539.376 : 539.384.6

Transient Creep of Shells of Revolution*

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This paper is concerned with the transient creep problems of shells of revolution under an axisymmetric load. The mathematical theory of creep as it applies to shells of revolution is developed for two hardening laws—the time and the strain hardening laws, and the basic differential equations derived are numerically solved by finite difference method. As a numerical example the torispherical head of a pressure vessel which is composed of a spherical, a toroidal and a cylindrical shell is analyzed with the computer program developed for two hardening laws.

1. Introduction

A survey of the literature reveals that the numerical analyses of creep problems that are related to shell analysis primarily were concerned with the steady creep of cylindrical shells⁽¹⁾⁽²⁾ and conical shells⁽³⁾ and next concerned with the transient creep of spherical shells⁽⁴⁾ and cylindrical shells⁽⁵⁾. While these efforts were useful in developing an understanding of the creep process in shells, they did not offer much to the designer of multicomponent shells undergoing the usual process of transient creep.

Therefore in this paper we treated transient creep problems of shells of revolution. We derived the basic differential equations of shells of revolution in transient creep and numerically solved by transforming them into difference equations and applying an elimination method. We analyzed the creep problem for two hardening theories—the time hardening and the strain hardening.

2. Notations

A, m, n : material constants
 C : tensile stiffness $= Eh$
 D : bending stiffness $= Eh^3/12(1-\nu^2)$
 E : Young's modulus
 H, V : radial and axial components of internal force per unit length, respectively
 h : thickness of shell
 M : bending moment per unit length
 N : normal force per unit length
 p : distributed load per unit area
 p_H, p_V, p_n, p_φ : radial, axial, normal and meridional components of p , respectively

Q : shearing force per unit length
 r, θ, z : cylindrical coordinates of middle surface
 s : meridional length along middle surface from boundary
 $t, \Delta t$: time and increment of time, respectively
 u, w : radial and axial components of displacement of middle surface, respectively
 δ : interval of mesh point
 ϵ : strain
 η : distance from middle surface
 φ : increment of φ due to deformation
 κ : change of curvature
 ν : Poisson's ratio
 σ : stress
 φ : angle between axis of symmetry and normal to middle surface

A, B, C, D, F : coefficients by shape of shell, and
 a, b, c, d, f : external load or creep strain rates
 Subscripts

c : refers to creep components
 e : refers to effective stress and strain
 j : refers to j -th mesh point
 m : refers to middle surface of shell
 θ, φ : refer to circumferential and meridional directions, respectively

Superscripts

' : differentiation with respect to s
 $\dot{}$: differentiation with respect to t

3. Creep theory of shells

In this section a mathematical theory of transient creep for two hardening laws—the time hardening and the strain hardening, as it applies to shells of revolution is developed.

The parametric expressions of a middle surface of shells of revolution may be written as

$$r=r(s), \quad z=z(s) \dots \dots \dots (1)$$

where s is the meridional length along the middle

* Received 30th July, 1970.

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surface from a boundary, and r and z are the co-ordinates of middle surface, as shown in Fig. 1. The relations between the co-ordinates r , z and the angle φ in Fig. 1 are as follows.

$$r' = \cos \varphi, \quad z' = \sin \varphi, \quad (r')^2 + (z')^2 = 1 \dots\dots (2)$$

where the prime denotes the differentiation with respect to s .

The relations among the internal forces per unit length, N_φ , Q , H and V are given by

$$\dot{N}_\varphi = r' \dot{H} + z' \dot{V}, \quad \dot{Q} = -z' \dot{H} + r' \dot{V} \dots\dots\dots (3)$$

where the dot denotes the differentiation with respect to time.

The equations of equilibrium are given by

$$\left. \begin{aligned} (r\dot{V})' + r\dot{p}_V &= 0, & (r\dot{H})' - \dot{N}_\theta + r\dot{p}_H &= 0 \\ (r\dot{M}_\varphi)' - r'\dot{M}_\theta - r\dot{Q} &= 0 \end{aligned} \right\} \dots\dots\dots (4)$$

where M_φ and M_θ are the meridional and circumferential bending moments per unit length, respectively, and p_H and p_V are the components of the distributed load per unit area of the middle surface, as shown in Fig. 1.

