Measurement of Residual Stresses in a Cylinder*

(3rd Report, On the Effect of Eccentricity of Holes Bored out in Sachs Method)

By Haruo KAWAGOE**

The Sachs boring-out method is used for determining axially symmetrical residual stresses in circular cylinders. However, when holes bored out in a circular cylinder layer by layer are not exactly concentric, the above method will not be satisfactorily applicable to estimation of the residual stresses in the cylinder on account of the eccentricity of the holes.

Since it is usually accepted that dimensional changes caused by any relaxation of stress are purely elastic, the author analyzes the effect of eccentricity on the deformation of a cylinder from purely elastic theory and gives the numerical evaluations.

1. Introduction

The Sachs boring-out method is widely used for determining residual stresses in a solid or hollow cylinder. The author reported, in the previous papers⁽¹⁾, the effect of the ratio ξ of length to diameter of a specimen taken from a long cylinder having residual stresses, on the discrepancy between the true residual stresses and the measured ones.

In the Sachs method, every time each thin layer is removed from a bore, longitudinal and tangential strains in the outer surface of a cylinder are measured. When the bore obtained is eccentric, in general it is expected that the above strains vary along the outer circular boundary. Then, in the case of the bore having a slight eccentricity, the author analyzed the deformation produced in subsequent boring-out process of a long solid cylinder having axially symmetrical and longitudinally uniform residual stresses under the fundamental assumption in a mechanical method of residual stress measurement in which all processes are purely elastic. Through this analysis, he investigated the difference between the true residual stresses and the measured ones when an eccentric bore was made.

2. Analysis

Consider a long solid cylinder of radius a in which the residual stresses are axially symmetrical and longitudinally uniform except for the regions near both ends of the cylinder. In the system of cylindrical coordinates (r, θ, z) where the sym-

metrical axis of the above cylinder is coincident with the z axis and the mid-point of the axis is the origin, three residual stress components are functions of r only and denoted by $\sigma_r^*(r)$, $\sigma_\theta^*(r)$, and $\sigma_z^*(r)$ respectively.

These are evidently principal stresses and can be expanded in a series of the form

$$\left. \begin{aligned} \sigma_z^*(r) &= \sum_{m=1}^{\infty} \sigma_{zm}^*(r) = \sum_{m=1}^{\infty} A_m J_0(p_m r) \\ \sigma_r^*(r) &= \sum_{n=1}^{\infty} \sigma_{rn}^*(r) = \sum_{n=1}^{\infty} B_n J_0(q_n r) \\ \sigma_\theta^*(r) &= \sum_{n=1}^{\infty} \sigma_{\theta n}^*(r) \\ &= \sum_{n=1}^{\infty} B_n \{J_0(q_n r) - q_n r J_1(q_n r)\} \end{aligned} \right\} \dots\dots (1)$$

respectively where A_m and B_n are now unknown coefficients, $p_m a$ is the m -th positive root of the equation $J_1(p_m a) = 0$, and $q_n a$ is the n -th root of the equation $J_0(q_n a) = 0$.

These residual stresses, of course, satisfy the necessary conditions of equilibrium for the above cylinder:

$$\int_0^a 2\pi r \sigma_z^*(r) dr = 0, \quad \int_0^a \sigma_\theta^*(r) dr = 0$$

Under the assumption described in Chapter I that all deformations produced in the boring-out process are purely elastic, the residual stresses can be divided into two residual stress systems

- [I] $\sigma_z^*(r)$ only exists and $\sigma_r^*(r) = \sigma_\theta^*(r) = 0$,
- [II] $\sigma_r^*(r)$, $\sigma_\theta^*(r)$ exist and $\sigma_z^*(r) = 0$

Now by evaluating the deformation produced in boring-out process when each of the above stress systems exists and by adding them together, we can

* Received 24th July, 1971.

** Professor, Faculty of Engineering, Tottori University, Tottori.

obtain the total deformation when both of the stress systems simultaneously exist. We now assume that all strain components are zero throughout the solid cylinder before the boring-out process.

2.1 The case with the residual stress system [I]

In the same manner as the above discussion, we can resolve the residual stress system [I] into an infinite multitude of residual stresses

$$\sigma_{zm}^*(r) = A_m J_0(p_m r) \quad m=1, 2, 3 \dots \quad (2)$$

Therefore, in order to obtain the deformation produced in the boring-out process in the case of a residual stress system [I], we have only to evaluate the deformation produced in that process, when a residual stress $\sigma_{zm}^*(r)$ for an arbitrary value of m exists.

