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Lateral Vibrations of Cantilevers Considering the Effects of Rotatory Inertia, Shear Flexibility and Solid Viscosity*

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Natural frequencies of short cantilevers with different slenderness ratios are calculated using damped Timoshenko beam equation. The existence of extrema is found in the relation between the frequencies and slenderness ratios, and between the frequencies and the damping coefficients belonging to the first spectrum.

1. Introduction

In the calculation of the natural frequencies of the lateral vibration of a short cantilever such as gear teeth, the effects of shear flexibility and rotatory inertia can not be overlooked. Several studies⁽¹⁾⁽²⁾⁽³⁾ have been made on this problem but there seems to be some questions about the selection of the shear coefficient and the approximation of the boundary condition and so on in them.

Examining those questions minutely and taking account of the effect of solid viscosity, the natural frequencies and the damping coefficients of short cantilevers having different slenderness ratios and viscous coefficients are given. The results are tabulated for practical use.

2. Fundamental equation of motion

The equations of motion obtained from Fig. 2 are

$$p + \frac{\partial v}{\partial x} = \rho A \frac{\partial^2 y}{\partial t^2} + c_1 \frac{\partial y}{\partial t} \quad (1)$$

$$v + \frac{\partial m}{\partial x} = \rho I \frac{\partial^2 \psi}{\partial t^2} + c_2 \frac{\partial \psi}{\partial t} \quad (2)$$

$$m = EI \left(1 + \varepsilon^* \frac{\partial}{\partial t} \right) \frac{\partial \psi}{\partial x} \quad (3)$$

$$v = kAG \left(1 + \varepsilon \frac{\partial}{\partial t} \right) \left(\frac{\partial y}{\partial x} - \psi \right) \quad (4)$$

where the dynamic model of solid viscosity is assumed to be Voigt-type which is most fundamental. There are some discussions⁽⁶⁾ that a three elements model is more reasonable than a Voigt model as the dynamic model for the lateral vibration of Bernoulli-Euler beam. But it is difficult to evaluate the elastic stiffness in the three elements model and there is no particular

difference between the two models in mathematical treatment as far as the natural frequencies are concerned, so the Voigt model is chosen here†.

In the above equations

y : lateral displacement cm

x : length coordinate cm

c_1, c_2 : transverse and rotatory external viscous dampings, respectively, kg sec/cm², kg sec

$\varepsilon^*, \varepsilon$: viscous coefficient sec

l : length of beam cm

p : distributed load per unit length kg/cm

ρ : mass density of beam material kg sec²/cm⁴

A : cross sectional area cm²

I : cross sectional moment of inertia, cm⁴

E, G : Young's modulus and rigidity, respectively kg/cm²

k : Timoshenko's shear modulus

Fig. 1 Three elements model

† In the case of a three elements model, the fundamental equations are derived as follows:

$$\left(1 + \frac{G_1}{G_2} + \frac{\eta_2}{kG_2} \frac{\partial}{\partial t} \right) p + \left(kAG_1 + A\eta_2 \frac{\partial}{\partial t} \right) \left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) = \left(1 + \frac{G_1}{G_2} + \frac{\eta_2}{kG_2} \frac{\partial}{\partial t} \right) \left(\rho A \frac{\partial^2}{\partial t^2} + c_1 \frac{\partial}{\partial t} \right) y \quad (5)'$$

$$\begin{aligned} & \left(\frac{1}{I} + \frac{E_1}{E_2 I} + \frac{\eta_1}{E_2 I} \frac{\partial}{\partial t} \right) \left(kAG_1 + A\eta_2 \frac{\partial}{\partial t} \right) \left(\frac{\partial y}{\partial x} - \psi \right) \\ & + \left(1 + \frac{G_1}{G_2} + \frac{\eta_2}{kG_2} \frac{\partial}{\partial t} \right) \left(E_1 + \eta_1 \frac{\partial}{\partial t} \right) \frac{\partial^2 \psi}{\partial x^2} \\ & = \left(\rho I \frac{\partial^2}{\partial t^2} + c_2 \frac{\partial}{\partial t} \right) \left(\frac{1}{I} + \frac{E_1}{E_2 I} + \frac{\eta_1}{E_2 I} \frac{\partial}{\partial t} \right) \\ & \times \left(1 + \frac{G_1}{G_2} + \frac{\eta_2}{kG_2} \frac{\partial}{\partial t} \right) \psi \quad (6)' \end{aligned}$$

Dynamic model in the form of a three elements model is shown in Fig. 1.

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r : cross sectional radius of gyration cm

a : phase velocity of longitudinal waves in bars, $=\sqrt{E/\rho}$ cm/sec

λ : slenderness ratio, $=l/r$

From Eq. 1 to Eq. 4, the following equations are obtained:

$$p+kAG\left(1+\varepsilon\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 y}{\partial x^2}-\frac{\partial \phi}{\partial x}\right) \\ =\rho A\frac{\partial^2 y}{\partial t^2}+c_1\frac{\partial y}{\partial t} \dots\dots\dots(5)$$

$$EI\left(1+\varepsilon^*\frac{\partial}{\partial t}\right)\frac{\partial^2 \phi}{\partial x^2}+kAG\left(1+\varepsilon\frac{\partial}{\partial t}\right) \\ \times\left(\frac{\partial y}{\partial x}-\phi\right)=\rho I\frac{\partial^2 \phi}{\partial t^2}+c_2\frac{\partial \phi}{\partial t} \dots\dots\dots(6)$$

As special cases, the equations of motion for shear beam and for Rayleigh beam are obtained by ignoring the term of rotatory inertia and the term of shear flexibility, respectively. Bernoulli-Euler model is derived by omitting both the terms of rotatory inertia and shear effect.