Denoting the radial component of a displacement by u and the change of the angle φ by ϑ , the time derivatives of the two strains and the two changes of curvature of a middle surface are expressed by

$$\left. \begin{aligned} \dot{\epsilon}_{\varphi m} &= \frac{\dot{u} + z'\dot{\vartheta}}{r'}, & \dot{\epsilon}_{\theta m} &= \frac{\dot{u}}{r} \\ \dot{\kappa}_\varphi &= \dot{\vartheta}', & \dot{\kappa}_\theta &= \frac{r'\dot{\vartheta}}{r} \end{aligned} \right\} \dots\dots\dots (5)$$

The strain rate components of the surface at the distance of η from the middle surface as shown in Fig. 1, $\dot{\epsilon}_\varphi$ and $\dot{\epsilon}_\theta$, are given by next equations.

$$\dot{\epsilon}_\varphi = \dot{\epsilon}_{\varphi m} - \eta \dot{\kappa}_\varphi, \quad \dot{\epsilon}_\theta = \dot{\epsilon}_{\theta m} - \eta \dot{\kappa}_\theta \dots\dots\dots (6)$$

In the theory of creep it is assumed that in a given increment of time the total strain increments are composed of an elastic part and a part due to creep. Since the elastic strain rates are directly proportional to the stress rates by Hooke's law, the

total strain rates may be expressed as follows.

$$\dot{\epsilon}_\varphi = \frac{1}{E}(\dot{\sigma}_\varphi - \nu \dot{\sigma}_\theta) + \dot{\epsilon}_{\varphi c}, \quad \dot{\epsilon}_\theta = \frac{1}{E}(\dot{\sigma}_\theta - \nu \dot{\sigma}_\varphi) + \dot{\epsilon}_{\theta c} \dots\dots\dots (7)$$

where $\dot{\epsilon}_{\varphi c}$ and $\dot{\epsilon}_{\theta c}$ are the creep strain rates, and E and ν are Young's modulus and Poisson's ratio, respectively.

Now, if it is assumed that in the transient creep the creep strain in uni-axial state for constant stress may be given by

$$\epsilon_c = A \sigma^n t^m \dots\dots\dots (8)^{(6)}$$

then Eq. (8) can be differentiated with respect to time to get expressions for the creep strain rate. In Eq. (8) A , n and m are obtained from a fit of uniaxial test data. If Eq. (8) is differentiated directly, the so-called time hardening creep law

$$\dot{\epsilon}_c = m A \sigma^n t^{m-1} \dots\dots\dots (9)$$

is obtained. If, on the other hand, the m -th root of Eq. (8) is taken before Eq. (8) is differentiated with respect to time, then the so-called strain hardening creep law

$$\dot{\epsilon}_c = m A^{1/m} \sigma^{n/m} \epsilon_c^{(m-1)/m} \dots\dots\dots (10)$$

is obtained.

If Eqs. (9) and (10) are extended to the plane stress state assumed in the ordinary shell theory under the assumptions that the solid is isotropic, that the material obeys the Von Mises flow rule and that the creep strains are incompressible, then the meridional and the circumferential components of creep strain rate, $\dot{\epsilon}_{\varphi c}$ and $\dot{\epsilon}_{\theta c}$, may be written as follows for the time hardening and the strain hardening theories.

$$\left. \begin{aligned} \dot{\epsilon}_{\varphi c} &= m A \sigma_s^{n-1} t^{m-1} \left(\sigma_\varphi - \frac{1}{2} \sigma_\theta \right) \\ \dot{\epsilon}_{\theta c} &= m A \sigma_s^{n-1} t^{m-1} \left(\sigma_\theta - \frac{1}{2} \sigma_\varphi \right) \end{aligned} \right\} \dots\dots\dots (11)$$

$$\left. \begin{aligned} \dot{\epsilon}_{\varphi c} &= m A^{1/m} \sigma_s^{(n-m)/m} \epsilon_{sc}^{(m-1)/m} \left(\sigma_\varphi - \frac{1}{2} \sigma_\theta \right) \\ \dot{\epsilon}_{\theta c} &= m A^{1/m} \sigma_s^{(n-m)/m} \epsilon_{sc}^{(m-1)/m} \left(\sigma_\theta - \frac{1}{2} \sigma_\varphi \right) \end{aligned} \right\} \dots\dots\dots (12)$$