Consider a sufficiently long solid cylinder of the radius a and the length $2L$ with the residual stress $\sigma_{zm}^*(r)$ except for the regions near both ends. When the above cylinder is bored out layer by layer until the radius of the bore with the distance c ($c=\beta a$, $0 < \beta < 1$) from the center O' of the outer boundary becomes ρ ($\rho=\alpha a$, $0 < \alpha < 1$), the residual shear stress $\tau_{rz}(\rho, z)$ acting on the cylindrical surface of radius ρ and rapidly decreasing with the distance from both ends is relaxed. By considering the condition of equilibrium of the portion bounded by the cylindrical surface of the radius ρ and planes $z=0$ and $z=L$, we have

$$\int_0^\rho \int_0^{2\pi} \sigma_{zm}^*(r) r' dr' d\varphi = \int_0^L \int_0^{2\pi} \tau_{rz}(\rho, z) \times \rho d\varphi dz \dots \dots \dots (3)$$

Namely, using the notation $F_m(\rho)$ for the resultant force of the above residual shear stress $\tau_{rz}(\rho, z)$ ($0 < z < L$), we have from Eq. (3)

$$F_m(\rho) = \int_0^\rho \int_0^{2\pi} \sigma_{zm}^*(r) r' dr' d\varphi \dots \dots \dots (4)$$

and also we find

$$g = \frac{\int_0^\rho \int_0^{2\pi} \sigma_{zm}^*(r) r'^2 \cos \varphi dr' d\varphi}{\int_0^\rho \int_0^{2\pi} \sigma_{zm}^*(r) r' dr' d\varphi} \dots \dots \dots (5)$$

where g gives the location of application of the above resultant force $F_m(\rho)$ measured from the center of the bore as shown in Fig. 1. Therefore, the stresses and strains caused by the above boring-out process in the region apart from both ends of the bored-out cylinder are mechanically equivalent to those caused by the normal forces $F_m(\rho)$ uniformly acting on both end planes and the bending moment $(c+e-g)F_m(\rho)$ longitudinally uniform in a hollow cylinder being the same in size and free from stress as the above cylinder. Denoting the values of strain components ϵ_z and ϵ_θ for $r=a$ in the middle region of a long hollow cylinder with the bore of radius ρ by $\lambda_I(\alpha, \beta, \theta)$ and $\mathcal{G}_I(\alpha, \beta, \theta)$ respectively, we have

$$\left. \begin{aligned} \lambda_I(\alpha, \beta, \theta) &= \frac{F_m(\rho)}{EA_\rho} \\ &+ \frac{F_m(\rho)(c+e-g)}{EI_G} (a \cos \theta - e) \end{aligned} \right\} \dots \dots \dots (6)$$

$$\mathcal{G}_I(\alpha, \beta, \theta) = -\nu \lambda_I(\alpha, \beta, \theta)$$

where A_ρ is the area of cross section of the cylinder, e is the distance between the centers of gravity of the section and of the outer boundary, I_G is the moment of inertia of the section with respect to x axis in Fig. 1, E is Young's modulus and ν is Poisson's ratio, and A_ρ , e , and I_G are given by the following equations:

$$A_\rho = \pi(a^2 - \rho^2), \quad e = c\rho^2/(a^2 - \rho^2)$$

$$I_G = (\pi/4)\{(a^4 - \rho^4) - 4a^2\rho^2c^2/(a^2 - \rho^2)\}$$

Substituting Eq. (2) into Eqs. (6), we have

$$\left. \begin{aligned} \lambda_I(\alpha, \beta, \theta) &= \frac{2A_m p_m \rho J_1(p_m \rho)}{E p_m^2 (a^2 - \rho^2)} J_0(p_m c) \\ &+ \frac{2\pi p_m (a \cos \theta - e)}{E p_m^4 I_G} p_m \rho J_1(p_m \rho) \\ &\times \left\{ p_m (c+e) J_0(p_m c) - J_1(p_m c) \right. \\ &\times \left. \left[2 - p_m \rho \frac{J_0(p_m \rho)}{J_1(p_m \rho)} \right] \right\} \\ \mathcal{G}_I(\alpha, \beta, \theta) &= -\nu \lambda_I(\alpha, \beta, \theta) \end{aligned} \right\} \dots \dots (7)$$

