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Research of Force-Fitted Gear*

(1st Report, On the Distribution of Stress along the Fillet Curve of a Spur Gear)

By Toshio AIDA**, Susumu SATO***, and Norihisa ARAI****

The present paper has been carried out for the purpose of calculating the distribution of stress along the fillet curve of a spur gear tooth with a force-fitted shaft in the rim.

Using the two-dimensional elastic theory and the transform function proposed by H. Neuber, the authors have introduced equations for the calculation of the stress at the fillet curve of a gear tooth caused by force-fitting.

Moreover, the authors have examined the stresses of various factors, that is, dimensions of gear, thickness of gear rim, conditions of fitting etc.

Consequently the practical formulas have been introduced for the stress at the weakest section of a gear tooth with a force-fitted shaft in the rim.

By the practical equations, the stress along the fillet curve of force-fitted gear can be calculated more easily than by theoretical equations.

1. Introduction

Many researches on the bending strength of spur gear, such as bending static and fatigue strength tests, photo-elastic experiments and theoretical analysis of bending stress have been made hitherto.

Aida and Terauchi⁽¹⁾ made an analysis of the stress at the root fillet of gear tooth theoretically by transforming a gear tooth shaped projection on the linear boundary of a semi-infinite plate and by applying the theory of stress using complex variables. But the theoretical studies of stresses at the root fillet of the force and shrinkage fitted gears seem to be very few.

H. Neuber⁽²⁾ transformed a figure having a circular curve projection around a disk, and introduced the theoretical equations of stress at the fillet curve of the circular projection from the theory of stress using complex variables, but in this case the theoretical equations were applied only for a general circular curve projection around the disk and not any consideration was made of the profile of the gear tooth formed by involute curve, and yet experiments for the

practical gear weren't done either.

So, the authors have tried to analyse the stress at the root fillet of a force-fitted gear with gear a tooth curve projection around the disk.

Moreover, the theoretically calculated results for practical gears are compared with the experimental results obtained by means of strain gauge and photo-elasticity, and the effectiveness of their theoretical analysis is verified by the results.

However, it is very troublesome and needs a lot of complicated calculations to determine a coefficient " a_m ", which decides the shape of the gear tooth and makes the theoretical curve shown in Eq. (3) follow the actual profile of the gear tooth, and also to determine some other coefficients in the theoretical equations in which stress at the root fillet of gear tooth is calculated. Consequently, it is inadequate to use this theoretical calculating method practically in factories.

So the authors have introduced a practical formula that can calculate simply and quickly the fillet stress at the weakest section of gear tooth and at the center of tooth root curve in case when the dimensions of gear and the force-fitted conditions are given.

That's to say, by the theoretical method of H. Neuber the fillet stresses at the root fillet of force-fitted gear tooth at the weakest section and at the center of gear root curve are calculated under various arbitrary combinations of number of teeth, module

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** Professor, Faculty of Engineering, Kyoto University.

*** Assistant Professor, Faculty of Engineering, Kyoto University.

**** Assistant Professor, Faculty of Engineering, Doshisha University, Kamikyo-ku, Kyoto.

and tooth profile, and the practical formulas are introduced by finding out the mutual relations between these stresses and force-fitted conditions or dimensions of the gear.

2. Theoretical equations and their applications for the practical gear

2.1 Theoretical equations

Let stress components on the $z(x, y)$ plane be σ_x , σ_y , τ_{xy} and Airy's stress function be U , then each stress will be given by the following equations.

$$\left. \begin{aligned} \sigma_x &= \partial^2 U / \partial y^2, & \tau_{xy} &= -\partial^2 U / (\partial x \partial y) \\ \sigma_y &= \partial^2 U / \partial x^2 \end{aligned} \right\} \dots\dots (1)$$

Besides $\Delta^2 U$ is equal to zero, and so U is expressed as follows by using $z = x + iy$, $\bar{z} = x - iy$.

$$U = z\bar{f}_1(\bar{z}) + \bar{z}f_1(z) + f_2(z) + \bar{f}_2(\bar{z}) \dots\dots\dots (2)$$

Where f_1 , f_2 are analytical functions of z and $\bar{f}_1$, $\bar{f}_2$ are conjugate functions of $f_1(z)$, $f_2(z)$.