where

$$\left. \begin{aligned} \sigma_s &= \sqrt{\sigma_\varphi^2 - \sigma_\varphi \sigma_\theta + \sigma_\theta^2} \\ \epsilon_{sc} &= \sqrt{\frac{4}{3} (\epsilon_{\varphi c}^2 + \epsilon_{\varphi c} \epsilon_{\theta c} + \epsilon_{\theta c}^2)} \end{aligned} \right\} \dots\dots\dots (13)$$

Next, by solving Eqs. (7) about $\dot{\sigma}_\varphi$ and $\dot{\sigma}_\theta$ and substituting Eqs. (6) into these, $\dot{\sigma}_\varphi$ and $\dot{\sigma}_\theta$ are expressed as

$$\left. \begin{aligned} \dot{\sigma}_\varphi &= \frac{E}{1-\nu^2} \{ \dot{\epsilon}_{\varphi m} + \nu \dot{\epsilon}_{\theta m} - \eta (\dot{\kappa}_\varphi + \nu \dot{\kappa}_\theta) - (\dot{\epsilon}_{\varphi c} + \nu \dot{\epsilon}_{\theta c}) \} \\ \dot{\sigma}_\theta &= \frac{E}{1-\nu^2} \{ \dot{\epsilon}_{\theta m} + \nu \dot{\epsilon}_{\varphi m} - \eta (\dot{\kappa}_\theta + \nu \dot{\kappa}_\varphi) - (\dot{\epsilon}_{\theta c} + \nu \dot{\epsilon}_{\varphi c}) \} \end{aligned} \right\} \dots\dots\dots (14)$$

By use of Eqs. (14) the time derivatives of the normal forces N_φ and N_θ and the moments M_φ and M_θ are obtained as follows.

Fig. 1 Notations of shells of revolution

$$\dot{N}_\varphi = \frac{C}{1-\nu^2} (\dot{\epsilon}_{\varphi m} + \nu \dot{\epsilon}_{\theta m}) - \dot{N}_{\varphi c}, \quad \dot{N}_\theta = \frac{C}{1-\nu^2} (\dot{\epsilon}_{\theta m} + \nu \dot{\epsilon}_{\varphi m}) - \dot{N}_{\theta c} \quad (15)$$

$$\dot{M}_\varphi = -D(\dot{\kappa}_\varphi + \nu \dot{\kappa}_\theta) - \dot{M}_{\varphi c}, \quad \dot{M}_\theta = -D(\dot{\kappa}_\theta + \nu \dot{\kappa}_\varphi) - \dot{M}_{\theta c} \quad (16)$$

where

$$\left. \begin{aligned} \dot{N}_{\varphi c} &= \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} (\dot{\epsilon}_{\varphi c} + \nu \dot{\epsilon}_{\theta c}) d\eta, & \dot{N}_{\theta c} &= \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} (\dot{\epsilon}_{\theta c} + \nu \dot{\epsilon}_{\varphi c}) d\eta \\ \dot{M}_{\varphi c} &= \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} (\dot{\epsilon}_{\varphi c} + \nu \dot{\epsilon}_{\theta c}) \eta d\eta, & \dot{M}_{\theta c} &= \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} (\dot{\epsilon}_{\theta c} + \nu \dot{\epsilon}_{\varphi c}) \eta d\eta \end{aligned} \right\} \quad (17)$$

and

$$C = Eh, \quad D = Eh^3/12(1-\nu^2) \quad (18)$$

Also in the above equations h is the thickness of a shell. Equations (15) may be transformed to the next equations.

$$\left. \begin{aligned} \dot{\epsilon}_{\varphi m} &= \frac{1}{C} \{ \dot{N}_\varphi - \nu \dot{N}_\theta + (\dot{N}_{\varphi c} - \nu \dot{N}_{\theta c}) \} \\ \dot{\epsilon}_{\theta m} &= \frac{1}{C} \{ \dot{N}_\theta - \nu \dot{N}_\varphi + (\dot{N}_{\theta c} - \nu \dot{N}_{\varphi c}) \} \end{aligned} \right\} \quad (19)$$