Eqs. (7) hold regardless of the magnitude of the distance c between the centers of the inner and outer boundaries. It is clear from Eqs. (7) that the mean values $\lambda_{I2}(\alpha, \beta, \theta)$ and $\mathcal{G}_{I2}(\alpha, \beta, \theta)$ of $\lambda_I(\alpha, \beta, \theta)$ and $\mathcal{G}_I(\alpha, \beta, \theta)$ for θ , and $\theta+\pi$ and the mean values $\lambda_{I4}(\alpha, \beta, \theta)$ and $\mathcal{G}_{I4}(\alpha, \beta, \theta)$ of those for θ , $\theta+\pi/2$, $\theta+\pi$, and $\theta+3\pi/2$ are independent of the angle θ and then $\lambda_{I2}(\alpha, \beta, \theta)$ and $\mathcal{G}_{I2}(\alpha, \beta, \theta)$ are equal to $\lambda_{I4}(\alpha, \beta, \theta)$ and $\mathcal{G}_{I4}(\alpha, \beta, \theta)$ respectively. Namely, we have

$$\left. \begin{aligned} \lambda_{I2}(\alpha, \beta, \theta) &= \lambda_{I4}(\alpha, \beta, \theta) = \frac{2A_m p_m \rho J_1(p_m \rho)}{E p_m^2 (a^2 - \rho^2)} \\ &\times J_0(p_m c) - \frac{2\pi p_m e p_m \rho J_1(p_m \rho)}{E p_m^4 I_G} \left\{ p_m (c+e) \right. \\ &\times J_0(p_m c) - J_1(p_m c) \left[2 - p_m \rho \frac{J_0(p_m \rho)}{J_1(p_m \rho)} \right] \left. \right\} \\ \mathcal{G}_{I2}(\alpha, \beta, \theta) &= \mathcal{G}_{I4}(\alpha, \beta, \theta) = -\nu \lambda_{I2}(\alpha, \beta, \theta) \end{aligned} \right\} \dots \dots \dots (8)$$

Fig. 1 Relative position of bore to outer boundary

Denoting the values of strain components $\varepsilon_z, \varepsilon_\theta$ for $r=a$ in the case of $c=0$ without eccentricity by $\lambda_I(\alpha), \mathcal{G}_I(\alpha)$, we have

$$\lambda_I(\alpha) = \frac{2A_m p_m \rho J_1(p_m \rho)}{E p_m^2 (a^2 - \rho^2)}, \quad \mathcal{G}_I(\alpha) = -\nu \lambda_I(\alpha) \quad \dots\dots\dots (9)$$

In the case of small c , $\lambda_{I4}(\alpha, \beta, \theta)$ and $\mathcal{G}_{I4}(\alpha, \beta, \theta)$ slightly differ from $\lambda_I(\alpha)$ and $\mathcal{G}_I(\alpha)$ respectively.

2.2 The case with the residual stress system [II]

In the same way of discussion as in section 2.1, we can resolve the residual stress system [II] into the infinite multitude of residual stresses

$$\left. \begin{aligned} \sigma_{rn}^*(r) &= B_n J_0(q_n r) \\ \sigma_{\theta n}^*(r) &= B_n \{J_0(q_n r) - q_n r J_1(q_n r)\} \\ n &= 1, 2, 3, \dots \end{aligned} \right\} \quad \dots\dots\dots (10)$$

Consequently, in order to obtain the deformation produced in the boring-out process in the case of a residual stress system [II], we have only to evaluate the deformation produced in that process when residual stresses $\sigma_{rn}^*(r)$ and $\sigma_{\theta n}^*(r)$ for an arbitrary value of n exist.