Eliminating the viscous damping forces proportional to shear strain rate and to rotatory angular velocity in Eq. 5 and Eq. 6, non-dimensional equations of motion are obtained as follows:

$$\bar{p}+\frac{1}{\gamma}\left(\frac{\partial^2 \bar{y}}{\partial \bar{x}^2}-\frac{\partial \bar{\phi}}{\partial \bar{x}}\right)-\left(\frac{\partial^2 \bar{y}}{\partial \bar{t}^2}+\bar{\beta}\frac{\partial \bar{y}}{\partial \bar{t}}\right)=0 \dots\dots(7)$$

$$\left(1+\bar{\varepsilon}\frac{\partial}{\partial \bar{t}}\right)\frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2}+\frac{\lambda^2}{\gamma}\left(\frac{\partial \bar{y}}{\partial \bar{x}}-\bar{\phi}\right)-\frac{\partial^2 \bar{\phi}}{\partial \bar{t}^2}=0 \dots\dots(8)$$

where

Fig. 2 Flexural vibration of cantilever

$$\bar{x}=x/l, \quad \bar{y}=y/l, \quad \bar{t}=at/l, \quad \gamma=E/kG, \\ \bar{\varepsilon}=\varepsilon^*a/l, \quad \bar{\beta}=lc_1/\rho aA, \quad \bar{p}=pl/AE$$

3. Frequency equation and formal solution

The problem of free vibration is dealt with in this paper, but from the mathematical point of view, it can be treated in the same manner as one of forced vibration. So, it is treated as a problem of forced vibration formally.

Assuming that the beam is not subjected to any distributed load and is excited by displacement only at its fixed end, the boundary and initial conditions are represented in the following nondimensional form

$$\bar{x}=1: \bar{y}=q(\bar{t}), \quad \bar{\phi}=0 \dots\dots\dots(9)$$

$$\bar{x}=0: \frac{\partial \bar{\phi}}{\partial \bar{x}}=0, \quad \frac{\partial \bar{y}}{\partial \bar{x}}-\bar{\phi}=0 \dots\dots\dots(10)$$

$$\bar{t}=0: \bar{y}=0, \quad \bar{\phi}=0 \dots\dots\dots(11)$$

$$\frac{\partial \bar{y}}{\partial \bar{t}}=0, \quad \frac{\partial \bar{\phi}}{\partial \bar{t}}=0 \dots\dots\dots(12)$$

where $q(\bar{t})$ means the exciting displacement.

From the Laplace transforms of Eq. 7 to Eq. 10 using the initial conditions Eq. 11 and Eq. 12, the equations are converted to the following equations in $Y(\bar{x}, s)$:

$$(1+\bar{\varepsilon}s)\frac{d^4 Y}{d\bar{x}^4}-\{\gamma(s^2+\bar{\beta}s)(\bar{\varepsilon}s+1)+s^2\} \\ \times \frac{d^2 Y}{d\bar{x}^2}+(s^2+\bar{\beta}s)(\gamma s^2+\lambda^2)Y=0 \dots\dots\dots(13)$$

The boundary conditions are

$$\bar{x}=1: Y(\bar{x}, s)=Q(s) \dots\dots\dots(14)$$

$$\frac{d^3 Y}{d\bar{x}^3}-\frac{\gamma^2(1+\bar{\varepsilon}s)(s^2+\bar{\beta}s)-\lambda^2}{\gamma(1+\bar{\varepsilon}s)}\frac{dY}{d\bar{x}}=0 \dots\dots\dots(15)$$

$$\bar{x}=0: \frac{d^2 Y}{d\bar{x}^2}-\gamma(s^2+\bar{\beta}s)Y=0 \dots\dots\dots(16)$$

$$\frac{d^3 Y}{d\bar{x}^3}-\frac{\gamma s(s+\bar{\beta})(\bar{\varepsilon}s+1)+s^2}{1+\bar{\varepsilon}s}\frac{dY}{d\bar{x}}=0 \dots\dots\dots(17)$$

where

$$Y(\bar{x}, s)=\int_0^\infty \bar{y}(\bar{x}, \bar{t})e^{-s\bar{t}}d\bar{t}$$

$$Q(s)=\int_0^\infty q(\bar{t})e^{-s\bar{t}}d\bar{t}$$

Let the solution of Eq. 13 be $Y=Ae^{m\bar{x}}$. Substitution of this solution into Eq. 13 leads to the following algebraic equations in m :

$$(1+\bar{\varepsilon}s)m^4-\{\gamma s(s+\bar{\beta})(\bar{\varepsilon}s+1)+s^2\}m^2 \\ +s(s+\bar{\beta})(\gamma s^2+\lambda^2)=0 \dots\dots\dots(18)$$

The roots of Eq. 18 are given by the following m_1 and m_2

$$m_1=[g_1(s)-\{g_2(s)\}^{1/2}]^{1/2} \dots\dots\dots(19)$$

$$m_2=[g_1(s)+\{g_2(s)\}^{1/2}]^{1/2} \dots\dots\dots(20)$$

where

$$g_1(s)=\frac{\gamma s(s+\bar{\beta})(\bar{\varepsilon}s+1)+s^2}{2(1+\bar{\varepsilon}s)} \dots\dots\dots(21)$$

$$g_2(s) = \frac{[\gamma s(s + \bar{\beta})(\bar{\epsilon}s + 1) + s^2]^2 - 4s(s + \bar{\beta})(\gamma s^2 + \lambda^2)(\bar{\epsilon}s + 1)}{4(1 + \bar{\epsilon}s)^2} \quad (22)$$