$$\begin{aligned} U = e^w \left\{ \left[1 + \sum_m a_m \coth^m (sw/2) \right] \left[\sum_{l=-\infty}^{+\infty} \bar{K}_l e^{ls\bar{w}} + \sum_{p=1}^n \bar{c}_p \coth^p (s\bar{w}/2) \right] \right. \\ \left. + \left[1 + \sum_p \bar{a}_p \coth^p (s\bar{w}/2) \right] \left[\sum_{l=-\infty}^{+\infty} K_l e^{ls w} + \sum_{m=1}^n c_m \coth^m (sw/2) \right] \right\} + f_2 + \bar{f}_2 \dots\dots\dots (5) \end{aligned}$$

Substituting the boundary conditions of $u = u_0$, $\bar{w} = 2u_0 - w$, $\mu = \bar{p} I_n^{(-1)}/4\pi$ (μ is a coefficient of complex number, $\mu\bar{z} = \partial U / \partial w$) for the equation, the differentiated Eq. (5) with respect to w is shown as follows.

$$\begin{aligned} \partial U / \partial w = e^w \left\{ \left[1 - \sum_m a_m g_m \right] \left[\sum_l \bar{K}_l \{ e^{ls\bar{w} + \bar{w}} - e^{(ls+1)(2u_0-w)} \} + \sum_p \bar{c}_p \{ e^{\bar{w}} \coth^p (s\bar{w}/2) - e^{2u_0-w} \coth^p (su_0 \right. \right. \\ \left. \left. - s\bar{w}/2) \} \right] + \left[e^{\bar{w}} - e^{2u_0-w} + \sum_p \bar{a}_p \{ e^{\bar{w}} \coth^p (s\bar{w}/2) - e^{2u_0-w} \coth^p (su_0 - s\bar{w}/2) \} \right] \left[\sum_l (ls+1) K_l e^{ls w} - \sum_m c_m g_m \right] \right\} \end{aligned}$$

g_m in above equation is as follows

$$g_m = sm [\coth^{m+1} (sw/2) - \coth^{m-1} (sw/2)] / 2 - \coth^m (sw/2) \dots\dots\dots (6)$$

On the other hand, the fundamental equation of two dimensional elasticity is as follows.

$$\sigma_v - \sigma_u + 2i\tau_{uv} = (4/\bar{z})(\partial/\partial w)\{ (1/\bar{z})(\partial U/\partial w) \} \dots\dots\dots (7)$$

Substituting z , $\partial U/\partial w$ and the boundary condition $u = u_0$ in Eq. (7), the stress at the root fillet of the gear tooth will be

$$\sigma_{\text{Rand}} = 8R_e \left\{ \left[\sum_{l=-\infty}^{+\infty} (ls+1) K_l e^{ls w} - \sum_{p=1}^n c_p g_p \right] / \left[1 - \sum_{m=1}^n a_m g_m \right] \right\} \dots\dots\dots (8)$$

In Eq. (8) R_e is a symbol showing the real number part, and the coefficient c_p is obtained by the following equation.

$$\sum_{p=1}^n \sum_{m=1}^n H_{m,p-t} (c_m \bar{a}_p + \bar{c}_p a_m) - \sum_{p=1}^n \sum_{l=-\infty}^{+\infty} (ls+1) K_l \bar{a}_p L_{l,p-t} - \bar{c}_t = 0 \dots\dots\dots (9)$$

Where $H_{m,p-t}$ and $L_{l,p-t}$ in Eq. (9) may be written in the following form.

$$g_m = \sum_{q=0}^{\infty} H_{m,q} \tanh^q (su_0 - sw/2), \quad e^{ls w} = \sum_{q=0}^{\infty} L_{l,q} \tanh^q (su_0 - sw/2) \dots\dots\dots (10)$$

Substituting the boundary conditions at the force-fitting part of shaft, theoretical stress at the circular projection induced by force-fitting of shaft could be calculated.

2.2 Applications of the calculating method for practical gear

To apply theoretical equation obtained in Chapter 2.1 for practical gear, the value of coefficient a_m in Eq. (3) should be chosen adequately so as to give the shape of transform function expressed by Eq. (3) which is as close as possible to the shape of practical gear tooth profile.

Of course, the larger the number of suffix m of a_m is, the more approximate to the shape of practical gear tooth profile the theoretical curve is, but a particularly complicated calculation has to be done.

So in this calculation the values of $m=1,2,3$ are adopted. Figure 1 shows a theoretical tooth profile curve described by Eq. (3), and Fig. 2 shows an example of how theoretical tooth profile varies with the value

Now, circular curve projections around the disk such as gears are expressed by the following transform function that H. Neuber proposed.

$$z = e^w \left[1 + \sum_{m=1}^n a_m \coth^m (sw/2) \right] \dots\dots\dots (3)$$

Where $z = x + iy$ (z plane), $w = u + iv$ (w plane) and s is the number of projections and a_m is a coefficient which decides the shape of the circular curve projections.