Consider a sufficiently long solid cylinder of the radius a in which residual stresses $\sigma_{rn}^*(r)$ and $\sigma_{\theta n}^*(r)$ exist. Expressing the residual stress components referred to the new coordinates (ξ, η) at the point P in Fig. 2 in terms of those referred to coordinates (r, θ) , we have

$$\left. \begin{aligned} \sigma_{\eta n}^*(\rho) &= \frac{1}{2} \{ \sigma_{rn}^*(\rho') + \sigma_{\theta n}^*(\rho') \} \\ &\quad - \frac{1}{2} \{ \sigma_{rn}^*(\rho') - \sigma_{\theta n}^*(\rho') \} \cos 2\alpha_\rho \\ \sigma_{\xi n}^*(\rho) &= \frac{1}{2} \{ \sigma_{rn}^*(\rho') + \sigma_{\theta n}^*(\rho') \} \\ &\quad + \frac{1}{2} \{ \sigma_{rn}^*(\rho') - \sigma_{\theta n}^*(\rho') \} \cos 2\alpha_\rho \\ \tau_{\xi\eta n}^*(\rho) &= -\frac{1}{2} \{ \sigma_{rn}^*(\rho') - \sigma_{\theta n}^*(\rho') \} \sin 2\alpha_\rho \end{aligned} \right\} \quad \dots\dots\dots (11)$$

From Fig. 2, we have easily

$$\cos \alpha_\rho = \frac{\rho - c \cos \varphi}{\sqrt{\rho^2 + c^2 - 2\rho c \cos \varphi}} \quad \dots\dots\dots (12)$$

Fig. 2 Transformation of coordinates (r, θ) into coordinates (ξ, η)

Substituting Eqs. (10) into Eqs. (11) and making use of Eq. (12), in the case of $\rho \gg c$, we have approximately

$$\left. \begin{aligned} \sigma_{\xi n}^*(\rho) &= B_n J_0(q_n \rho) + (c/\rho) B_n q_n \rho J_1(q_n \rho) \cos \varphi \\ \tau_{\xi\eta n}^*(\rho) &= -(c/\rho) B_n q_n \rho J_1(q_n \rho) \sin \varphi \end{aligned} \right\} \quad \dots\dots\dots (13)$$

When the above cylinder is bored out layer by layer until the radius of the bore with the distance c from the center of the outer boundary becomes ρ , the residual stresses $\sigma_{\xi n}^*(\rho)$ and $\tau_{\xi\eta n}^*(\rho)$ in Eqs. (13) are relaxed. Therefore, the strains produced in the above bored-out hollow cylinder are mechanically equivalent to those in a hollow cylinder the same in size and free from stress when that cylinder without longitudinal constraint is submitted to longitudinally uniform and circumferentially varying forces equal to $\sigma_{\xi n}^*(\rho)$, and $\tau_{\xi\eta n}^*(\rho)$ in magnitude and opposite in direction on the inner surface. Namely, we have only to obtain the solutions of two problems of a thick walled cylinder satisfying the following boundary conditions:

$$\left. \begin{aligned} (1) \quad &\text{on the outer surface } \sigma_\xi = \tau_{\xi\eta} = 0 \\ &\text{on the inner surface } \sigma_\xi = -B_n J_0(q_n \rho), \quad \tau_{\xi\eta} = 0 \end{aligned} \right\} \quad \dots\dots\dots (14)$$

$$\left. \begin{aligned} (2) \quad &\text{on the outer surface } \sigma_\xi = \tau_{\xi\eta} = 0 \\ &\text{on the inner surface } \sigma_\xi = -(c/\rho) B_n q_n \rho J_1(q_n \rho) \cos \varphi \\ &\quad \tau_{\xi\eta} = (c/\rho) B_n q_n \rho J_1(q_n \rho) \sin \varphi \end{aligned} \right\} \quad \dots\dots\dots (15)$$

Using a bipolar coordinates (ξ, η) defined by

$$z' = il \coth(\zeta/2), \quad \zeta = \xi + i\eta \quad \dots\dots\dots (16)$$

we can solve these problems mentioned above⁽²⁾. In these bipolar coordinates, it is clear that curves $\xi = \text{constant}$ are a family of circles as shown in Fig. 3. Let the outer boundary be the circle $\xi = \xi_0$, and let the bore be the circle $\xi = \xi_1$, and we obtain

$$\left. \begin{aligned} a &= l \operatorname{cosech} \xi_0 \\ \rho &= l \operatorname{cosech} \xi_1 \\ c &= l (\coth \xi_0 - \coth \xi_1) \end{aligned} \right\} \quad \dots\dots\dots (17)$$

Therefore, provided that the values of a , ρ , and c are given, ξ_0 , ξ_1 , and l are calculated from Eqs. (17).