The solution is written in the form

$$Y = A_1 e^{m_1 \bar{x}} + A_2 e^{-m_1 \bar{x}} + A_3 e^{m_2 \bar{x}} + A_4 e^{-m_2 \bar{x}} \quad (23)$$

From the boundary conditions Eq. 14 to Eq. 17, the constants A_1 to A_4 in Eq. 24 are determined. Then the Laplace transform of the solution is obtained as follows:

$$Y(\bar{x}, s) = \frac{X(\bar{x}, s)}{C(s)} Q(s) \quad (24)$$

where

$$C(s) = -m_1^2(m_1^2 - F)(m_2^2 - G) - m_2^2(m_2^2 - F) \times (m_1^2 - G) + \{m_2^2(m_2^2 - F)(m_2^2 - G) + m_1^2(m_1^2 - F)(m_1^2 - G)\} \cosh m_1 \cosh m_2 - m_1 m_2 \{ (m_2^2 - F)(m_1^2 - G) + (m_1^2 - F)(m_2^2 - G) \} \sinh m_1 \sinh m_2 \quad (25)$$

$$X(\bar{x}, s) = -m_1^2(m_2^2 - G)(m_1^2 - F) \cosh(1 - \bar{x}) m_1 - m_2^2(m_1^2 - G)(m_2^2 - F) \cosh(1 - \bar{x}) m_2 + m_2^2(m_2^2 - G)(m_2^2 - F) \cosh m_2 \cosh m_1 \bar{x} + m_1^2(m_1^2 - G)(m_1^2 - F) \cosh m_1 \cosh m_2 \bar{x} - m_1 m_2 \{ (m_1^2 - G)(m_2^2 - F) \sinh m_2 \sinh m_1 \bar{x} + (m_2^2 - G)(m_1^2 - F) \sinh m_1 \sinh m_2 \bar{x} \} \quad (26)$$

$$F = \frac{\gamma^2(1 + \bar{\epsilon}s)(s^2 + \bar{\beta}s) - \lambda^2}{\gamma(1 + \bar{\epsilon}s)} \quad (27)$$

$$G = \gamma(s^2 + \bar{\beta}s) \quad (28)$$

Consequently, the frequency equation takes the following form using Eqs. 18, 27, 28 and 25:

$$(m_1^2 - G)(m_2^2 - G) [2m_1 m_2 + (m_1^2 + m_2^2) \times \sinh m_1 \sinh m_2] - m_1 m_2 \{ (m_1^2 - G)^2 + (m_2^2 - G)^2 \} \cosh m_1 \cosh m_2 = 0 \quad (29)$$

The values of s which satisfy Eq. 29 give the damping coefficients and frequency; that is, putting $s = -\xi_k \pm i\eta_k$, ξ_k and η_k are the values of damping coefficient and frequency, respectively.

As $X(\bar{x}, s)/C(s)$ is analytic and $O(s^{-k})$, $k=0$, and $Q(s)$ is ordinarily analytic and $O(s^{-k})$, $k>1$, the inverse transform of Y exists uniquely and is given by the following form:

$$\bar{y}(\bar{x}, \bar{t}) = \sum_{j=1}^n \frac{X(\bar{x}, a_j) P(a_j)}{C(a_j) R'(a_j)} \exp(a_j \bar{t}) + 2 \sum_{k=1}^{\infty} \frac{\exp(-\xi_k \bar{t})}{\sqrt{U_k^2 + V_k^2}} \{ \Phi_k(\bar{x}) \sin(\eta_k \bar{t} + \alpha_k) + \Psi_k(\bar{x}) \cos(\eta_k \bar{t} + \alpha_k) \} \quad (30)$$

where $Q(s) = P(s)/R(s)$; and $R(s) = 0$ has simple roots a_i .

$$\left. \begin{aligned} R'(s) &= \frac{dR(s)}{ds}, \quad C'(s) = \frac{dC(s)}{ds} \\ X(\bar{x}, s) P(s) \Big|_{s=-\xi_k \pm i\eta_k} &= \Phi_k(\bar{x}) \pm i\Psi_k(\bar{x}) \\ C'(s) R(s) \Big|_{s=-\xi_k \pm i\eta_k} &= U_k \pm iV_k \\ \alpha_k &= \tan^{-1} \left(\frac{U_k}{V_k} \right), \quad i = \sqrt{-1} \end{aligned} \right\} \quad (31)$$

4. Frequency equation for special cases

4.1 Case of neglecting the rotatory inertia and shear effect

Considering the limit of $\gamma \rightarrow 0$ and $1/\lambda \rightarrow 0$ in Eqs. 19, 20 and 29, Eq. 19 and Eq. 20 satisfy the following equation

$$m_1^4 = m_2^4 = -\frac{\lambda^2(s^2 + \bar{\beta}s)}{1 + \bar{\epsilon}s} (\equiv \mu^4) \quad (32)$$

and Eq. 29 satisfies the following equation for Bernoulli-Euler beam:

$$1 + \cosh \mu \cos \mu = 0 \quad (33)$$

4.2 Case of neglecting damping forces

Taking the limit $\bar{\beta} \rightarrow 0$ and $\bar{\epsilon} \rightarrow 0$ in Eqs. 19, 20 and 29, and putting $s = i\eta$, the frequency equation is simplified as follows

