced can differ from that when the rotor is in actual operation, but care should be taken so that bearing stiffness may not be too soft as the rotor supporting condition approaches free-free state, nor be too hard as the h -th critical speed ω_h exceeds the maximum operational speed.

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Discussion

K. SHIRAKI (Mitsubishi Heavy Industry, Ltd.):

(1) We approve your way of analysing vectors of unbalances and that of the shaft deflection as components in a plane including the shaft axis. In this connection, the assumption of no coupling action of bearing stiffness between two radial directions perpendicular to each other should also be stated.

(2) For a multi-rotor system on multi-bearings we consider that your theory will be applicable, where dynamic balance at low speed is pre-requisite. In practice, however, a coupled rotor cannot be mounted on a conventional low-speed balancing machine, nor balanced so accurately in final mounting condition owing to low-sensitivity at low speed operation. Further, a coupled rotor, each unit rotor of

which has been balanced separately on low-speed balancing machine, guarantees only inferior state of dynamic balance owing to coupling errors. How is the accurate dynamic balance of a coupled rotor obtained?

G. SHIMOMURA (Akashi Seisakusho, Ltd.):

(3) When deriving Eq. (15), both end conditions are assumed to be restrained as $r(0)=a$ and $r(l)=a+b$ in stationary function. But the kinetic energy of bearing motion should be dealt with, because the bearing stiffness is of certain finite magnitude.

(4) Three corrections for balancing a rotor up to the first mode have been illustrated independently of the magnitude α . It is also desirable that another example be given of balancing up to the second mode, assuming e.g. $u(z)=u_0(1-2z/l)$ or $u(z)=u_0\sin(2\pi z/l)$.

T. IWATSUBO (Kobe University):

(5) Balancing of a rotor supported on soft bearing at site to the h -th mode under the condition of nearly pinned-pinned ends compels the rotor into the state of 'excess balance'. Hence, $(n+2)$ -plane-balancing on nearly hard bearing seems inadequate in this case (cf. the case of $\omega_1/\sigma_1=0.6$ in Fig. 7). Without such excess balancing it cannot be concluded directly that $\int_0^l \bar{u}(z)\phi_n(z)dz \cong 0$ and $\int_0^l \bar{u}(z)\phi_n(z)dz \cong 0$ are equivalent to each other.

A. TAMURA (Tokyo Institute of Technology):

(6) According to your previous paper⁽⁴⁾ the shaft deflection $r(z)$ vanishes, if the modal mass eccentricity $\bar{\epsilon}_n$ or the component of unbalance $\bar{u}_n$ becomes zero for $n=1, 2, \dots$. In case where $\mu(z)$ and $EI(z)$ are constant for any z , $\bar{\epsilon}_n=0$ and $\bar{u}_n=0$ are interchangeable, otherwise $\bar{\epsilon}_n=0$ is the only condition of reducing $r(z)$ to zero.

H. ÔTA (Nagoya University):

(7) Are there any experiments where the theories appearing in the series of your papers on balancing a flexible rotor have been practiced?

Author's closure

(1) In the system shown in Fig. 1, the bearing stiffness of constant magnitude acts only in the direction of radial displacement of the bearing, which the author defined as 'bearing isotropy'. Hence, it is unnecessary to assume non-coupling of two restoring forces in a bearing perpendicular to each other. But there is a case where bearing isotropy means such configuration that several springs of the same stiffness are arranged radially with the same angular distances. In this case, the assumption of non-coupling of the bearing stiffness is generally needed.

(2) The following procedure is recommended for dynamic balancing of a coupled rotor. (i) Each of individual rotors is preliminarily balanced on low-speed balancing machine and then they are coupled in such a manner that the state of dynamic balance will not be disturbed. As you suggested, care should be taken on the concentricity of shaft axes of both rotors to be coupled, concentricity of coupling adapters and parallelity of both coupling surfaces. (ii) When any difficulty arises in practice of (i), stop the low-speed dynamic balancing and balance the rotor system by n -plane-method at high speed (s). If the rotor system is balanced on their own bearings, it may be of no use to obtain the universal state of balance, so that n -plane-balancing will be sufficient for the present purpose. It would be inevitable that residual dynamic forces acting on bearings are inferior to those obtained through $(n+2)$ -plane-method.