The solution of a boundary value problem satisfying the condition (14) has been obtained on the basis of a combination of complex potentials

Fig. 3 Bipolar coordinates in eccentric cylinder

$$\left. \begin{aligned} \phi(z') &= Az' \\ \phi(z') &= iB \cosh \zeta, \quad \chi(z') = aB \sinh \zeta \\ \phi(z') &= iC \sinh \zeta, \quad \chi(z') = aC \cosh \zeta \\ \chi(z') &= aD\zeta \end{aligned} \right\} \dots\dots\dots (18)$$

where A, B, C and D are real constants. Therefore, we have only to obtain the solution of a boundary value problem satisfying the condition (15).

On the circle $\xi = \xi_1$ in Fig. 3, we have

$$\cos \eta = \frac{\cos \varphi + \operatorname{sech} \xi_1}{1 + \cos \varphi \operatorname{sech} \xi_1} \dots\dots\dots (19)$$

In the case of $\operatorname{sech} \xi_1 \ll 1$, Eq. (19) reduces approximately to

$$\eta = \varphi \dots\dots\dots (20)$$

Therefore, provided Eq. (20) holds, the solution of the problem under consideration can be obtained, without much difficulty, by making use of an another combination of the complex potentials of Eqs. (19).

Then solving Eq. (17) with respect to ξ_1 , we have

$$\cosh \xi_1 = (a^2 - \rho^2 - c^2) / 2\rho c \dots\dots\dots (21)$$

By using Eq. (21), the variation of $\alpha (= \rho/a)$ with $\beta (= c/a)$ for several values of $\cosh \xi_1$ are shown in Fig. 4. From this figure, we can determine the region in which the approximate relation (20) holds.

Adding the solutions satisfying the boundary conditions (14) and (15) by the method described above, we get the values $\mathcal{G}_{II}(\alpha, \beta, \theta)$ and $\lambda_{II}(\alpha, \beta, \theta)$ of ε_θ and ε_z on the outer surface when the residual stresses $\sigma_{\xi n}^*(\rho)$ and $\tau_{\xi \eta n}^*(\rho)$ in Eqs. (13) are relaxed:

$$\left. \begin{aligned} \mathcal{G}_{II}(\alpha, \beta, \theta) &= \frac{B_n J_0(q_n \rho)}{E \{ \sinh^2 \xi_0 + \sinh^2 \xi_1 \}} \\ &\times \{ 1 + \sinh 2\xi_0 \coth(\xi_1 - \xi_0) - 2 \sinh \xi_1 \\ &\times \operatorname{cosech}(\xi_1 - \xi_0) \cos \eta + \cos 2\eta \} - \frac{1}{2} \left(\frac{c}{\rho} \right) \\ &\times \frac{B_n q_n \rho J_1(q_n \rho)}{E \cosh \xi_1 \cosh(\xi_1 - \xi_0)} \left\{ \frac{1}{\sinh(\xi_1 - \xi_0)} \right. \\ &\left. - \frac{\sinh 2\xi_0 \cosh(\xi_1 - \xi_0) + \sinh(\xi_1 - \xi_0)}{\sinh^2 \xi_0 + \sinh^2 \xi_1} \right\} \\ &+ \left(\frac{c}{\rho} \right) \frac{B_n q_n \rho J_1(q_n \rho)}{E \cosh \xi_1 \cosh(\xi_1 - \xi_0)} \\ &\times \left\{ \frac{\cosh \xi_0}{\sinh(\xi_1 - \xi_0)} - \frac{\sinh \xi_1}{\sinh^2 \xi_0 + \sinh^2 \xi_1} \right\} \\ &\times \cos \eta - \frac{1}{2} \left(\frac{c}{\rho} \right) \frac{B_n q_n \rho J_1(q_n \rho)}{E \cosh \xi_1 \cosh(\xi_1 - \xi_0)} \\ &\times \left\{ \frac{1}{\sinh(\xi_1 - \xi_0)} - \frac{\sinh(\xi_1 - \xi_0)}{\sinh^2 \xi_0 + \sinh^2 \xi_1} \right\} \cos 2\eta \\ \lambda_{II}(\alpha, \beta, \theta) &= -\nu \mathcal{G}_{II}(\alpha, \beta, \theta) \end{aligned} \right\} \dots\dots\dots (22)$$