$$\eta^2 < \lambda^2/\gamma$$

$$2 - \frac{(\gamma+1)\eta^2}{\sqrt{\lambda^2\eta^2 - \gamma\eta^4}} \sin \mu_1 \sinh \mu_2 + \left[2 + \frac{(\gamma-1)^2}{\lambda^2} \eta^2 \right] \cos \mu_1 \cosh \mu_2 = 0 \quad (34)$$

$$\mu_1 = \sqrt{\frac{(\gamma+1)}{2} \eta^2 + \sqrt{\frac{(\gamma-1)^2 \eta^4 + 4\lambda^2 \eta^2}{4}}} \quad (35)$$

$$\mu_2 = \sqrt{-\frac{(\gamma+1)}{2} \eta^2 + \sqrt{\frac{(\gamma-1)^2 \eta^4 + 4\lambda^2 \eta^2}{4}}} \quad (36)$$

$$\eta^2 > \lambda^2/\gamma$$

$$2 - \frac{(\gamma+1)\eta^2}{\sqrt{\gamma\eta^4 - \lambda^2\eta^2}} \sin \mu_1 \sin \mu_2' + \left[2 + \frac{(\gamma-1)^2}{\lambda^2} \eta^2 \right] \cos \mu_1 \cos \mu_2' = 0 \quad (37)$$

$$\mu_2' = \sqrt{\frac{(\gamma+1)}{2} \eta^2 - \sqrt{\frac{(\gamma-1)^2 \eta^4 + 4\lambda^2 \eta^2}{4}}} \quad (38)$$

which is obviously a frequency equation for Timoshenko beam. In order to examine the effect of rotatory inertia and shear flexibility, let's put $b^2 = \lambda^2 \eta^2$ in Eqs. 34 to 36. Then one obtains the following equations

$$2 - \frac{b^2(\gamma/\lambda^2 + 1/\lambda^2)}{\sqrt{b^2 - (\gamma/\lambda^2)(1/\lambda^2)b^4}} \sin \mu_1 \sinh \mu_2 + \left[2 + \left(\frac{\gamma}{\lambda^2} - \frac{1}{\lambda^2} \right) b^2 \right] \cos \mu_1 \cosh \mu_2 = 0 \quad (34)'$$

$$\mu_1 = \sqrt{\frac{1}{2} \left[\left(\frac{\gamma}{\lambda^2} + \frac{1}{\lambda^2} \right) b^2 + \sqrt{\left(\frac{\gamma}{\lambda^2} - \frac{1}{\lambda^2} \right)^2 b^4 + 4b^2} \right]} \quad (35)'$$

$$\mu_2 = \sqrt{\frac{1}{2} \left[-\left(\frac{\gamma}{\lambda^2} + \frac{1}{\lambda^2} \right) b^2 + \sqrt{\left(\frac{\gamma}{\lambda^2} - \frac{1}{\lambda^2} \right)^2 b^4 + 4b^2} \right]} \quad (36)'$$

Considering the limit $1/\lambda^2 \rightarrow 0$ in the above equations, the frequency equation and eigen value for shear beam are obtained; and the limit $\gamma/\lambda^2 \rightarrow 0$ for Rayleigh beam and the limit $1/\lambda^2 \rightarrow 0$, $\gamma/\lambda^2 \rightarrow 0$ for Bernoulli-Euler beam, respectively. For example, in the case of shear beam, after setting the limit $1/\lambda^2 \rightarrow 0$, the following equations are obtained using variable η

$$2 - \frac{\gamma\eta}{\lambda} \sin \mu_1 \sin \mu_2 + \left[2 + \frac{\gamma^2}{\lambda^2} \eta^2 \right] \cos \mu_1 \cosh \mu_2 = 0 \quad \dots\dots\dots (34)''$$

$$\mu_1 = \sqrt{\frac{1}{2} [\gamma\eta^2 + \sqrt{\gamma^2\eta^4 + 4\lambda^2\eta^2}]} \quad \dots\dots\dots (35)''$$

$$\mu_2 = \sqrt{\frac{1}{2} [-\gamma\eta^2 + \sqrt{\gamma^2\eta^4 + 4\lambda^2\eta^2}]} \quad \dots\dots\dots (36)''$$

It is already shown in Chapter 3 that the consideration of the effect of rotatory inertia and shear influences the order of time derivatives in the fundamental equations. Now, comparing Eq. 34 to Eq. 36 with Eq. 34'' to Eq. 36'', those effects are exhibited as the change in the form of frequency equation and eigen value function according to whether η exceeds $\lambda/\sqrt{\gamma}$ or not*. Furthermore, whether those effects are considered or not also decides

Fig. 3 Eigen values for different slenderness ratios

- (a) Phase velocity belonging to the second spectrum ($a\eta/\mu_2'c_s$)
 (b) Phase velocity for pure rotatory wave $\sqrt{\gamma}/\sqrt{1-(1/Y)^2}$
 (c) Phase velocity corresponding to the first spectrum ($a\eta/\mu_1c_s$), ($c_s = \sqrt{kG/\rho}$, $Y = \eta\sqrt{\gamma}/\lambda$)

Fig. 4 Variation of phase velocity with frequency

* R.W. Trail-Nash shows that there is a possibility of some singular motion occurring at the frequency $\eta = \lambda/\sqrt{\gamma}$ in simple supported beams.

the character of η itself; that is, η has only one character or two. H. Schirmer⁽⁷⁾ examined these two characters of motion and showed the dependencies of phase velocities of the two on frequency as shown by curves (a) and (c) in Fig. 4. According to Fig. 4, (c) approaches the phase velocity of shear wave and (a) and (b) approach that of moment wave with an increasing frequency. Moreover, (a) and (b) never lie below the value of $\eta = \lambda/\sqrt{\gamma}$. For later discussion let's denote these two phase velocities as belonging to the first and the second spectrum, respectively.