(3) The total energy of the system under free

vibration is considered as the functional, i.e.

$$I(y) = \frac{1}{2} \int_0^l [EI(y'')^2 + \mu \dot{y}^2] dz + \frac{1}{2} K_A y_A^2 + \frac{1}{2} K_B y_B^2 \quad \dots\dots\dots (i)$$

where the stiffnesses of both end bearings are dealt with. Stationary function of Eq. (i) is assumed as

$$\left. \begin{aligned} y &= y(z) e^{j\omega t} \\ y(z) &= a + b(z/l) + \sum_{j=1}^{\infty} c_j \phi_j(z) \end{aligned} \right\} \quad \dots\dots\dots (ii)$$

and the unknown coefficients a , b and c_j are to be determined. Since a and b are expressed by c_j as seen in boundary conditions, Eqs. (13) and (14), it is sufficient to determine only $c_j (j=1, 2, \dots)$. The coefficients c_j shall be of such values that make $I(y)$ as the extremum, which leads us to the condition of

$$\partial I(y) / \partial c_j = 0, \quad (j=1, 2, \dots)$$

Substituting Eq. (ii) into Eq. (i) and calculating gives

$$\begin{aligned} \partial I / \partial c_j &= \int_0^l \left[EI y'' \frac{\partial y''}{\partial c_j} - \mu \omega^2 y \frac{\partial y}{\partial c_j} \right] dz \\ &+ K_A y_A (\partial y / \partial c_j)_A + K_B y_B (\partial y / \partial c_j)_B \quad \dots\dots\dots (iii) \end{aligned}$$

here y in this expression means $y(z)$.

On the other hand, the following relations are readily seen in Eq. (ii):

$$\begin{aligned} \partial y'' / \partial c_j &= \phi_j''(z), \quad \partial y / \partial c_j = \frac{\partial a}{\partial c_j} + \frac{z}{l} \frac{\partial b}{\partial c_j} + \phi_j(z) \\ (\partial y / \partial c_j)_A &= \partial a / \partial c_j, \quad (\partial y / \partial c_j)_B = \frac{\partial a}{\partial c_j} + \frac{\partial b}{\partial c_j} \end{aligned}$$

Substituting them into Eq. (iii) and re-arranging gives

$$\begin{aligned} \partial I(y) / \partial c_j &= \int_0^l [EI y'' \phi_j'' - \mu \omega^2 y \phi_j] dz \\ &+ \left[-\omega^2 \int_0^l \mu y dz + K_A y_A + K_B y_B \right] \partial a / \partial c_j \\ &+ \frac{1}{l} \left[-\omega^2 \int_0^l \mu y z dz + l K_B y_B \right] \partial b / \partial c_j \quad \dots\dots\dots (iv) \end{aligned}$$

In this right hand side of Eq. (iv), the factors multiplied to $\partial a / \partial c_j$ and $\partial b / \partial c_j$ in the second and third terms illustrate the total force of the system and the total moment of the forces about the point A, respectively. The system being under free vibration, the total force and the total moment given here are in equilibrium, i.e. they are equal to zero. Next, partially integrating the first term of RHS of Eq. (iv) with respect to $EI y'' \phi_j''$ and considering that ϕ_j as well as ϕ_j'' is zero at both ends $z=0$ and l , then

$$\begin{aligned} \partial I(y) / \partial c_j &= \int_0^l [(EI y'')' - \mu \omega^2 y] \phi_j dz = 0 \\ (j &= 1, 2, \dots) \end{aligned}$$

is derived, which coincides with Eq. (15).

