539.4.019.1 : 539.313 : 539.376

Viscous Property of Steel and Its Effect on Strain Wave Front*

By Kichinosuke TANAKA** and Tomoaki KUROKAWA***

When an axial step load is applied experimentally at one end of a semi-infinite bar, the stress wave front is not an exact step but is accompanied with some finite rising time. In this paper, the effect of the viscous property of the material on the shape of the wave front is considered.

To express the viscoelastic property of the material, a Voigt model is used for the analysis. The approximate solutions for the strain wave and the stress wave are obtained and it is clarified that the rising time is proportional to $\sqrt{\tau t}$, where t is the elapsed time after the impact and τ is the relaxation time.

The relaxation times for steel and stainless steel are also measured experimentally and it is revealed that the values of the relaxation time of steels are less than 10^{-8} sec.

1. Introduction

When one end of a bar is impacted, according to the one-dimensional theory of elasticity, a step wave travels in the bar. The wave front observed actually is not such an ideal step wave but is accompanied with a finite rising time. Several factors may be the cause of existence of this finite rising time. Substantial factors are radial inertia, which is neglected by one-dimensional elasticity, and internal viscosity of material itself. And as for the factors concerning the experimental technique, the following are considered: a given impact has not a complete stepped form and also the measurement by strain gauge may have some error. A solution including the radial inertia only was presented by Skalak⁽¹⁾ and a solution taken into account the internal viscosity was obtained by Morrison⁽²⁾. However, the latter gave a solution only for stress distribution and did not give one for strain waves which can be observed directly by experiment.

Although it is difficult to determine the internal viscosity of solid by means of one model, the most simple model which is not accompanied by plastic deformation, a so-called Voigt model (Fig. 1), is assumed in this paper. It is composed of parallel combination of spring and dashpot. Assuming this model, the solution for the strain wave which was produced by stepped impact and travelled along the bar was obtained. On the other hand, the viscosity

coefficient of steel was obtained experimentally with the aid of an assumptive Voigt model. The strain wave form theoretically obtained using the viscosity coefficient was compared with the observed wave form produced experimentally by the impact of steel bars. And the effect of internal viscosity on rising time of wave front was studied. In this paper, the measurement of viscosity coefficient is described at first, and the strain wave form is analyzed subsequently. The results are discussed in conclusion.

2. Measurement of viscosity coefficient

As for measurement of internal viscosity of solid, two methods are usually applied; one is to measure the damping of transverse vibration of a thin beam and the other is to measure the damping of a longitudinal wave propagating along the bar. The latter method was applied in our experiments. The experimental set-up is shown in Fig. 2. Test specimen I is held horizontally by strings approximately 4 meters long hung from the ceiling. The diameter d of test specimen is 17 mm and the length l is between 182~

Fig. 1

* Received 24th March, 1971.

** Professor, Faculty of Engineering, Kyoto University, Sakyo-ku, Kyoto.

*** Research Fellow, Central Research Laboratories, Sumitomo Metal Industries.

660 mm. Therefore, the aspect ratio l/d is above 10, and it can be considered that this problem can be well treated as one of a one-dimensional longitudinal wave. Impact bar II having the same material and diameter as the test specimen, is put on the rollers and set on the same axis of the test specimen. The length l' of the impact bar is approximately one-half of the length l . Upon the collision of the two bars, I and II, compression waves are produced in each bar, and after the elapsed time of $2l/c$ (c : velocity of longitudinal wave of test specimen) the two bars are separated. Longitudinal wave having a period of $2l/c$ is produced in the test specimen I, and the test specimen continues free vibration afterwards. Vibration of the test specimen is detected by means of two strain gauges pasted symmetrically on the surface of the center of the test specimen ($x=l/2$) and a damping curve is then displayed on the synchroscope through a bridge circuit. The impact velocity is 0.1 m/sec, and the magnitude of strain generated in the test specimen is 2×10^{-5} at the most.

Taking x -axis along the axis of the test specimen and denoting the displacement by u and the density by ρ , an equation of motion of the one-dimensional longitudinal wave is given as

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} \quad (1)$$

In the case of a Voigt model, stress σ has the following relation with strain $\epsilon = \partial u / \partial x$.

$$\sigma = E\epsilon + \eta \frac{\partial \epsilon}{\partial t} \quad (2)$$

where E =modulus of longitudinal elasticity,

η =viscosity coefficient.

Substituting Eq. (2) into Eq. (1), the following equation is obtained:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(1 + \tau \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial x^2} \quad (3)$$

where $c = \sqrt{E/\rho}$ (velocity of one-dimensional longitudinal wave),

$\tau = \eta/E$ (relaxation time).