On the other hand, elimination of u from Eqs. (5) gives

$$(r\dot{\epsilon}_{\theta m})' - r'\dot{\epsilon}_{\varphi m} = -z'\dot{\varphi} \quad (20)$$

Now, from Eqs. (3) to (5) and Eqs. (16) to (20) the following simultaneous differential equations of second order may be derived.

$$\left. \begin{aligned} \dot{\varphi}'' + \frac{(Dr)'}{Dr} \dot{\varphi}' - \left\{ \left(\frac{r'}{r} \right)^2 - \nu \frac{(Dr)'}{Dr} \right\} \dot{\varphi} - \frac{z'}{rD} (r\dot{H}) &= -\frac{r'}{rD} (r\dot{V}) + \frac{r'\dot{M}_{\theta c}}{rD} - \frac{(r\dot{M}_{\varphi c})'}{rD} \\ (r\dot{H})'' + \frac{(r/C)'}{r/C} (r\dot{H})' - \left\{ \left(\frac{r'}{r} \right)^2 + \nu \frac{(r/C)'}{r/C} \right\} (r\dot{H}) + \frac{z'}{r/C} \dot{\varphi} &= \left\{ \frac{z'r'}{r^2} \right. \\ &+ \nu \frac{(z'/C)'}{r/C} \left. \right\} (r\dot{V}) + \nu \frac{z'}{r} (r\dot{V})' - \left\{ \frac{(r/C)'}{r/C} + \nu \frac{r'}{r} \right\} (r\dot{p}_H) - (r\dot{p}_H)' \\ &+ \left\{ \frac{r'}{r} + \nu \frac{(r/C)'}{r/C} \right\} \dot{N}_{\varphi c} + \nu \dot{N}_{\varphi c}' - \left\{ \nu \frac{r'}{r} + \frac{(r/C)'}{r/C} \right\} \dot{N}_{\theta c} - \dot{N}_{\theta c}' \end{aligned} \right\} \quad (21)$$

Equations (21) are the fundamental equations of shells of revolution for creep problems. The time derivatives of the components of internal forces and displacements are related to the solutions $\dot{\varphi}$ and $r\dot{H}$ in Eqs. (21) by the following equations.

$$\left. \begin{aligned} \dot{N}_\varphi &= r'\dot{H} + z'\dot{V}, & \dot{N}_\theta &= (r\dot{H})' + r\dot{p}_H, & \dot{Q} &= -z'\dot{H} + r'\dot{V} \\ \dot{M}_\varphi &= -D \left(\dot{\varphi}' + \nu \frac{r'}{r} \dot{\varphi} \right) - \dot{M}_{\varphi c}, & \dot{M}_\theta &= -D \left(\frac{r'}{r} \dot{\varphi} + \nu \dot{\varphi}' \right) - \dot{M}_{\theta c} \\ \dot{u} &= \frac{r}{C} \{ \dot{N}_\theta - \nu \dot{N}_\varphi + (\dot{N}_{\theta c} - \nu \dot{N}_{\varphi c}) \}, & \dot{w}' &= \frac{z'}{C} \{ \dot{N}_\varphi - \nu \dot{N}_\theta + (\dot{N}_{\varphi c} - \nu \dot{N}_{\theta c}) \} + r'\dot{\varphi} \end{aligned} \right\} \quad (22)$$

If in Eqs. (21) and (22) the dots are ignored and the terms of creep neglected, they become the elastic equations of shells of revolution derived by E. Reissner⁽⁷⁾.