Analytical function f_1 in case of force fitting is written as follows

$$f_1 = \sum_{l=-\infty}^{+\infty} K_l e^{ls w + w} + \sum_{p=1}^n c_p e^w \coth^p (sw/2) \dots\dots (4)$$

In Eq. (4), the first term expresses the influence of internal pressure caused by force fitting and the second term indicates the influence of a circular projection around the disk. If analytical functions are expressed by shapes of equation mentioned above, Airy's function U is given by Eq. (5).

of a_m .

In Fig. 2(a), (b), (c), tooth profiles are described respectively in respective cases when (a) only a_1 , (b) a_1, a_2 , (c) a_1, a_2, a_3 are adopted.

As obvious from these figures, the larger the number of suffix m of a_m is, the greater the whole depth of tooth becomes.

As above mentioned, the shape of theoretical tooth profile varies with the values of a_1, a_2 and a_3 , so the calculation of tooth profile was carried out using an electronic computer for combinations of several hundred values of a_1, a_2, a_3 , and the calculated results were plotted on graph and were approximated to the practical tooth profile as much as possible by repeating this calculation many times. In this calculation, pole-coordinate of Eq. (3) was transformed to x - y coordinate as follows.

$$\left. \begin{aligned} x &= e^u [\cos v + a_1 [\sinh(su_0) \cos v \\ &\quad + \sin(sv) \sin v] / [\cosh(su_0) - \cos(sv)] \\ &\quad + a_2 [\cos v \{ \sinh^2(su_0) - \sin^2(sv) \} \\ &\quad + 2 \sin v \sin(sv) \sinh(su_0)] / [\cosh(su_0) \\ &\quad - \cos(sv)]^2 + a_3 [\cos v \{ \sinh^3(su_0) \\ &\quad - 3 \sinh(su_0) \sin^2(sv) \} \\ &\quad + \sin v \{ 3 \sinh^2(su_0) \sin(sv) \\ &\quad - \sin^3(sv) \}] / [\cosh(su_0) - \cos(sv)]^3] \\ y &= e^u [\sin v + a_1 [\sinh(su_0) \sin v \\ &\quad - \sin(sv) \cos v] / [\cosh(su_0) - \cos(sv)] \\ &\quad + a_2 [\sin v \{ \sinh^2(su_0) - \sin^2(sv) \} \\ &\quad - 2 \cos v \sin(sv) \sinh(su_0)] / [\cosh(su_0) \\ &\quad - \cos(sv)]^2 + a_3 [\sin v \{ \sinh^3(su_0) \\ &\quad - 3 \sinh(su_0) \sin^2(sv) \} \\ &\quad - \cos v \{ 3 \sinh^2(su_0) \sin(sv) \\ &\quad - \sin^3(sv) \}] / [\cosh(su_0) - \cos(sv)]^3] \end{aligned} \right\} \dots\dots\dots (11)$$

Variations of theoretical tooth profiles with values of a_1, a_2 and a_3 are shown in Fig. 3. In these cases, any one of a_1, a_2 and a_3 was varied and the other two coefficients were chosen to approximate the curve to the practical gear tooth profile as much as possible.

Whole depth of tooth is proportional to the absolute value of a_1, a_2 or a_3 respectively in each case when a_1 in Fig. 3(a), a_2 in Fig. 3(b) or a_3 in Fig. 3(c) is varied.

The same tendency can also be observed with

Fig. 1 Theoretical circular wave fringe of a disk

swelling of gear tooth.

The influence of these coefficients on the shape of the tooth profile becomes larger in the order of a_1, a_2 and a_3 as shown in Fig. 3.

The curvature of the root fillet curve of gear tooth shows the same tendency as the depth and swelling of tooth profile. a_1, a_2 and a_3 were determined by the above mentioned method, but as shown in Fig. 4 the whole depth of theoretical gear tooth is small in comparison with that of practical one. However, it⁽³⁾ is well known that the curvature at the root fillet of gear tooth has large influence on the fillet stress and the whole depth of gear tooth gives hardly any influence on it. Especially when the shaft is force-fitted into the gear rim, that is, in case when the load works as an internal pressure, the influence of whole depth of tooth on the fillet stress seems to be very small.