The impact end and the other end of the specimen are expressed as $x=0$ and $x=l$ respectively, and the

Fig. 2

displacement is expressed by the following equation:

$$u = \sum_{n=0}^{\infty} \bar{u}_n \cos \lambda_n x + \sum_{n=1}^{\infty} \bar{u}_n \sin \lambda_n x \quad \left. \begin{array}{l} \lambda_n = \frac{n\pi}{l} \end{array} \right\} \dots\dots\dots (6)$$

Substituting Eq. (6) into Eq. (3),

$$\sum_{n=0}^{\infty} (\ddot{\bar{u}}_n + \lambda_n^2 c^2 \tau \dot{\bar{u}}_n + \lambda_n^2 c^2 \bar{u}_n) \cos \lambda_n x + \sum_{n=1}^{\infty} (\ddot{\bar{u}}_n + \lambda_n^2 c^2 \tau \dot{\bar{u}}_n + \lambda_n^2 c^2 \bar{u}_n) \sin \lambda_n x = 0 \dots\dots\dots (7)$$

is obtained. Taking each term enclosed with brackets equal to zero, and substituting $\bar{u}_n = U e^{p t}$ and $\bar{u}_n = \bar{U} e^{p t}$ into each equation, we obtain

$$p = -\frac{\lambda_n^2 c^2 \tau}{2} \pm i \lambda_n c \sqrt{1 - \left(\frac{\lambda_n c \tau}{2} \right)^2} \dots\dots\dots (8)$$

Equation (8) can be approximated as Eq. (8)' within the range of $\lambda_n c \tau / 2 \ll 1$

$$p \approx -\frac{\lambda_n^2 c^2 \tau}{2} \pm i \lambda_n c \dots\dots\dots (8)'$$

Using Eq. (8)', strain wave is expressed finally as follows.

$$\epsilon = \frac{\partial u}{\partial x} = - \sum_{n=1}^{\infty} \exp \left\{ - \left(\lambda_n^2 c^2 \tau / 2 \right) t \right\} \left\{ (v_n \sin \lambda_n c t + w_n \cos \lambda_n c t) \sin \lambda_n x - (v_n' \sin \lambda_n c t + w_n' \cos \lambda_n c t) \cos \lambda_n x \right\} \dots\dots\dots (9)$$

This equation means that a wave of n th mode damps as an exponential function of the time t with a different attenuation constant

$$\lambda_n^2 c^2 \tau / 2 \dots\dots\dots (10)$$

Wave form which travels along the test specimen resembles more a rectangular wave than a sinusoidal one. Figure 3 shows an example of wave form observed at $x=l/2$, ($l=660$ mm in this case), after the incident wave reflected several times along the test specimen. The abscissa represents time and one division corresponds to 50 μ sec. Fourier's sinusoidal coefficients calculated for this wave form are shown graphically in Fig. 4. If we neglect the damping within the measured interval, the coefficients correspond to terms

$$v_n \sin \lambda_n c t + w_n \cos \lambda_n c t$$

in Eq. (9). It can be seen from this figure that the

Fig. 3

values of v_n of the high modes are very much smaller than v_1 . Furthermore, as it is found immediately from the exponential term in Eq. (9), waves of high modes damp quickly with the time. Therefore, the damping curve represents finally the damping curve for the wave of the first mode. Figure 5 shows a damping curve obtained from the same specimen as in Fig. 3. (Figure 5 is equivalent to Fig. 3 whose axis of time is shortened.) Figure 6 shows the relation between the magnitude of the envelope of the damping curve ε and the time. The ordinate is $\varepsilon/\varepsilon_0$ (where ε_0 is the value at $t=0$). It is found from this figure that logarithm of $\varepsilon/\varepsilon_0$ changes linearly with the time. This means that, when some time has elapsed after the impact, a wave of only the first mode is left. Therefore, the damping envelope, except for the period of just after the impact, is expressed, by taking only $n=1$ and putting $x=l/2$ and $t=(2k+1/2)l/c$ (where k is integer) in Eq. (9), as

$$\frac{\varepsilon}{\varepsilon_0} = \exp\left(-\frac{\lambda_1^2 c^2 \tau}{2} t\right) \dots\dots\dots (11)$$

Fig. 4

Fig. 5

The gradient of straight line in Fig. 6 corresponds to this logarithmic decrement $\alpha = \lambda_1^2 c^2 \tau / 2$. The decrement α was obtained in this manner for several specimens made of stainless steel or S45C. Figure 7 and Fig. 8 illustrate the relations between α and λ_1 for stainless steel and S45C respectively. Some of the measurements were carried out using one specimen which was cut successively at both ends to change its length. These are shown by same marks in the

Fig. 6

Fig. 7 Stainless steel

Fig. 8 S45C

figures. Further, measured points with different marks mean the results obtained from different specimens. It is apparent that measured points obtained from the same specimen shortened gradually by cutting itself are on the straight lines having a gradient of 2.