And then on the outer boundary surface $\xi = \xi_0$, from geometrical consideration we have

$$\cos \eta = \frac{\cos \theta + \operatorname{sech} \xi_0}{1 + \cos \theta \operatorname{sech} \xi_0} \dots\dots\dots (23)$$

Denoting the mean values of $\cos \eta$ and of $\cos 2\eta$ for θ , and $\theta + \pi$ by the notation $(\cos \eta)_2$ and $(\cos 2\eta)_2$

respectively and also denoting the mean values of $\cos \eta$ and of $\cos 2\eta$ for θ , $\theta + \pi/2$, $\theta + \pi$, and $\theta + 3\pi/2$ by the notation $(\cos \eta)_4$ and $(\cos 2\eta)_4$ respectively, we have

$$\left. \begin{aligned} (\cos \eta)_2 &= \frac{\sin^2 \theta}{2 \cosh^2 \xi_0} \\ (\cos 2\eta)_2 &= \cos 2\theta \\ (\cos \eta)_4 &= \frac{1}{2 \cosh^2 \xi_0} \\ (\cos 2\eta)_4 &= 0 \end{aligned} \right\} \dots\dots\dots (24)$$

Consequently, the mean values $\mathcal{G}_{II}(\alpha, \beta, \theta)$ and $\lambda_{II}(\alpha, \beta, \theta)$ of $\mathcal{G}_{II}(\alpha, \beta, \theta)$ and $\lambda_{II}(\alpha, \beta, \theta)$ for θ , $\theta + \pi$ are dependent on the angle θ , while the mean values $\mathcal{G}_{II4}(\alpha, \beta, \theta)$ and $\lambda_{II4}(\alpha, \beta, \theta)$ for θ , $\theta + \pi/2$, $\theta + \pi$, and $\theta + 3\pi/2$ are independent of the angle θ . From Eq. (20) and Eq. (24), we have

$$\left. \begin{aligned} \mathcal{G}_{II4}(\alpha, \beta, \theta) &= \frac{B_n J_0(q_n \rho)}{E \{ \sinh^2 \xi_0 + \sinh^2 \xi_1 \}} \\ &\times \left\{ \frac{\cosh(\xi_1 - \xi_0)}{\sinh(\xi_1 - \xi_0)} \sinh 2\xi_0 + 1 - \frac{1}{\sinh(\xi_1 - \xi_0)} \right. \\ &\times \left. \frac{\sinh \xi_1}{\cosh \xi_0} \right\} + \frac{1}{2} \left(\frac{c}{\rho} \right) \frac{B_n q_n \rho J_1(q_n \rho)}{E \{ \sinh^2 \xi_0 + \sinh^2 \xi_1 \}} \\ &\times \frac{\sinh 2\xi_0}{\cosh \xi_1} \\ \lambda_{II4}(\alpha, \beta, \theta) &= -\nu \mathcal{G}_{II4}(\alpha, \beta, \theta) \end{aligned} \right\} \dots\dots\dots (25)$$

Of course, $\mathcal{G}_{II4}(\alpha, \beta, \theta)$ and $\lambda_{II4}(\alpha, \beta, \theta)$ slightly differ from values $\mathcal{G}_{II}(\alpha)$ and $\lambda_{II}(\alpha)$ of strain components ε_θ and ε_z in the case of $c=0$ without eccentricity

$$\left. \begin{aligned} \mathcal{G}_{II}(\alpha) &= \frac{2B_n(q_n \rho)^2 J_0(q_n \rho)}{E q_n^2 (a^2 - \rho^2)} \\ \lambda_{II}(\alpha) &= -\nu \mathcal{G}_{II}(\alpha) \end{aligned} \right\} \dots\dots\dots (26)$$

3. Numerical calculation

The values of $\lambda_{II4}(\alpha, \beta, \theta)$ taking β as a parameter and of $\lambda_{II}(\alpha)$ were numerically calculated as functions of α for $m=1, 2$, and 3 . The plots of $\{\lambda_{II4}(\alpha, \beta, \theta) - \lambda_{II}(\alpha)\}$ vs. α for $\beta=0.05, 0.025$, and 0.015 are shown in Fig. 5, and the plots of $\lambda_{II}(\alpha)$ vs. α are shown in Fig. 6. The values of $\{\lambda_{II4}(\alpha, \beta, \theta) - \lambda_{II}(\alpha)\}$ for $\beta=0.005$ are about one-tenth