4. Numerical procedure

In converting Eqs. (21) and (22) into difference equations, the approximations

$$\left. \begin{aligned} \left(\frac{d\Phi}{ds} \right)_j &= \frac{\Phi_{j+1} - \Phi_{j-1}}{2\delta} \\ \left(\frac{d^2\Phi}{ds^2} \right)_j &= \frac{\Phi_{j+1} - 2\Phi_j + \Phi_{j-1}}{\delta^2} \end{aligned} \right\} \quad (23)$$

are used for every mesh point except the boundary points of the shell, where the subscript j refers to the j -th mesh point. And for the boundary points ($j=1$ and N)

$$\left. \begin{aligned} \left(\frac{d\Phi}{ds} \right)_1 &= \frac{-\Phi_3 + 4\Phi_2 - 3\Phi_1}{2\delta} \\ \left(\frac{d\Phi}{ds} \right)_N &= \frac{3\Phi_N - 4\Phi_{N-1} + \Phi_{N-2}}{2\delta} \end{aligned} \right\} \quad (24)$$

are used. The mesh points are taken at equal distance δ in the direction of s on the middle surface of the shell.

By using Eqs. (23), Eqs. (21) and (22) are transformed into the following equations.

$$\begin{aligned}
& \left\{ 1 - \frac{D_{j+1}/D_j + (r_{j+1} - r_{j-1})/r_j - D_{j-1}/D_j}{4} \right\} \dot{\varphi}_{j-1} + \left\{ -2 - \frac{(r_{j+1} - r_{j-1})^2}{4r_j^2} \right. \\
& \quad + \nu \frac{(r_{j+1} - r_{j-1})D_{j+1} + 4(r_{j+1} - 2r_j + r_{j-1})D_j - (r_{j+1} - r_{j-1})D_{j-1}}{4D_j r_j} \left. \right\} \dot{\varphi}_j \\
& \quad + \left\{ 1 + \frac{D_{j+1}/D_j + (r_{j+1} - r_{j-1})/r_j - D_{j-1}/D_j}{4} \right\} \dot{\varphi}_{j+1} - \frac{(z_{j+1} - z_{j-1})\delta \dot{H}_j}{2D_j} \\
& = -\frac{(r_{j+1} - r_{j-1})\delta}{2D_j} \dot{V}_j + \left\{ \frac{r_{j+1} - r_{j-1}}{2r_j D_j} (\dot{M}_{\theta c j} - \dot{M}_{\varphi c j}) - \frac{\dot{M}_{\varphi c j+1} - \dot{M}_{\varphi c j-1}}{2D_j} \right\} \delta \\
& \quad \left\{ r_j - \frac{r_{j+1} - r_{j-1}}{2} - \frac{(r_{j+1} - r_{j-1})C_j - (C_{j+1} - C_{j-1})r_j}{4C_j} \right\} \dot{H}_{j-1} \\
& \quad + \left\{ r_{j+1} - 4r_j + r_{j-1} + \frac{(r_{j+1} - r_{j-1})^2 C_j - (C_{j+1} - C_{j-1})(r_{j+1} - r_{j-1})r_j}{4r_j C_j} \right. \\
& \quad \left. - \frac{(r_{j+1} - r_{j-1})^2}{4r_j} - \nu \frac{4(r_{j+1} - 2r_j + r_{j-1})C_j - (r_{j+1} - r_{j-1})(C_{j+1} - C_{j-1})}{4C_j} \right\} \dot{H}_j \\
& \quad + \left\{ r_j + \frac{r_{j+1} - r_{j-1}}{2} + \frac{(r_{j+1} - r_{j-1})C_j - (C_{j+1} - C_{j-1})r_j}{4C_j} \right\} \dot{H}_{j+1} + \frac{(z_{j+1} - z_{j-1})\delta C_j}{2r_j} \dot{\varphi}_j \\
& = \left\{ (1 + \nu) \frac{(r_{j+1} - r_{j-1})(z_{j+1} - z_{j-1})}{4r_j} + \nu(z_{j+1} - 2z_j + z_{j-1}) - \nu \frac{(z_{j+1} - z_{j-1})(C_{j+1} - C_{j-1})}{4C_j} \right\} \dot{V}_j \\
& \quad + \nu \frac{(z_{j+1} - z_{j-1})(\dot{V}_{j+1} - \dot{V}_{j-1})}{4} - \left\{ (2 + \nu) \frac{(r_{j+1} - r_{j-1})\delta}{2} - \frac{r_j(C_{j+1} - C_{j-1})\delta}{2C_j} \right\} \dot{p}_H \\
& \quad - \frac{r_j(\dot{p}_{Hj+1} - \dot{p}_{Hj-1})\delta}{2} + \left\{ (1 + \nu) \frac{r_{j+1} - r_{j-1}}{2r_j} - \nu \frac{C_{j+1} - C_{j-1}}{2C_j} \right\} \delta \dot{N}_{\varphi c j} \\
& \quad - \left\{ (1 + \nu) \frac{r_{j+1} - r_{j-1}}{2r_j} - \frac{C_{j+1} - C_{j-1}}{2C_j} \right\} \delta \dot{N}_{\theta c j} + \left\{ \nu \frac{\dot{N}_{\varphi c j+1} - \dot{N}_{\varphi c j-1}}{2} - \frac{\dot{N}_{\theta c j+1} - \dot{N}_{\theta c j-1}}{2} \right\} \delta
\end{aligned} \tag{25}$$