(4) A uniform and symmetrical rotor/bearing system is to be balanced up to its second mode for an assumed distribution of initial unbalance $u(z)=u_0 \sin(2\pi z/l)$. Four correction planes are needed and

Append.-Fig. 1 4 corrections for balancing up to second mode

arranged symmetrically at axial stations of $z_1=0$, $z_2=\xi l$, $z_3=(1-\xi)l$ and $z_4=l$. Characteristic functions are approximated by neglecting the terms higher than the third order as

$$\phi_1(z) = 1 + \gamma_1 \sin(\pi z/l)$$

$$\phi_2(z) = (1 - 2z/l) + \gamma_2 \sin(2\pi z/l)$$

γ_1 and γ_2 in these functions are the dependents of bearing stiffness as shown in Eqs. (21) and (27). Substituting them into the conditional equation (9-a) gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \xi & 1-\xi & 1 \\ 1 & 1+\gamma_1 \sin \pi \xi & 1+\gamma_1 \sin \pi \xi & 1 \\ 1 & 1-2\xi+\gamma_2 \sin 2\pi \xi & -(1-2\xi+\gamma_2 \sin 2\pi \xi) & -1 \end{pmatrix} \times \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = -u_0 l \begin{pmatrix} 0 \\ -(1/2\pi) \\ 0 \\ (1/\pi) + (\gamma_2/2) \end{pmatrix}$$

Solving this, four correction unbalances are obtained as

$$\left. \begin{aligned} U_1 &= \frac{u_0 l}{4 \sin 2\pi \xi} \left\{ (1-2\xi) - \frac{2}{\pi} \sin 2\pi \xi \right\} \\ U_2 &= -\frac{u_0 l}{4 \sin 2\pi \xi} \\ U_3 &= -U_2 \\ U_4 &= -U_1 \end{aligned} \right\}$$

see Append.-Fig. 1

As obviously seen, these corrections contain neither

γ_1 nor γ_2 in them, so they are valid irrespective of bearing stiffness.

(5) The universal state of balance no more disturbed by the variation of bearing stiffness K for whole range of $0 \leq K \leq \infty$ is dealt with in the paper, where a rotor shall be balanced for the mode ϕ_n to the order of $n=1, 2, \dots, h$. Your question suggests a special case where K does not exceed certain finite amount, say K_a , i.e. the case of 'practically universal balance' in the region of $0 \leq K \leq K_a$. The characteristic function ϕ_n has the form of Eq. (19) also for $K \leq K_a$, but some higher terms of ϕ_n may be negligible. Consider, for example, the highest term $c_{hh}\phi_h$ appearing in RHS of Eq. (19) for $j=n=h$. The factor c_{hh} becomes very small if K_a is small, because $(\omega_h/\sigma_h)^2$ in Eq. (17'a) is fairly smaller than unity in this case

(see Fig. 6). Hence, the h -th modal balance $\int_0^l \bar{u}(z) \times \phi_h dz = 0$ is in fact satisfied without the condition of excess balance of $\int_0^l \bar{u}(z) \phi_h dz = 0$ if c_{hh} is nearly zero.

To what degree should j of ϕ_j in RHS of Eq. (32) be taken up for a given value of K_a to get the 'practically' universal balance can be determined by considering the tolerable error of balance, the measure of which will be given by the magnitude of c_{nj} in Eq. (17'a).

(6) I quite agree with your opinion. In the theory of balancing a flexible rotor, only the relation of the sequence of concepts of $\bar{u}(z) \rightarrow \bar{e}_n \rightarrow r_n$ is important and has meaning, so that the concept of $\bar{u}_n$ (the modal component of unbalance) is of no use in the analysis. Every expression on $\bar{u}_n$ shall be deleted out of the part of the previous paper.

(7) As far as we know, no report is given on the field of rotating machines. As examples of laboratory scale, however, brief reports are available (see following literatures^(*)(^{*}2)). A detailed report on the series of experiments of balancing a flexible rotor with several disks is to be published in the near future.

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