5. Result of calculation

The eigen values for $\bar{\epsilon} = 0$ and $\bar{\beta} = 0$ are shown as a function of $1/\lambda$ in Fig. 3. Numerical results of the eigen value and the damping coefficient are tabulated in Table 1 for practical use. As for higher modes, those eigen values larger than $\lambda/\sqrt{\gamma}$ are not shown in Fig. 3; for they are not useful for practical purpose and it is difficult to distinguish to which

Fig. 5 Example of roots of frequency equation ($\lambda=50$) Fig. 6 Example of roots of frequency equation ($\lambda=25$)

spectrum they belong.

The eigen values for shear beam are shown by a broken line in Fig. 3. It seems that the natural frequency increases with an increasing λ . On the

other hand, the natural frequency has extrema in the neighbourhood of $\lambda/\sqrt{\gamma}$ in the case of considering the effects of rotatory inertia and shear flexibility. Such property was already reported by T. C. Huang, whose

Table 1 Eigen values and damping coefficients (Part 1)

 $(s = -\xi \pm i\eta)$

$1/\lambda$	Order of mode		$\bar{\epsilon}$						
			0	0.1	0.2	0.4	0.6	0.8	1.0
0.02	1	ξ η	0 0.070 07	0.000 24 0.070 05	0.000 48 0.070 05	0.000 97 0.070 04	0.001 46 0.070 04	0.001 94 0.070 02	0.002 43 0.070 01
	2	ξ η	0 0.429 72	0.008 88 0.429 62	0.017 78 0.429 39	0.035 60 0.428 45	0.053 46 0.426 88	0.071 41 0.424 65	0.089 48 0.421 74
	3	ξ η	0 1.164 89	0.062 38 1.163 80	0.125 12 1.160 53	0.253 12 1.147 01	0.387 33 1.122 62	0.532 25 1.083 83	0.694 89 1.023 27
	4	ξ η	0 2.186 51	0.207 50 2.182 88	0.421 17 2.171 59	0.900 47 2.119 18	1.580 74 1.986 83	3.582 85 2.883 37	2.842 56 4.326 84
	5	ξ η	0 3.439 19	0.483 80 3.438 16	1.014 35 3.437 10	2.750 55 3.627 53	3.000 81 5.991 24	2.517 99 6.630 66	2.219 60 7.046 73
	6	ξ η	0 4.868 40	0.916 74 4.900 19	2.047 50 5.071 05	3.203 55 7.705 11	2.518 46 8.806 30	1.979 16 9.254 55	1.618 50 9.459 04
	7	ξ η	0 6.430 06	1.517 51 6.581 48	3.363 25 7.758 37	2.708 46 10.567 06	1.892 32 11.170 97	1.450 48 11.375 51	1.180 53 11.472 85
	8	ξ η	0 8.089 94	2.263 01 8.553 81	3.521 99 11.057 92	2.101 89 12.886 90	1.445 00 13.203 58	1.107 93 13.320 30	0.902 57 13.379 61
	9	ξ η	0 9.822 28	3.014 22 10.933 98	3.058 92 13.867 82	1.660 09 14.929 57	1.140 87 15.124 05	0.876 60 15.199 29	0.716 54 15.238 89
	10	ξ η	0 11.607 65	3.459 15 13.687 44	2.528 82 16.198 14	1.348 09 16.864 49	0.929 10 16.999 77	0.714 52 17.055 32	0.583 38 17.085 28
	11	ξ η	0 13.431 45	3.421 03 16.467 03	2.106 20 18.288 65	1.120 21 18.748 16	0.775 22 18.847 50	0.599 72 18.890 30	0.492 77 18.914 95
	12	ξ η	0 15.282 47	3.101 06 18.977 20	1.782 81 20.262 92				
	13	ξ η	0 17.151 93	2.735 27 21.230 40					
	14	ξ η	0 19.032 63						
0.04	1	ξ η	0 0.138 61	0.000 93 0.138 59	0.001 87 0.138 58	0.003 75 0.138 55	0.005 63 0.138 49	0.007 51 0.138 41	0.009 39 0.138 31
	2	ξ η	0 0.802 65	0.028 09 0.802 43	0.056 29 0.801 83	0.113 45 0.799 37	0.172 40 0.795 08	0.234 30 0.788 68	0.300 68 0.779 68
	3	ξ η	0 2.032 16	0.157 80 2.033 78	0.320 61 2.039 10	0.688 26 2.070 69	1.197 32 2.217 84	1.519 34 2.878 48	1.358 47 3.263 65
	4	ξ η	0 3.551 98	0.419 59 3.580 09	0.871 91 3.686 60	1.622 05 4.512 46	1.521 72 5.253 22	1.291 79 5.650 12	1.079 05 5.848 12
	5	ξ η	0 5.243 45	0.797 72 5.375 64	1.542 36 5.925 92	1.503 19 7.293 55	1.093 47 7.689 79	0.858 27 7.832 96	0.710 50 7.906 28
	6	ξ η	0 7.027 65	1.219 40 7.414 45	1.708 78 8.595 36	1.073 53 9.499 95	0.752 13 9.676 45	0.583 66 9.742 50	0.481 45 9.775 70
	7	ξ η	0 8.859 50	1.528 64 9.685 78	1.400 90 10.930 50	0.826 61 11.429 88	0.592 11 11.549 13	0.467 81 11.598 59	0.392 80 11.625 63
	8	ξ η	0 10.704 77	1.539 84 12.009 86	1.074 11 12.930 48	0.613 07 13.190 39	0.480 62 13.247 43	0.433 07 13.289 01	0.413 00 13.333 07
	9	ξ η	0 12.529 13	1.294 47 14.104 55	0.499 78 14.539 24	0.447 67 14.531 31	0.501 97 14.535 33	0.605 51 14.527 75	0.746 03 14.507 94
	10	ξ η	0 14.206 02	0.382 29 14.708 89					
	11	ξ η	0 14.957 91						