On the other hand, according to the Voigt model, it is found that from Eq. (11) the logarithmic decrement is proportional to λ_1^2 . Hence assuming that materials obey the Voigt model, the relaxation time τ , which represents the viscosity of material, can be obtained from straight lines having a gradient of 2 shown in Figs. 7 and 8 which connect measured points. Although some deviations are observed generally, relaxation times τ obtained in this manner are:

$$\tau = (1.2 \sim 3.5) \times 10^{-8} \text{ sec (for strainless steel),}$$

$$\tau = (1.9 \sim 6.4) \times 10^{-8} \text{ sec (for S45C).}$$

From this result the value of $\lambda_1 C \tau$ in this experiment is about $5 \times 10^{-5} \sim 3 \times 10^{-4}$ and is far smaller than unity. It can be said that approximation adopted to reduce Eq. (8)' is quite appropriate.

3. Analysis of wave form

One end of a semi-infinite bar is impacted and is moved with a constant velocity v at the time $t \geq 0$. This problem is equivalent to one of the impact of two bars which have a semi-infinite length, same diameter and same material and strike each other along the axial direction with a relative velocity $2v$ at the time $t=0$ as shown in Fig. 9. The state of strain

$$\bar{\varepsilon} = \frac{iv}{\sqrt{2\pi c}} \exp\left(-\frac{\lambda^2 c^2 \tau t}{2}\right) \frac{\exp(i\lambda ct \sqrt{1-(\lambda c \tau/2)^2}) - \exp(-i\lambda ct \sqrt{1-(\lambda c \tau/2)^2})}{\lambda \sqrt{1-(\lambda c \tau/2)^2}} \quad (17)$$

Therefore, the solution of ε is obtained as follows by inverse transformation.

$$\varepsilon = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\varepsilon} e^{-i\lambda x} d\lambda = \frac{iv}{2\pi c} \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda^2 c^2 \tau t}{2}\right) \times \frac{\exp(i\lambda ct \sqrt{1-(\lambda c \tau/2)^2}) - \exp(-i\lambda ct \sqrt{1-(\lambda c \tau/2)^2})}{\lambda \sqrt{1-(\lambda c \tau/2)^2}} e^{-i\lambda x} d\lambda \quad (18)$$

Here in the case of $t/\tau \gg 1$, the following approximation can be made. The term $\exp(-\lambda^2 c^2 \tau t/2)$ which is in the integrand of Eq. (18) can be written as $\exp\{-2(\lambda c \tau/2)^2(t/\tau)\}$. Therefore, with the condition of $t/\tau \gg 1$, integration of Eq. (18) over the range, where λ is large, has slight influence on the value integrated over the total range, and Eq. (18) is approximated as follows.

$$\varepsilon = \frac{iv}{2\pi c} \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda^2 c^2 \tau t}{2}\right) \frac{e^{i\lambda ct} - e^{-i\lambda ct}}{\lambda} e^{-i\lambda x} d\lambda \quad (19)$$

Carrying out the integration, (refer to appendix), we obtain

$$\varepsilon = \begin{cases} -\frac{v}{2c} \left\{ \operatorname{erf}\left(\frac{x+ct}{c\sqrt{2\tau t}}\right) - \operatorname{erf}\left(\frac{x-ct}{c\sqrt{2\tau t}}\right) \right\} : x > ct \\ -\frac{v}{2c} \left\{ \operatorname{erf}\left(\frac{ct+x}{c\sqrt{2\tau t}}\right) + \operatorname{erf}\left(\frac{ct-x}{c\sqrt{2\tau t}}\right) \right\} : |x| \leq ct \\ -\frac{v}{2c} \left\{ \operatorname{erf}\left(\frac{ct-x}{c\sqrt{2\tau t}}\right) - \operatorname{erf}\left(\frac{-ct-x}{c\sqrt{2\tau t}}\right) \right\} : x < -ct \end{cases} \quad (20)$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2} dr$.

The solution of strain wave is thus obtained, and the strain distribution calculated by Eq. (20) is shown in

waves produced in this case is such that, according to one-dimensional theory of elasticity, step compression waves with a strain magnitude of $-v/c$ travel to the right and left sides. However, in the case of a visco-elastic body, a strict step wave is not produced.

