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The Meshing Test on Helical Gears under Load Transmission*

(2nd Report, The Approximate Formula for Bending-Moment Distribution of Gear Tooth)

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An approximate formula is given for the distribution of bending-moment of gear tooth with finite width under the condition that a concentrated load acts on any location of its surface. This approximate formula includes the effect of the pressure angle of gear.

This formula coincides with the numerical solutions obtained by applying the calculus of finite differences to the basic differential equation and with the results from strain-gages on the gear tooth.

The effect of the number of teeth on the bending-moment distribution is not found in the results from straining investigation.

1. Introduction

The stresses at the fillet of spur gear have been studied and these are evaluated theoretically.

In a helical gear, the stresses at the fillet have not been obtained theoretically for the reasons that the contact line is not parallel to the tip edge, and that the distribution of the stresses along the fillet due to a concentrated load have not been obtained.

The distribution of the bending-moments of a cantilever plate loaded by a concentrated force has been discussed by MacGregor⁽¹⁾. His solution of the problem was obtained by applying the thin-plate theory to a cantilever plate with finite width and constant thickness, only in the case when the load was applied at the top free edge of the plate.

An exact solution in terms of improper integrals for the distribution of bending-moments due to a transverse concentrated load acting at an arbitrary point of an infinite width cantilever with constant thickness was obtained by Jaramillo⁽²⁾ and Fujita⁽⁴⁾. Fujita checked the solutions with the results of strain-gages.

The theory of a constant thickness plate does not suffice for the problems of gear. The distribution of the moments of infinite rectangular plates of varying thickness simulating the gear tooth has been solved by Aida and Fujio⁽⁷⁾. Wellauer and Seireg⁽⁸⁾ gave a semiempirical solution for a finite cantilever plate under transverse load at any location on its

surface. The solution which was based on the principle of superposition and a proposed "moment-image" method, showed agreement with the results of strain-gages investigations on cantilever plates simulating a gear tooth.

To make use of this "moment-image" method, Kugimiya⁽⁵⁾⁽⁶⁾ studied practically the distribution of bending-moments of a helical gear.

The author⁽⁶⁾ gave the numerical solutions for the distribution of bending-moments of a rack-shaped (the rack profile of JIS) cantilever with finite width under the condition that the transverse load acts at any location on the surface. These solutions were obtained using the method of finite differences and new boundary conditions that the built-in edge is a semi-elastic built-in edge. These solutions were checked experimentally with good agreement.

But when the distribution of bending-moments is to evaluate in practice,

(1) Owing to the ratio of the width and the length, the side profile of a rack-shaped cantilever and loaded points, many kinds of big simultaneous equations will have to be evaluated.

(2) The numerical solutions are evaluated only on the fixed net-points (the co-ordinate). The formula of numerical integral can not be used, because such a formula needs the numerical solutions on the points of special co-ordinate.

(3) Practically the approximate equation for the distribution of bending-moments is very convenient.

In this paper, under the condition that the cutter pressure angle is common to gear tooth, the common characters have been found in the numerical solutions.

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which were obtained with the method of finite differences. An approximate equation for the distribution of the bending-moments has been built up with the use of these common characters. And the approximate equation is checked with numerical solutions and the strain-gage investigation.

2. Notations

- ξ, η : the co-ordinates of the loaded point
 x, y : the co-ordinates of the measured point of bending-moments
 $\bar{x}, \bar{y}, \bar{\xi}, \bar{\eta}$: letter symbols with bar mean divided by the height of cantilever ($\bar{\eta} = \eta/h$)
 b : one half of gear width
 h : whole depth of gear
 h_g : the depth between tip and strain-gage
 m_n : normal module
 α_n : normal pressure angle
 T : thickness of gear tooth
 T_g : thickness at strain-gage
 z : number of teeth
 P : a concentrated load
 $M(x, \xi, \eta)$: bending-moment along the fillet
 $\beta(x, \xi, \eta)$: effect function of bending-moment along the fillet, when a concentrated load acts on (ξ, η)
 $\beta(x, \xi)$: effect function of bending moment along the fillet, when a concentrated load acts on the free edge.
 σ_α : coefficient of pressure angle
 E : modulus of elasticity

3. The approximate equation for the effect function of the distribution of bending-moments

The co-ordinates are taken on the middle surface of the rack-shaped cantilever as shown in Fig. 1. When a concentrated load acts on the co-ordinate (ξ, η) , the bending-moment along the fillet $M(x, \xi, \eta)$ is given as,

$$M(x, \xi, \eta) = \beta(x, \xi, \eta)P \quad (1)$$

where β is named the effect function of bending-moments along the fillet. This function is determined

Fig. 1 Symbols of cantilever

with the profile of the cantilever, the value of $2b/h$ and the modulus of elasticity.