Table 1 (Part 2)

$1/\lambda$	Order of mode		$\bar{\epsilon}$						
			0	0.1	0.2	0.4	0.6	0.8	1.0
0.06	1	ξ/η	0 0.204 31	0.001 98 0.204 28	0.003 97 0.204 25	0.007 95 0.204 16	0.011 94 0.204 01	0.015 93 0.203 79	0.019 93 0.203 50
	2	ξ/η	0 1.093 85	0.045 64 1.094 09	0.091 71 1.094 85	0.186 90 1.098 22	0.289 90 1.105 24	0.407 34 1.119 44	0.550 84 1.151 68
	3	ξ/η	0 2.593 23	0.210 61 2.605 29	0.427 80 2.645 76	0.870 89 2.888 44	1.046 57 3.329 84	1.039 61 3.632 65	0.990 31 3.869 05
	4	ξ/η	0 4.292 02	0.472 83 4.368 50	0.910 32 4.638 17	1.102 16 5.499 74	0.834 53 5.864 14	0.657 52 5.987 46	0.548 77 6.044 91
	5	ξ/η	0 6.090 58	0.774 63 6.337 98	1.134 68 7.034 33	0.804 49 7.707 60	0.589 22 7.862 01	0.478 94 7.923 38	0.415 33 7.960 80
	6	ξ/η	0 7.882 36	0.902 09 8.433 43	0.756 66 9.116 18	0.505 15 9.269 45	0.505 20 9.306 86	0.572 65 9.343 66	0.685 55 9.390 64
	7	ξ/η	0 9.566 44	0.341 16 10.036 28	0.448 93 10.218 99	0.399 55 10.392 74	0.400 95 10.391 70	0.428 78 10.344 22	0.454 05 10.266 12
	8	ξ/η	0 10.200 81						
0.08	1	ξ/η	0 0.266 12	0.003 25 0.266 11	0.006 52 0.266 07	0.013 05 0.265 91	0.019 59 0.265 65	0.026 16 0.265 28	0.032 76 0.264 80
	2	ξ/η	0 1.307 47	0.056 45 1.308 77	0.113 52 1.312 81	0.231 92 1.331 02	0.359 70 1.370 56	0.493 35 1.453 50	0.584 84 1.601 33
	3	ξ/η	0 2.950 14	0.230 37 2.972 40	0.459 05 3.045 66	0.816 89 3.379 23	0.905 00 3.785 63	0.799 99 4.071 64	0.673 21 4.209 11
	4	ξ/η	0 4.688 69	0.450 80 4.796 90	0.779 81 5.134 68	0.697 08 5.725 98	0.551 69 5.877 94	0.489 96 5.945 18	0.464 64 5.996 43
	5	ξ/η	0 6.458 69	0.607 84 6.760 00	0.563 55 7.279 37	0.439 73 7.350 28	0.495 34 7.361 24	0.601 44 7.353 22	0.742 47 7.324 65
	6	ξ/η	0 7.619 61	0.323 78 7.716 89	0.660 61 8.022 95	0.408 30 8.338 00	0.338 57 8.345 81	0.308 94 8.319 68	0.285 19 8.285 39
	7	ξ/η	0 8.607 80	0.829 73 8.820 99					
0.1	1	ξ/η	0 0.323 46	0.004 61 0.323 43	0.009 24 0.323 39	0.018 52 0.323 20	0.027 85 0.322 88	0.037 23 0.322 44	0.046 71 0.321 86
	2	ξ/η	0 1.459 12	0.061 28 1.461 57	0.122 90 1.469 17	0.247 56 1.503 39	0.367 28 1.574 47	0.449 43 1.687 54	0.469 56 1.797 61
	3	ξ/η	0 3.183 50	0.236 53 3.212 99	0.463 04 3.307 13	0.765 20 3.694 71	0.704 92 4.070 88	0.577 10 4.220 43	0.494 83 4.291 06
	4	ξ/η	0 4.854 06	0.369 45 4.973 31	0.539 73 5.261 95	0.488 55 5.515 53	0.514 42 5.570 97	0.599 29 5.597 69	0.729 15 5.613 15
	5	ξ/η	0 6.339 59	0.228 15 6.440 74	0.327 03 6.580 50	0.332 96 6.710 01	0.344 61 6.706 02	0.366 21 6.659 90	0.377 21 6.592 91
	6	ξ/η	0 6.890 75						
0.12	1	ξ/η	0 0.375 98	0.005 96 0.375 96	0.011 94 0.375 93	0.023 93 0.375 79	0.036 01 0.375 55	0.048 24 0.375 21	0.060 65 0.374 76
	2	ξ/η	0 1.564 87	0.062 60 1.568 12	0.124 96 1.578 18	0.245 73 1.621 99	0.345 11 1.701 30	0.398 33 1.794 97	0.417 39 1.868 17
	3	ξ/η	0 3.339 10	0.236 20 3.374 25	0.455 34 3.485 78	0.650 05 3.904 30	0.550 75 4.130 06	0.499 21 4.223 84	0.480 47 4.304 59
	4	ξ/η	0 4.787 59	0.237 72 4.871 69	0.336 49 5.007 34	0.404 67 5.115 25	0.480 41 5.118 52	0.579 70 5.068 96	0.702 97 4.972 94
	5	ξ/η	0 5.960 53	0.313 05 6.085 82	0.366 98 6.354 29	0.259 43 6.469 73	0.220 31 6.500 13	0.203 77 6.487 82	0.192 79 6.469 18
	6	ξ/η	0 6.660 23						