Moreover, in the photo-elastic experimental results shown later, stresses caused by force-fitting are hardly found in the addendum part of tooth profile. For these reasons, the authors paid much attention to the coincidence of the curvature at the root fillet of gear tooth with that of practical gear tooth profile, even if the whole depth of theoretical gear tooth profile is smaller than that of practical one. Symbol s in Fig. 4 is the chordal thickness in Hofer's weakest section of gear tooth. Symbol ρ is the curvature radius of tooth profile curve at the point A, that is, at the weakest section of gear tooth. Thus transform function can be chosen and the stress at the root fillet of force-fitted gear tooth can be also calculated by Eq. (8), but before these calculations K_I in Eq. (8) has to be determined. The outside circumference of the practical force-fitted gear has a tooth profile, but inside one is circular in general cases. But when the tooth profile on the outside circumference is expressed by Eq. (3), the inside one must also be expressed

Fig. 2 Representation of tooth profile by a_m

Fig. 3 Theoretical profile of gear tooth in case when one of a_1, a_2, a_3 changes

by a tooth profile. So, the approximating method was adopted as follows. That's to say, the average of unevenness in inside one was regarded as the inside diameter. Moreover, the disk was assumed to be formed by the average diameter of unevenness on inside and outside circumferences, and a constant pressure P_0 was assumed to act at the inside circumferences, so K_0 is calculated by the following equation and is considered approximately equal to K_l in Eq. (8).

$$K_0 = P_0/4[e^{2(u_0' - u_1')} - 1] \quad \dots\dots\dots(12)$$

The constant pressure P_0 occurring when the shaft is force-fitted for gear rim (outside radius $e^{u_0'}$, inside radius $e^{u_1'}$) with the interference δ , can be obtained from Eq. (13).

$$P_0 = \delta E/2[e^{u_1'}[(e^{2(u_0' - u_1')} + 1)/(e^{2(u_0' - u_1')} - 1) + 1]] \quad \dots\dots\dots(13)$$

Substituting the conditions mentioned above such as $m=1, 2, 3$ and $K_l=K_0$ into Eq. (8), we have the following equation.

$$\sigma_{Rand} = 8R_e \left\{ \left[K_0 - \sum_{p=1}^3 c_p g_p \right] / \left[1 - \sum_{m=1}^3 a_m g_m \right] \right\} \quad \dots\dots\dots(14)$$

c_p in Eq. (14) is determined by solving the following three simultaneous equations introduced from Eq. (9).

(i) In case of $t=1$

$$\sum_{m=1}^3 \sum_{q=0}^2 H_{m,q} (c_m \bar{a}_{q+1} + \bar{c}_{q+1} a_m) - K_0 (\bar{a}_1 L_{0,0} + \bar{a}_2 L_{0,1} + \bar{a}_3 L_{0,2}) - \bar{c}_1 = 0$$

(ii) In case of $t=2$

$$\sum_{m=1}^3 \sum_{q=0}^1 H_{m,q} (c_m \bar{a}_{q+2} + \bar{c}_{q+2} a_m) - K_0 (\bar{a}_2 L_{0,0} + \bar{a}_3 L_{0,1}) - \bar{c}_2 = 0$$

(iii) In case of $t=3$

$$\sum_{m=1}^3 \sum_{q=0}^0 H_{m,q} (c_m \bar{a}_{q+3} + \bar{c}_{q+3} a_m) - K_0 \bar{a}_3 L_{0,0} - \bar{c}_3 = 0$$

... (15)

Tooth profile: B type ($c=0.25$ mm), Pressure angle: $\alpha=20^\circ$, Module: $m=4$, Number of teeth: $z=30$

Fig. 4 Comparison between practical tooth profile and theoretical one

$H_{m,0}$, $H_{m,1}$, $H_{m,2}$ in Eq. (15) are decided from Eq. (6), (10).

Expanding Eq. (6), (10) in Taylor series

$$g_m = (sm/2) [\coth^{(m+1)}(su_0) + (m+1) \{ \coth^{(m+2)}(su_0) - \coth^m(su_0) \} \{ su_0 - (sw/2) \} \\ + \{ (m+1)/2 \} \{ (m+2) \coth^{(m+3)}(su_0) - (2m+2) \coth^m(su_0) + m \coth^{m-1}(su_0) \} \{ su_0 - (sw/2) \}^2 \\ - \coth^{(m-1)}(su_0) - (m-1) \{ \coth^m(su_0) - \coth^{(m-2)}(su_0) \} \{ su_0 - (sw/2) \} \\ - \{ (1-m)/2 \} \{ -m \coth^{(m+1)}(su_0) + (2m-2) \coth^{(m-1)}(su_0) - (m-2) \coth^{(m-3)}(su_0) \} \{ su_0 - (sw/2) \}^2 \\ - \coth^m(su_0) - m \{ \coth^{(m+1)}(su_0) - \coth^{(m-1)}(su_0) \} \{ su_0 - (sw/2) \} + (m/2) \{ 2m \coth^m(su_0) \\ - (m+1) \coth^{(m+2)}(su_0) - (m-1) \coth^{(m-2)}(su_0) \} (su_0 - sw/2)^2]$$