The wave form produced for a Voigt model is analyzed hereinafter. Denoting time by t , coordinate by x and displacement by u , and assuming that the struck end of the bar is $x=0$, the initial conditions are expressed by the following equations.

$$t=0 : u(x)=0, \quad \frac{\partial u}{\partial t} = v\{1-2H(x)\} \quad (12)$$

where $H(x)$ is a Heaviside step function. Differentiating the equation of motion (3) and initial conditions (12) with respect to x , we obtain

$$\frac{\partial^2 \varepsilon}{\partial t^2} = c^2 \left(1 + \tau \frac{\partial}{\partial t}\right) \frac{\partial^2 \varepsilon}{\partial x^2} \quad (13)$$

$$t=0 : \varepsilon(x)=0, \quad \frac{\partial \varepsilon}{\partial t} = -2v\delta(x) \quad (14)$$

where $\delta(x)$ is a Dirac delta function. Applying Fourier transformation

$$\tilde{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx$$

to Eqs. (13) and (14), we obtain

$$\frac{d^2 \tilde{\varepsilon}}{dt^2} + \lambda^2 c^2 \tau \frac{d\tilde{\varepsilon}}{dt} + \lambda^2 c^2 \tilde{\varepsilon} = 0 \quad (15)$$

$$t=0 : \tilde{\varepsilon}=0, \quad \frac{d\tilde{\varepsilon}}{dt} = -\frac{2v}{\sqrt{2\pi}} \quad (16)$$

Solving this equation, we obtain the solution,

Fig. 9

Fig. 10. According to Eq. (20), the magnitude of the produced strain in a sufficiently inside part from the wave front is equal to $-v/c$, which is the same as in the elastic solution, while it does not change suddenly and strictly from zero to 1 as found in the case of elastic solution, but approaches gradually from zero to 1. Just half strain of $-(1/2)(v/c)$ travels at the same velocity of c as the velocity of elastic wave front, and a rising part expressed by error functions is observed before and after it. Further, it is shown that this range will spread in proportion to $\sqrt{t}$ as time elapses.

Substituting Eq. (20) into (2), the following equation is obtained as the stress distribution.

$$\sigma = \begin{cases} -\frac{vE}{2c} \left[\operatorname{erf} \left(\frac{x+ct}{c\sqrt{2\tau t}} \right) - \operatorname{erf} \left(\frac{x-ct}{c\sqrt{2\tau t}} \right) \right] \\ - \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{(x+ct)^2}{2\tau t c^2} \right\} \frac{x-ct}{ct\sqrt{2t/\tau}} + \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{(x-ct)^2}{2\tau t c^2} \right\} \frac{x+ct}{ct\sqrt{2t/\tau}} \right], & x > ct \\ -\frac{vE}{2c} \left[\operatorname{erf} \left(\frac{ct+x}{c\sqrt{2\tau t}} \right) + \operatorname{erf} \left(\frac{ct-x}{c\sqrt{2\tau t}} \right) \right] \\ - \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{(x+ct)^2}{2\tau t c^2} \right\} \frac{x-ct}{ct\sqrt{2t/\tau}} + \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{(ct-x)^2}{2\tau t c^2} \right\} \frac{x+ct}{ct\sqrt{2t/\tau}} \right], & |x| \leq ct \\ -\frac{vE}{2c} \left[\operatorname{erf} \left(\frac{ct-x}{c\sqrt{2\tau t}} \right) - \operatorname{erf} \left(\frac{-ct-x}{c\sqrt{2\tau t}} \right) \right] \\ + \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{(ct-x)^2}{2\tau t c^2} \right\} \frac{x+ct}{ct\sqrt{2t/\tau}} - \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{(x+ct)^2}{2\tau t c^2} \right\} \frac{x-ct}{ct\sqrt{2t/\tau}} \right], & x < -ct \end{cases} \quad \dots\dots(21)$$

Eqs. (20) and (21) are approximate solutions in case of $t/\tau \gg 1$. As for stress, this can be well compared with the solution by Morrison. This comparison is shown in Fig. 11, from which it can be seen that the error is small even in the case that t/τ is not so large compared with 1.

4. Considerations

As mentioned previously in section 2, the viscosity coefficient for steel assuming that it consists of a Voigt element is obtained and the relaxation time τ is found to be $10^{-8} \sim 10^{-7}$ sec. According to other

investigations using the same model, the relaxation time is 10^{-5} sec by Sezawa and Kubo⁽³⁾, and 10^{-7} sec by Rakhmatulin⁽⁴⁾. The values obtained in our experiments are small compared with those presented by others. Because the loss of energy due to air and strings which suspend the test specimen is not considered in our experiments, the value of the relaxation time τ will probably be smaller than the value obtained here.