3.1 The approximate equation for infinite width cantilever

When a concentrated load acts on the origin of a rack-shaped cantilever whose whole depth is $2.25 m_n$, the pressure angle is 20 degrees and $2b/h=4.0$, from the reference (9), the numerical solutions of the effect function are shown in Fig. 2 as black points.

When a concentrated load acts on the free edge, the integral of this effect function in widthwise direction must be 1.0 under the condition that the whole depth is unit length. The approximate function must suffice this nature of the effect function and must simulate the black points and line of Fig. 2.

The approximate function is obtained as,

$$\beta(\bar{x}, 0) = \frac{1}{\sqrt{2\pi}\sigma_\alpha} e^{-\bar{x}^2/2\sigma_\alpha^2} \quad (2)$$

where $(\bar{x}, \bar{\xi}, 0) = (\bar{x}, \bar{\xi})$ when a concentrated load acts on the free edge and σ_α is the co-efficient of the pressure angle.

When the pressure angle is 20 degrees, from $\beta(0,0)=0.58$, the co-efficient of pressure angle is obtained as $\sigma_\alpha=0.69$. So the approximate equation of a rack-shaped cantilever with 20 degrees is shown as,

$$\beta_{\alpha_n=20^\circ}(\bar{x}, 0) = \frac{1}{\sqrt{2\pi} \times 0.69} e^{-\bar{x}^2/2(0.69)^2} \quad (3)$$

The values of Eq. (3) are shown in Fig. 2 with white dots and dotted line. These values of the approximate equation almost coincide with the numerical solutions in all the region and the integral of Eq. (3) is 1.0.

In Fig. 2, the numerical solutions and the values of approximate equation are shown for a thick plate with $2b/h=6.0$. However the numerical solutions in Fig. 2 are extrapolated approximately with the numerical solutions of a thick plate with $2b/h=4.0$ under the condition that the integral of the solutions is 1.0. The coefficient of pressure angle has been calculated to be 1.10 from the condition $\beta_{\alpha_n=0}(0,0)=0.36$.

That is, the effect function of bending-moment can be shown as Eq. (2).

The coefficient of pressure angle was calculated

Fig. 2 Numerical solutions and the values of approximate equation (2)

from the numerical solutions and shown in Fig. 3.

3.2 The approximate equation for finite width cantilever

The width of gear is practically finite. So the approximate equation for a finite width cantilever must be considered. That is, the effect of the infinite width being cut by the side free edge of the cantilever, must be estimated.

As shown in Fig. 4, let us consider the condition that the effect function of bending-moments is limited by the side free edge of $x = \pm n$ (where $2n = 2b/h$). In Fig. 4, the area in the region where x is larger than n , is shown as

$$\int_{n-\bar{\xi}_i}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_\alpha} e^{-(\bar{x}-\bar{\xi}_i)^2/2\sigma_\alpha^2} d\bar{x} = \operatorname{erfc}\left(\frac{n-\bar{\xi}_i}{\sigma_\alpha}\right) \quad (4)$$

For the method of superposition, we make the assumption that the area $\operatorname{erfc}\{(n-\bar{\xi}_i)/\sigma_\alpha\}$ is changed parabolically to accommodate for the inside in the region $2h$, as shown in Fig. 4.

Under the above assumptions, the value of superposition $\beta'(\bar{x}, \bar{\xi}_i)$ is shown as,

$$\beta'(\bar{x}, \bar{\xi}_i) = \frac{3}{8} \operatorname{erfc}\left(\frac{n-\bar{\xi}_i}{\sigma_\alpha}\right) \{\bar{x} - (n-2)\}^2 \quad (5)$$

So the approximate equation for an infinite width cantilever is shown as,

Fig. 3 The coefficient of pressure angle

Fig. 4 The method of superposition

$$\beta_{2n}(\bar{x}, \bar{\xi}) = \frac{1}{\sigma_\alpha} \varphi\left(\frac{\bar{x}-\bar{\xi}}{\sigma_\alpha}\right) + a_1\{\bar{x} - (n-2)\}^2 + a_2\{\bar{x} + (n-2)\}^2 \quad (6)$$

where

$$a_1 = \begin{cases} \frac{3}{8} \operatorname{erfc}\left(\frac{n-\bar{\xi}}{\sigma_\alpha}\right) & (-n \leq \bar{x} \leq 2-n) \\ 0 & (2-n < \bar{x}) \end{cases}$$

$$a_2 = \begin{cases} \frac{3}{8} \operatorname{erfc}\left(\frac{n+\bar{\xi}}{\sigma_\alpha}\right) & (n-2 \leq \bar{x} \leq n) \\ 0 & (\bar{x} < n-2) \end{cases}$$

$$2n = 2b/h$$

$$\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$$

With the pressure angle 20 degrees and $2b/h = 3.0$, the values of the approximate equation (6) and the numerical solutions with the references (8) are shown in Fig. 5.