Fig. 4 Permissible limit of α and β for several $\cosh \xi_1$

of those for $\beta=0.015$ and are not shown in Fig. 5. The inspection of Fig. 5 and Fig. 6 shows that for a given α , $\{\lambda_{I4}(\alpha, \beta, \theta) - \lambda_I(\alpha)\}$ approximately varies with β^2 . This inspection also shows that the variations of the ratios of $\{\lambda_{I4}(\alpha, \beta, \theta) - \lambda_I(\alpha)\}$ to $\lambda_I(\alpha)$ are nearly independent of α for every β and these ratios increase with m . As a result, $\{\lambda_{I4}(\alpha, \beta, \theta) - \lambda_I(\alpha)\}$ are less than 2~3 per cent of $\lambda_I(\alpha)$ for $\beta=0.025$ and $m=3$, and in practice we can neglect the effect of eccentricity for $\beta < 0.01$.

Fig. 5 Variation of $\{\lambda_{I4}(\alpha, \beta, \theta) - \lambda_I(\alpha)\}$ with α for $m=1, 2$, and 3

Fig. 6 Variation of $\lambda_I(\alpha)$ with α for $m=1, 2$, and 3

The values of $\vartheta_{II4}(\alpha, \beta, \theta)$ taking β as a parameter and of $\vartheta_{II}(\alpha)$ also were numerically calculated as functions of α for $m=1, 2$, and 3. The plots of $\{\vartheta_{II4}(\alpha, \beta, \theta) - \vartheta_{II}(\alpha)\}$ vs. α for $\beta=0.05, 0.025, 0.015$, and 0.005 are shown in Fig. 7 and the plots of $\vartheta_{II}(\alpha)$ vs. α are shown in Fig. 8. We temporarily assumed that for ξ_1 being greater than arccosh 10, Eq. (20) holds and determined for every β the upper limit of α below which the curves are shown by the solid lines. $\vartheta_{II4}(\alpha, \beta, \theta)$ varies with β^2 , the ratios of $\{\vartheta_{II4}(\alpha, \beta, \theta) - \vartheta_{II}(\alpha)\}$ to $\vartheta_{II}(\alpha)$ rapidly increase with α , and these ratios increase with n . As a result, $\{\vartheta_{II4}(\alpha, \beta, \theta) - \vartheta_{II}(\alpha)\}$ shown by the solid lines is less than 2~3 per cent of $\vartheta_{II}(\alpha)$ even

Fig. 7 Variation of $\{\vartheta_{II4}(\alpha, \beta, \theta) - \vartheta_{II}(\alpha)\}$ with α for $m=1, 2$, and 3

Fig. 8 Variation of $\vartheta_{II}(\alpha)$ with α for $m=1, 2$, and 3

for $\beta=0.025$ and $m=3$, and in practice we can neglect the effect of eccentricity for $\beta<0.01$.

In general, all residual stress components actually existing in a cylinder have many terms as shown in Eqs. (1). Then, for example, considering the residual stresses

$$\left. \begin{aligned} \sigma_z^*(r) &= \sigma_{z0}^* \{1 - 2(r/a)^2\} = \sum_{m=1}^{\infty} A_m J_0(p_m r) \\ \sigma_r^*(r) &= \sigma_{r0}^* \{1 - (r/a)^2\} = \sum_{n=1}^{\infty} B_n J_0(q_n r) \end{aligned} \right\} \dots\dots\dots (27)$$

we have two sets of the first three values of A_m and of B_n

$$A_1 = 1.35\sigma_{z0}^*, \quad A_2 = -0.542\sigma_{z0}^*, \quad A_3 = 0.309\sigma_{z0}^*$$

$$B_1 = 1.11\sigma_{r0}^*, \quad B_2 = -0.140\sigma_{r0}^*, \quad B_3 = 0.046\sigma_{r0}^*$$

Therefore, to obtain the effect of eccentricity, we must take into consideration the first several terms for $\sigma_z^*(r)$ because A_m 's for $m=1, 2$, and 3 are comparable with each other, while we have only to consider the first term for $\sigma_r^*(r)$ because B_1 is much greater than B_n for $n \geq 2$.