Figure 12 shows the state of wave front which is produced by impact in the direction of bar axis. The wave is observed at the point 1 m apart from the impacted end of a stainless steel bar having 12 mm diameter. The rising time of approximately $20 \mu\text{sec}$ is observed in Fig. 12.

As can be understood in section 3, the rising time of wave front in the case of a Voigt model is approximately $2\sqrt{2\tau t}$. Using the values $\tau = (1 \sim 3) \times 10^{-8}$ sec for stainless steel obtained in Section 2 and $t = 200 \mu\text{sec}$ (wave requires $200 \mu\text{sec}$ after the impact to reach the observation point with the velocity of $c = 5000$ m/sec), the approximate rising time is calculated as

Fig. 10

Fig. 11

Fig. 12

several μsec . The rising time of an observed wave form shown in Fig. 12 is longer than this value. Perhaps this is caused by other factors.

Strain gauges used in experiments are of 10 mm length and their influence, according to the report by Oi⁽⁵⁾, is considered to be below 2 μsec . The influence of radial inertia is calculated to be about $4(\nu^2 a^2 t / c^2)^{1/3}$ (where ν is Poisson's ratio, a is diameter of the bar, c is velocity of one-dimensional elastic wave and t is time elapsed after the impact) by using the solution of Skalak. Therefore, the influence of radial inertia is approximately 10 μsec . Furthermore, it must be remembered that impact will not take place in a complete manner to produce a strict step wave. However, the influence of this factor is not estimated numerically. The superposed results of all these factors are the observed rising time. The influence of internal viscosity is several μsec at the most, as stated before, and is considered not to exceed this value. The existence of the rising time is inevitable under the influences of internal viscosity, radial inertia and others. In addition to these factors, the fact that the experimental input wave form differs from an ideal step wave form is considered as the cause of the existence of a longer rising time.

5. Conclusions

(1) Approximate analytical solutions of stress and strain in a bar are obtained under the following conditions in the range of $t \gg \tau$. The conditions are that the stress-strain relation of the bar is represented by a Voigt model and one end of the bar begins to move suddenly with a constant velocity.

(2) When a Voigt model is adopted to express the mechanical property of steel, the relaxation time τ , the ratio of viscosity coefficient to modulus of longitudinal elasticity, is order of 10^{-8}sec at the most.

Appendix

Denoting

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda^2 c^2 \tau t}{2}\right) e^{-i\lambda x} d\lambda \quad \dots\dots(22)$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\lambda ct} - e^{-i\lambda ct}}{\lambda} e^{-i\lambda x} d\lambda \quad \dots\dots(23)$$

Equation (19) can be written, with the aid of convolution, as

$$\varepsilon = \frac{iv}{2\pi c} \int_{-\infty}^{\infty} f(x-y)g(y) dy \quad \dots\dots(24)$$

Using the formula of Fourier transformation, we obtain

$$f(x) = \frac{1}{c\sqrt{\tau t}} \exp\left(-\frac{x^2}{2c^2\tau t}\right) \quad \dots\dots(25)$$

$$g(x) = \begin{cases} i\sqrt{2\pi} & -ct < x < ct \\ 0 & x < -ct, x > ct \end{cases} \quad \dots\dots(26)$$

Substituting Eqs. (25) and (26) into Eq. (24), the following is obtained.

$$\begin{aligned} \varepsilon &= \frac{-v}{\sqrt{2\pi} c^2 \sqrt{\tau t}} \int_{-ct}^{ct} \exp\left\{-\frac{(x-y)^2}{2c^2\tau t}\right\} dy \\ &= -\frac{v}{2c} \frac{2}{\sqrt{\pi}} \int_{(x-ct)/c\sqrt{2\tau t}}^{(x+ct)/c\sqrt{2\tau t}} \exp(-r^2) dr \quad \dots\dots(27) \end{aligned}$$

Using an error function which is denoted as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2} dr$$

Eq. (27) can be written as Eq. (20).

References

- (1) Skalak, R., *Jour. Appl. Mech.*, Vol. 24, (1957), p. 59.
- (2) Morrison, J., *Quart. Appl. Math.*, Vol. 14, (1956), p. 153.
- (3) Sezawa, K. and Kubo, K., *Rep. Aeron. Res. Inst.*, No. 89 (1932), p. 195.
- (4) Rakhmatulin, Kh. A. and Dem'yanov, Yu. A., *Strength under High Transient Loads*, (1966), p. 279, Israel Program for Scientific Translations, Jerusalem.
- (5) Oi, K., *Jour. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 69, No. 575 (1966-12), p. 1602.