The values evaluated by the moment-image method⁽³⁾ under the condition that the effect function of infinite width is used in the numerical solution in Fig. 2, are shown in Fig. 5.

On the assumption of this superposition, the approximate equation (6) is used for the cantilever whose width is greater than or equal to 2. When the width is $2b/h = 2.0$, that is, when it is smallest in this equation, the values of the approximate equation (6) and the numerical solutions are shown in Fig. 6.

As shown in Figs. 5 and 6, the values of the approximate equation (6) agree well with the numerical solutions in all the region under loads at any location on the free edge.

3.3 When a concentrated load acts on the inside surface from the free edge

When a concentrated load acts on the axis $y = \bar{y}$, we make the assumption that the effect function

Fig. 5 The effect function of the rack-shaped cantilever with pressure angle 20 degrees and $2b/h = 3.0$

$(\bar{x}, \bar{\xi}, \bar{\eta})$ is shown as

$$\beta(\bar{x}, \bar{\xi}, \bar{\eta}) = \beta(\bar{x}, \bar{\xi}) \times (1 - \bar{\eta}) \dots \dots \dots (7)$$

The relation of Eq. (7) is checked with the numerical solutions as shown in Fig. 7. In Fig. 7, when a concentrated load acts on the axis from $\bar{\eta}=0$ to $\bar{\eta}=0.5$, the values of the approximate equation agree well with the numerical solutions. When a concentrated load acts on the axis $\bar{\eta}=0.75$, practically the values of the approximate equation agree well with numerical solutions.

When a concentrated load acts on the side free

Fig. 6 The effect function of the rack-shaped cantilever with pressure angle 20 degrees and $2b/h=2.0$

Fig. 7 The effect function when a load acts on y-axis of cantilever with pressure angle 20 degrees and $2b/h=3.0$ (points show numerical solutions)

edge, the relation of Eq. (7) is checked with the pressure angle 20 degrees and $2b/h=2.0$, as shown in Fig. 8.

In the study of gear tooth, the maximum value of $\bar{\eta}$ is 0.89 when the whole depth is $2.25 m_n$. So practically Eq. (7) is a good approximate equation.

Finally the approximate equation for the effect function of the bending-moments is shown as

$$\beta(\bar{x}, \bar{\xi}, \bar{\eta}) = \left[\frac{1}{\sigma_a} \varphi \left(\frac{\bar{x} - \bar{\xi}}{\sigma_a} \right) + a_1 (\bar{x} - (n-2))^2 + a_2 (\bar{x} + (n-2))^2 \right] (1 - \bar{\eta}) \dots \dots \dots (8)$$

4. Experiment and the effect of number of teeth

The numerical solutions are in satisfactory agreement with experimental results, as shown in the reference (8). But the experimental results in the reference (8) were obtained with only a rack-shaped cantilever. So a few experiments were performed to discuss the effect of number of teeth. The sections of cantilevers to be used in these experiments are the Japanese Industrial Standard gear profiles ($h=2.25 m_n$, $t=\pi m_n/2$) with $z=20$ and 36 ($m_n=8$) and rack ($m_n=11$), where t is a circular pitch.

4.1 The experiments

The tested cantilevers whose profiles are gear teeth are illustrated in Table 1. The profiles of gear teeth are ground with the Maag grinder HSS-30-BC. The tooth profile errors of tested gear are less than $5 \mu m$. These experimental blocks are cut out and ground as shown in Fig. 17 of the reference (9).

Fig. 8 The effect function when a load acts on free edge of cantilever with pressure angle 20 degrees and $2b/h=2.0$ (points show numerical solutions)

The strain-gages are located outside the fillet. So the stress concentration is not considered in these experimental results.

4.2 The experimental illustration and the loaded direction

With width 50 mm and side profile rack, an experimental example is shown in Fig. 9 when a concentrated load acts on the center of the width and 1.5 mm inside from the free edge. When a concentrated load acts in the perpendicular direction to the middle surface as shown by A in Table 1, the experimental results are shown by ● marks and the line as shown in Fig. 9.

In the perpendicular direction to the tooth surface, the experimental results are shown by ○ marks. These experimental results contain the effects of some component of the load acting in the height-

Fig. 9 The experimental results when a load on the center of cantilever with pressure angle 20 degrees and width 50 mm

wise direction. For the estimation of this effect, the experimental results are shown by ○ marks when a concentrated load acts in the heightwise direction as shown by C in Table 1. These results have been multiplied by $\sin 20^\circ$ which equals to the component of the height-wise direction.