$$\begin{aligned}
& \dot{N}_{\varphi j} = \frac{r_{j+1} - r_{j-1}}{2\delta} \dot{H}_j + \frac{z_{j+1} - z_{j-1}}{2\delta} \dot{V}_j, \quad \dot{N}_{\theta j} = \frac{r_{j+1} - r_{j-1}}{2\delta} \dot{H}_j + \frac{\dot{H}_{j+1} - \dot{H}_{j-1}}{2\delta} r_j + r_j \dot{p}_{Hj} \\
& \dot{Q}_j = -\frac{z_{j+1} - z_{j-1}}{2\delta} \dot{H}_j + \frac{r_{j+1} - r_{j-1}}{2\delta} \dot{V}_j, \quad \dot{M}_{\varphi j} = -D_j \left\{ \frac{\dot{\varphi}_{j+1} - \dot{\varphi}_{j-1}}{2\delta} + \nu \frac{r_{j+1} - r_{j-1}}{2r_j \delta} \dot{\varphi}_j \right\} - \dot{M}_{\varphi c j} \\
& \dot{M}_{\theta j} = -D_j \left\{ \frac{r_{j+1} - r_{j-1}}{2r_j \delta} \dot{\varphi}_j + \nu \frac{\dot{\varphi}_{j+1} - \dot{\varphi}_{j-1}}{2\delta} \right\} - \dot{M}_{\theta c j}, \quad \dot{u}_j = \frac{r_j}{C_j} \{ \dot{N}_{\theta j} - \nu \dot{N}_{\varphi j} + (\dot{N}_{\theta c j} - \nu \dot{N}_{\varphi c j}) \} \\
& \dot{w}'_j = \frac{z_{j+1} - z_{j-1}}{2C_j \delta} \{ \dot{N}_{\varphi j} - \nu \dot{N}_{\theta j} + (\dot{N}_{\varphi c j} - \nu \dot{N}_{\theta c j}) \} + \frac{r_{j+1} - r_{j-1}}{2\delta} \dot{\varphi}_j
\end{aligned} \tag{26}$$

Equations (25) are summarized into

$$\begin{aligned}
& \mathbf{A}_j \dot{\varphi}_{j-1} + \mathbf{B}_j \dot{\varphi}_j + \mathbf{C}_j \dot{\varphi}_{j+1} + \mathbf{D}_j \dot{H}_j = \mathbf{F}_j \\
& \mathbf{a}_j \dot{H}_{j-1} + \mathbf{b}_j \dot{H}_j + \mathbf{c}_j \dot{H}_{j+1} + \mathbf{d}_j \dot{\varphi}_j = \mathbf{f}_j
\end{aligned} \tag{27}$$

The coefficients in Eqs. (27), A_j, B_j , etc., are calculated from the shape of the shell and the value of external loads or creep strain rates at the j -th mesh point. Equations (27) represent a pair of simultaneous equations for every mesh point along the shell except the boundary points. Therefore, Eqs. (27) and the equations of boundary conditions constitute simultaneous equations with $2N$ unknowns, $\dot{\varphi}_j$ and $\dot{H}_j$ for $j=1, 2, \dots, N$. Since these equations usually have many unknowns, it is more advantageous than solving them simultaneously to solve by employing the procedures which have been proposed by W. K. Sepetoski and others⁽⁶⁾. Once the values of $\dot{\varphi}_j$ and $\dot{H}_j$ for every point are obtained, the components of internal force and displacements for every point are calculated from Eqs. (26).