result is shown by a two-dotted line in Fig. 3. It may be difficult to explain this phenomenon quantitatively from the physical point of view. Probably, it may be due to the fact that the motion comes to be strongly influenced by the rotatory inertia near and above the frequency of $\lambda/\sqrt{\gamma}$ as shown in Fig. 4 and to the peculiarity of nonsymmetric boundary condition of cantilever. (In the case of a simple supported beam or a free-free bar, it is easy to distinguish to which spectrum a vibration belongs⁽⁵⁾.)

The results presented here fail to agree with T. C. Huang's results at higher mode and at small values of slenderness ratios; which seems to be due to

the difference in the value of Timoshenko's shear modulus employed in the calculation. In this paper k is taken as 0.85 which is found to be reasonable experimentally and theoretically⁽⁴⁾. The value corresponds to a rectangular beam of those materials whose Poisson's ratio is 0.3.

Approximate solutions were given by one of the authors twenty years ago⁽²⁾, and his results show considerably good agreement with the present results in the range of $1/\lambda < 0.12$ for fundamental mode and of $1/\lambda < 0.1$ for second order mode.

Some examples for the calculated roots of the frequency equation are plotted on s -plane in Figs. 5

Table 1 (Part 3)

1/λ	Order of mode		ε						
			0	0.1	0.2	0.4	0.6	0.8	1.0
0.14	1	$\frac{\xi}{\eta}$	0 0.423 70	0.007 20 0.423 68	0.014 43 0.423 68	0.028 95 0.423 69	0.043 63 0.423 70	0.058 55 0.423 71	0.073 82 0.423 74
	2	$\frac{\xi}{\eta}$	0 1.637 52	0.062 51 1.641 13	0.124 20 1.652 18	0.239 30 1.697 97	0.326 97 1.769 97	0.378 70 1.842 13	0.413 18 1.896 77
	3	$\frac{\xi}{\eta}$	0 3.438 98	0.227 37 3.479 19	0.424 16 3.606 12	0.505 55 3.937 78	0.493 79 4.041 47	0.558 01 4.095 62	0.697 50 4.130 62
	4	$\frac{\xi}{\eta}$	0 4.557 00	0.176 13 4.600 45	0.291 55 4.685 23	0.365 02 4.812 95	0.385 62 4.809 48	0.402 26 4.741 82	0.379 78 4.648 16
	5	$\frac{\xi}{\eta}$	0 5.823 03	0.389 91 6.050 80					
0.16	1	$\frac{\xi}{\eta}$	0 0.466 75	0.008 30 0.466 76	0.016 62 0.466 82	0.033 37 0.467 08	0.050 36 0.467 54	0.067 72 0.468 21	0.085 61 0.469 13
	2	$\frac{\xi}{\eta}$	0 1.686 49	0.062 29 1.690 09	0.123 41 1.700 95	0.235 53 1.743 73	0.322 18 1.804 37	0.384 72 1.861 20	0.440 40 1.906 02
	3	$\frac{\xi}{\eta}$	0 3.489 76	0.203 20 3.533 08	0.350 36 3.657 27	0.402 98 3.831 01	0.473 22 3.857 86	0.588 79 3.842 76	0.730 08 3.788 13
	4	$\frac{\xi}{\eta}$	0 4.326 00	0.181 43 4.361 69	0.325 78 4.473 06	0.323 12 4.663 50	0.287 81 4.670 50	0.263 10 4.640 42	0.235 01 4.608 78
	5	$\frac{\xi}{\eta}$	0 5.791 09	0.363 37 6.159 10					
0.18	1	$\frac{\xi}{\eta}$	0 0.505 46	0.009 22 0.505 49	0.018 47 0.505 64	0.037 11 0.506 27	0.056 05 0.507 36	0.075 50 0.508 98	0.095 65 0.511 21
	2	$\frac{\xi}{\eta}$	0 1.718 54	0.062 54 1.721 86	0.123 82 1.731 78	0.236 65 1.769 21	0.329 47 1.818 80	0.408 67 1.864 31	0.490 39 1.902 11
	3	$\frac{\xi}{\eta}$	0 3.489 60	0.161 46 3.528 19	0.258 33 3.610 57	0.343 18 3.676 10	0.444 58 3.661 58	0.562 41 3.608 03	0.683 45 3.516 34
	4	$\frac{\xi}{\eta}$	0 4.168 34	0.218 54 4.216 81	0.370 93 4.406 45	0.273 58 4.602 52	0.220 46 4.614 27	0.191 14 4.602 29	0.168 61 4.587 69
	5	$\frac{\xi}{\eta}$	0 5.758 67	0.788 63 5.760 12					
0.2	1	$\frac{\xi}{\eta}$	0 0.540 15	0.009 97 0.540 23	0.019 97 0.540 50	0.040 12 0.541 58	0.060 65 0.543 47	0.081 77 0.546 30	0.103 69 0.550 26
	2	$\frac{\xi}{\eta}$	0 1.738 44	0.063 45 1.741 34	0.125 76 1.749 94	0.242 40 1.781 22	0.345 61 1.820 68	0.444 81 1.856 07	0.557 28 1.884 62
	3	$\frac{\xi}{\eta}$	0 3.442 96	0.120 52 3.467 73	0.198 61 3.508 35	0.303 42 3.527 39	0.408 90 3.491 20	0.514 76 3.417 62	0.608 82 3.307 00
	4	$\frac{\xi}{\eta}$	0 4.089 90	0.257 48 4.164 83	0.368 41 4.421 62	0.228 31 4.576 18	0.175 16 4.587 80	0.147 96 4.582 64	0.129 46 4.574 58
	5	$\frac{\xi}{\eta}$	0 5.696 57	0.797 38 5.591 95					