4. Conclusions

A few conclusions obtained from the analysis and the numerical calculation are summarized below:

(1) The mean values of the longitudinal strains and of the tangential ones at two points at regular intervals of angle π on the outer boundary vary with the position of a set of the two points.

(2) The mean values of those at four points at regular intervals of angle $\pi/2$ on the outer boundary are constant regardless of the position of a set of the four points and slightly differ from the values of those without eccentricity. The differences between the mean values of those with and without eccentricity depend on only β and are less than 2~3 per cent of those without eccentricity for $\beta<0.025$. And we can neglect the effect of eccentricity for $\beta<0.01$.

References

- (1) Kawagoe, H., *Bull. JSME*, Vol. 10, No. 40 (1967-8), p. 588, 595.
- (2) Timoshenko, S.P. and Goodier, J.N., *Theory of Elasticity*, 3rd ed., (1970), p. 196, McGraw-Hill.

Discussion

K. HATA (Hokkaido University):

The problems of eigen stresses are difficult and troublesome and, needless to say, the problem of residual stresses is one of the eigen stress problems, therefore, it seems inevitable to us to apply an approximate method suitable to these problems dealt with in this paper, and yet we venture to make a few comments on the author's paper.

(1) In case of axially non-symmetrical problems dealt with by the author, the method in which residual stresses in a cylinder are expanded in series of Bessel functions and for each term of the series the effect of the eccentricity of a bore is separately obtained would not be admissible, even if the residual stress components are axially symmetrical and functions of r only.

(2) In derivation of Eq. (6), the author has used elementary theory of strength of materials. However, it seems evident that, in discussing the bending of bars with cross-sections which are doubly connected, the above treatment would be unsatisfactory.

(3) The author has expressed λ in terms of ϑ with Poisson's ratio ν and vice versa. I would like to know the reason why for example, the expression $\lambda_{II}(\alpha) = -\nu\vartheta_{II}(\alpha)$ is permissible.

Y. MURAKAMI (Ritsumeikan University):

(4) The author has stated that on the outer boundary the conditions $\sigma_\xi = \tau_{\xi\eta} = 0$ should be used and therefore, it is advisable to write the conditions that on the outer boundary $\sigma_\xi \doteq \tau_{\xi\eta} \doteq 0$ in the case of $\rho \gg c$.

S. KODAMA (Tokyo Metropolitan University):

I have two comments to make on the author's paper.

(5) I would like to know the reason why the author assumed that ξ_1 is greater than $\cosh^{-1} 10$ for Eq. (20) holds.

(6) It is evident from Fig. 5 and Fig. 6 that the effects of eccentricity rapidly increase as ρ/a approaches 1. Therefore, it seems to me that when the residual stresses in the range near the outer surface are required, there arise serious problems to be solved.

Authors' closure

(1) As far as the assumption that all deformations produced in the boring-out process are purely elastic, by evaluating the deformations produced in the boring-out process in the case with each term of the residual stresses expanded in series and adding them together, we may obtain total deformation when the residual stresses exist.

(2) It is known that in discussing pure bending of double connected bars, the Bernoulli-Euler law is valid(*1).

(3) For example, we consider the case of the residual stresses $\sigma_r^*(r)$ and $\sigma_\theta^*(r)$ being present. By the relaxation of the residual stresses in the boring-out process, stress components σ_r and σ_θ change and always $\sigma_z=0$. Therefore, on the outer boundary, we have

$$\lambda_{II}(\alpha) = -\nu g_{II}(\alpha)$$

because of $\sigma_r=0$.

(4) The bipolar coordinate system (ξ, η) is

(*1) Muskhelishvili, N.I., *Some Basic Problems of the Mathematical Theory of Elasticity*, (1963), p. 629, p. Noordhoff.

one of the orthogonal coordinate systems and curves $\xi=\text{const.}$ are a family of circles. Therefore, the conditions $\sigma_\xi = \tau_{\xi\eta} = 0$ are exact, because the cylinder under consideration is free from stress on the outer boundary $\xi=\xi_0$.

(5) The reason for this is that the approximation (20) is somewhat good for $\cosh \xi_1=10$.

(6) For the complete determination of the residual stresses in the whole range of α , we should use a combined boring-out and tuning-off method. In the present study, we treated only the effects of eccentricity in the range of applicabilities of the Sachs boring-out method.