Now, the above numerical method is concerned with the handling of the spatial aspect of the problem. However, in transient creep the temporal aspect also must be considered. The latter is usually handled with an incremental procedure. Thus, the numerical

solution in a transient creep of shells of revolution proceeds as follows.

Step 1: Determine the elastic solution in the shell by removing the dots and ignoring the terms of creep in Eqs. (25)~(27).

Step 2: Assume that the stresses obtained in Step 1 remain constant over a very small initial increment of time Δt_0 and calculate the increments of creep strains that occur in the increment time Δt_0 by use of the time hardening law, Eq. (11). Next by substitution of the increments of creep strains into Eqs. (17) obtain the values of $\dot{N}_{\varphi c j}$, $\dot{N}_{\theta c j}$, $\dot{M}_{\varphi c j}$ and $\dot{M}_{\theta c j}$, and by use of Eqs. (25)~(27) calculate the stresses and displacements at $t = \Delta t_0$ for time hardening law. Equations (17) are integrated numerically by dividing the thickness h of the shell into ten and employing Simpson's 1/3 rule. Since in the case of using the strain hardening law, Eqs. (12), $\dot{\epsilon}_{\varphi c}$ and $\dot{\epsilon}_{\theta c}$ diverge because of $\epsilon_{sc} = 0$, in the initial routine Eqs. (12) are not employed. The validity of this procedure for strain hardening law may be examined by considering the state of convergency of stress distributions and displacements.

Step 3: Choose a suitable increment of time

Δt and calculate the increment of components of stresses and displacements. This involves the same numerical technique as in Step 2.

Step 4: Determine the stresses at the end of the interval Δt by adding the stress increments from Step 3 to the previous stresses.

Fig. 2 Torispherical head of pressure vessels

Fig. 3 Creep curve of material

Fig. 4 Deformations of pressure vessel head with time

Step 5: Add another increment and repeat Steps 2, 3 and 4. Continue in this manner until a desired time interval has been achieved or until a steady state has been reached, whichever occurs first.

Finally the boundary conditions are discussed. At each boundary, the following boundary conditions are considered.

(a) Clamped edge restrained in r direction:

$$\dot{\varphi} = 0, \quad \dot{u} = 0$$

(b) Clamped edge movable in r direction:

$$\dot{\varphi} = 0, \quad \dot{H} = 0$$

(c) Pinned support restrained in r direction:

$$\dot{M}_\varphi = 0, \quad \dot{u} = 0$$

(d) Pinned support movable in r direction:

$$\dot{M}_\varphi = 0, \quad \dot{H} = 0$$

(e) M_φ, H specified:

$$\dot{M}_\varphi = \dot{M}_\varphi, \quad \dot{H} = \dot{H}$$

(f) φ, u specified:

$$\dot{\varphi} = \dot{\varphi}, \quad \dot{u} = \dot{u}$$

In addition to above conditions (a)~(f), there exists a case when the shell is closed at the initial boundary. Then one of the boundary conditions becomes $\dot{\varphi} = 0$ at the vertex. Another condition may be obtained directly from the second of the basic equations (21). For simplicity assuming that the vertex is closed flatly and no concentrated force acts on the vertex, the second of Eqs. (21) is reduced to

$$\dot{H}' = 0 \quad \dots \dots \dots (28)$$

5. Numerical results and discussions

In this section as a numerical example, the problem of a torispherical pressure vessel head which is composed of a spherical, a toroidal and a cylindrical shell as shown in Fig. 2, is treated.

It is assumed that the characteristic constants of the material of the torispherical head are

$$\left. \begin{aligned} \nu &= 0.3, \quad E = 18400 \text{ kg/mm}^2 \\ A &= 4.36 \times 10^{-7} (\text{kg/mm}^2)^{-3.1} (h)^{-0.218} \\ &\times (h)^{-0.218} \\ m &= 0.218, \quad n = 3.1 \end{aligned} \right\} \quad \dots \dots \dots (29)$$

Figure 3 shows the creep curve of material which has the characteristic constants as indicated by Eqs. (29).