and 6. The curves approach ξ axis with an increasing $\bar{\epsilon}$, and the value of damping coefficient has extrema with respect to frequency. The roots in those figures belong to the first spectrum. Consequently, as shown in Fig. 4, only shear motion becomes prominent at higher mode of vibration and the damping effect caused by flexural vibration may be decreased; which in turn may bring out the extrema in damping coefficients. In the case of second spectrum, the motion approaches the moment wave with an increasing frequency, so the damping coefficient may never decrease at higher mode.

Conclusions

The results of the present analysis are summarized as follows:

- (1) The natural frequencies and damping coefficients of short cantilevers are calculated for different slenderness ratios and solid viscosities.
- (2) The existence of extrema is recognized in

the relation between natural frequencies and slenderness ratios.

(3) The damping coefficient has extrema with respect to frequency.

(4) The results are compared with those obtained in the past.

Numerical calculation was done at Tohoku University Computation Center.

References

- (1) Huang, T.C., *Trans. ASME, Ser. E*, Vol. 28, No. 4 (1961-12), p. 579.
- (2) Tobe, T., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 17, No. 57 (1951), p. 14.
- (3) Newman, M.K., *Trans. ASME, Ser. E*, Vol. 26, No. 3 (1959-9), p. 367.
- (4) Cowper, G.R., *Trans. ASME, Ser. E*, Vol. 33, No. 2 (1966-6), p. 335.
- (5) Trail-Nash, R.W., *Quart. Jour. Mech. and Appl. Math.*, (1953), p. 186.
- (6) Suzuki, S., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol. 33, No. 249 (1967-5), p. 686.
- (7) Schirmer, H., *Ing.-Arch.*, Bd. 20 (1952), p. 247.

Discussion

S. MAEZAWA (Yamanashi University):

(1) It may be useful to compare the authors' results with experimental ones. Please show some experimental data, if any.

(2) In the flexural vibration of actual short cantilevers such as gear teeth, it seems to be more reasonable to consider that the fixed end is elastically supported rather than rigid. Isn't it necessary to take into account such an effect?

S. SUZUKI (Nagoya University):

(3) The authors assume the dynamic model of solid viscosity as a Voigt type on the basis of the following remarks: when a three elements model is to be employed, the evaluation of elastic stiffness is difficult and there is no particular difference between the two models.

This discussor believes that a three elements model is more suitable. Voigt model is only a special case of the three elements model and differs from it essentially. Moreover it isn't unable to evaluate the elastic stiffness.

(4) Term of damping effect for shear motion $\bar{\epsilon}$ is neglected in this paper. Could the authors state how $\bar{\epsilon}$ is negligible compared with ϵ^* ? The discussor's experimental results, which will be presented in the near future, shows that $\bar{\epsilon}$ is 7% of ϵ^* .

(5) The validity of the application of Timoshenko beam to such a short cantilever is questionable. So far as there is no experimental proof for the authors' concept, it seems more reasonable to deal with such

a problem as two-dimensional viscoelastic vibration rather than flexural vibration of beam.

Authors' closure

(1) No experimental result corresponding to the authors' theoretical result has been reported so far. Some examples of T. C. Huang's numerical solution, where the effect of $\bar{\epsilon}$ is not considered, are shown in Fig. 3 for comparison.

(2) The effect of elastic support at fixed end of cantilevers under statical loadings has been studied theoretically by C. Weber and experimentally by W. J. O'Donnell^(*). It may be important to take into account this effect in dynamic problem. But the main purpose of the paper is the analysis using Timoshenko beam equation; and such an effect has been neglected.

(3) It is one of the most important matters which model is most suitable when solid viscosity is dealt with. There is an obvious difference between Voigt model and three elements model; and the authors are well aware of the validity of the latter by Suzuki's paper⁽⁶⁾. However, the authors are concerned with the examination of the effect of solid viscosity on the frequency equation of Timoshenko beam in this paper. So, Voigt model is chosen because of it having been used by many authors and its viscous coefficient being considerably well known.

(*) O'Donnell, W.J., *Trans. ASME, Ser. E*, Vol. 27, No. 3 (1960-9), p. 461.

(4) According to Ono^(*), the solid viscosity for shear motion may be about one-tenth as large as that for tension-compression. So ε is neglected in the paper.

(5) Obviously it is doubtful that very short

cantilevers with very small λ can be treated as Timoshenko beams. However, the calculation of natural frequency of Timoshenko beam and the comparison of the result with previous works are intended here. It is another problem to decide by which method the vibration of such a beam should be treated. Further analysis will make clear the validity or the limitation of the present analysis.

(*) Ono, A., *Strength of Materials* (in Japanese), (1967), p. 284, Iwanami Shoten.

Errata (Vol. 16, No. 92, February, 1973)

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