The remainder after taking the experimental results in the direction C from the experimental results in the direction B, are the experimental results when the component in the perpendicular direction to the middle surface acts on the gear tooth as shown by ○ marks in Fig. 9.

The mean value of the ratios of these experimental results to the results in the direction A is 0.902. This value is near $\cos 20^\circ$ which is 0.940.

When a concentrated load acts on the side free edge, the experimental results are shown in Fig. 10. Also the mean value of the ratios of the experimental results of D to the result in the direction A is 0.948. Practically this value equals $\cos 20^\circ$.

We conclude from the experiments described above that the components in the perpendicular direction to the middle surface affect the data with strain-gages when a concentrated load acts on the tooth surface. But when the distributions of bending-moments are measured by strain-gages, the effect of the components in the heightwise direction can not be neglected as shown in Fig. 10. So the measured results must be corrected with these effects.

4.3 Experimental results

Experimental values are measured by strain-gages under the condition that a concentrated load acts in the direction perpendicular to the middle surface.

Table 1 The tested cantilevers

Number of teeth	Rack ($Z = \infty$)	$z = 36$	$z = 20$
Pressure angle		20°	
Module	11	8	
Side profile			
Width	50 mm ($2b/h = 2.0$) 75 mm ($2b/h = 3.0$)	54 mm ($2b/h = 3.0$)	
Strain-gages (Gage-length) (Base width)	KB-1-A1 (1 mm) (5 mm)	KF-03-C1-11 (0.3 mm) (2.5 mm)	
Max. load	720 kg	480 kg	

The effect function β_{gi} is calculated from the strain reading ε_i by Eq. (9).

$$\beta_{gi} = \frac{\varepsilon_i}{P} \times E \times \frac{T_g^2}{6} \dots\dots\dots (9)$$

where E is modulus of elasticity (2.1×10^4 kg/mm²).

The effect function along the fillet is obtained on the assumption that there is a simple relation of the direct proportion between β_i and β_{gi} as shown in Eq. (10).

$$\beta_i = \beta_{gi} \times \left(\frac{h-1.5}{h_g-1.5} \right) \times \left(\frac{h}{h-1.5} \right) \dots\dots\dots (10)$$

When a concentrated load acts on the center of width and $\bar{\eta}=0.0$ and $\bar{\eta}=0.5$, with $2b/h=3.0$ and pressure angle 20 degrees, the experimental results are shown in Fig. 11. And the values of the approximate equation (8) are shown as the lines in the same figure.

When a concentrated load acts on the side free edge, the experimental results and the values of the approximate equation are shown in Fig. 12.

Within the limit of these experiments, practically the effect function of bending-moments are not affected by the number of gear teeth under the condition that all cutter pressure angles are equal.

5. Conclusions

With the use of the numerical solutions which are evaluated with the method of finite differences, and the experimental results with the gear tooth profile cantilever, the following conclusions are

Fig. 10 The experimental results when a load on the side free edge of cantilever with pressure angle 20 degrees and width 50 mm

gained:

(1) The approximate equation (8) is obtained. This equation can be applied to a cantilever with any pressure angle and agrees well with the experimental results.

(2) Practically the effect of number of teeth for the approximate equation is not observed with experimental results under the condition that all cutter pressure angles are equal.

(3) When the load acts on the surface in the perpendicular direction, the measured results by strain-gages must be corrected with the heightwise component of the load.

Fig. 11 The experimental results with pressure angle 20 degrees and $2b/h=3.0$ when a load acts on the center of width [lines show the values of approx. Eq. (8)]

Fig. 12 The experimental results with pressure angle 20 degrees and $2b/h=3.0$ when a load acts on the side free edge [lines show the values of approx. Eq. (8)]

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Errata (Vol. 15, No. 90, December, 1972)

Author	Page	Column	Line	For	Read
T. ÔYA	1530	left	1	Y_s	Y_S
	〃	〃	3	Y_s	Y_S
	1542	right	12 from the bottom	that "	that "
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H. KATO et al.	1556	〃	12	Ref. No. (15)	(14)
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	1559	left	Fig. 3	Temp. 110°C	11.0°C
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S. NAMIE et al.	1569	〃	Fig. 1	Experimental appartaus	Experimental apparatus
	1570	〃	Fig. 2(a)	Liquid flow of droplt	Liquid flow of droplet
	〃	〃	Table 1	0.3 $\mathcal{L}$ ~0.51	0.37~0.51
T. ITO et al.	1597	right	28	n -pentan	n -pentane
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