The boundary conditions at point A ($j=1$) are

$$\dot{\varphi}_1 = 0, \quad \dot{H}_1' = 0 \quad \dots \dots \dots (30)$$

At point D far from point C ($z_1 = 1.25R_3$) the boundary conditions

$$\dot{\varphi}_N = 0, \quad \dot{H}_N = 0 \quad \dots \dots \dots (31)$$

are employed. The value of z_1

is evaluated from the solution⁽⁶⁾ of the elastic problem of a semi-infinite cylindrical shell on whose edge the moment or the shearing force acts and from the solution⁽⁶⁾ of the creep of a cylindrical shell under an internal pressure.

A hundred mesh points are taken at equal distance for AD in Fig. 2. It is found from the calculations for various values of N that the error in employing $N=100$ is less than 0.5 percent.

Some of the essential features of the solutions are shown in Figs. 4~11. The deformations of the pressure vessel head with time are given in Fig. 4. It is interesting to see that the spherical part is largely deformed outwards and the toroidal part inwards with the lapse of time. Figures 5~8 illustrate the spatial variation of bending moments and normal forces with time. From these figures it is found that the changes of distributions of the bending moments and normal force except N_φ are large in the toroidal part and in its vicinity, and they are not greatly changed from the elastic solutions in other parts. Figures 9 show the variation of stresses σ_φ and σ_θ through the thickness of the shell at the boundary between the spherical and the toroidal parts, and Figs. 10 show ones at point F of the toroidal part. In the neighbourhood of point F the internal forces except N_φ become maximum. Figure 11 shows the variation of the bending

moments and normal forces at point F with time. It is seen from this and in view of Figs. 5~10 that the stresses greatly change in the initial ten hours, and in about one hundred hours converge to constant values, that is, become ones in the steady creep state. In Figs. 5~10 the results from the strain hardening law are illustrated as the values in steady creep state. As shown in these figures the results obtained by the strain hardening law agree with ones by the time hardening law within a few percent. The computing time for the creep example is about thirty minutes by FACOM 230-60 in Data Processing center, Kyoto University.

6. Conclusions

In this paper a numerical analysis of shells of

(a) Meridional stress σ_φ

(b) Circumferential stress σ_θ

Fig. 9 Stress distributions through the thickness of the shell at the boundary between the spherical and the toroidal parts (point B)

(a) Meridional stress σ_φ

(b) Circumferential stress σ_θ

Fig. 10 Stress distributions through the thickness of the shell at point F

Fig. 11 Variation of internal force at point F with time

revolution under axi-symmetric load in the transient creep was presented. The basic differential equations were derived and numerically solved by the finite difference method. The computer programs for two hardening laws—the time and the strain hardening laws were developed. As a numerical example a torispherical pressure vessel head composed of a spherical, a toroidal and a cylindrical shell was analyzed with the computer programs. The analysis of shells in steady creep becomes a subcase of the more general transient creep process described here.

References

- (1) Cozzarelli, F.A., et al., *AIAA J.*, Vol. 3, No. 7 (1965-7), p. 1298.
- (2) Byrne, T.P. and Mackenzie, A.C., *Jour. Mech. Engng. Sci.*, Vol. 8, No. 2 (1966-6), p. 215.
- (3) Cozzarelli, F.A. and Vito, R.P., *AIAA J.*, Vol. 7, No. 3 (1969-3), p. 412.
- (4) Penny, R.K., *Int. Jour. Mech. Sci.*, Vol. 9, No. 6 (1967-6), p. 373.
- (5) Murakami, S. and Iwatsuki, S., *Int. Jour. Mech. Sci.*, Vol. 11, No. 11 (1969-11), p. 897.
- (6) Finnie, I. and Heller, W.R., *Creep of Engineering Materials*, (1959), McGraw-Hill.
- (7) Reissner, E., *Proc. 3rd U.S. Nat. Congr. Mech.*, (1958), p. 51.
- (8) Sepetoski, W.K., et al., *Trans. ASME, Ser. E*, Vol. 29, No. 4 (1962-12), p. 655.
- (9) Timoshenko, S.P. and Krieger, S.W., *Theory of Plates and Shells*, 2nd ed., (1959), p. 469, McGraw-Hill.