and

$$\sum_{q=0}^{\infty} H_{m,q} \tanh^q(su_0 - sw/2) \\ = H_{m,0} + H_{m,1}(su_0 - sw/2) - (1/3)H_{m,1}(su_0 - sw/2)^3 + \dots \\ + H_{m,2}(su_0 - sw/2)^2 - (2/3)H_{m,2}(su_0 - sw/2)^4 + \dots \\ + H_{m,3}(su_0 - sw/2)^3 + \dots$$

are obtained, $H_{m,0}$, $H_{m,1}$ and $H_{m,2}$ in Eq. (15) become as follows.

$$\left. \begin{aligned} H_{m,0} &= (sm/2) \{ \coth^{(m+1)}(su_0) - \coth^{(m-1)}(su_0) \} - \coth^m(su_0) \\ H_{m,1} &= (sm/2) \{ (m+1) \coth^{(m+2)}(su_0) - 2m \coth^m(su_0) \\ &\quad + (m-1) \coth^{(m-2)}(su_0) \} + m \{ \coth^{(m-1)}(su_0) - \coth^{(m+1)}(su_0) \} \\ H_{m,2} &= (sm/4) \{ (m^2+3m+2) \coth^{(m+3)}(su_0) + (-3m^2-3m-2) \coth^{(m+1)}(su_0) \\ &\quad + (3m^2-3m+2) \coth^{(m-1)}(su_0) + (-m^2+3m-2) \coth^{(m-3)}(su_0) \} \\ &\quad + (m/2) \{ 2m \coth^m(su_0) - (m+1) \coth^{(m+2)}(su_0) - (m-1) \coth^{(m-2)}(su_0) \} \end{aligned} \right\} \dots \quad (16)$$

Where $m=1, 2, 3$. Similarly

$$L_{l,0} = e^{2lsu_0}, \quad L_{l,1} = -2le^{2lsu_0}, \quad L_{l,2} = 2l^2e^{2lsu_0} \dots \quad (17)$$

Here, because $l=0$

$$L_{l,0} = L_{0,0} = 1, \quad L_{l,1} = L_{0,1} = 0, \quad L_{l,2} = L_{0,2} = 0 \dots \quad (18)$$

Each term in above equations is replaced by the following symbols

$$\begin{aligned} A &\equiv \cosh(su) - \cos(sv) \\ B &\equiv \cosh^3(su) \sinh(su) - 3 \cosh^2(su) \sinh(su) \cos(sv) \\ &\quad + 3 \cosh(su) \sinh(su) \cos^2(sv) - \sinh(su) \cos^3(sv) \\ C &\equiv -\cosh^3(su) \sin(sv) + 3 \cosh^2(su) \cos(sv) - 3 \cosh(su) \cos^2(sv) \sin(sv) + \cos^3(sv) \sin(sv) \\ D &\equiv \cosh^2(su) \sinh^2(su) - 2 \cosh(su) \sinh^2(su) \cos(sv) + \sinh^2(su) \cos^2(sv) \\ &\quad - \cosh^2(su) \sin^2(sv) + 2 \cosh(su) \cos(sv) \sin^2(sv) - \sin^2(sv) \cos^2(sv) \\ E &\equiv -2 \cosh^2(su) \sinh(su) \sin(sv) + 4 \cosh(su) \sinh(su) \cos(sv) \sin(sv) - 2 \sinh(su) \cos^2(sv) \sin(sv) \\ F &\equiv \cosh(su) \sinh^3(su) - 3 \cosh(su) \sinh(su) \sin^2(su) - \sinh^3(su) \cos(sv) + 3 \sinh(su) \cos(sv) \sin^2(sv) \\ G &\equiv -3 \cosh(su) \sinh^2(su) \sin(sv) + 3 \cosh^2(su) \cos(sv) \sin(sv) + \cosh(su) \sin^3(sv) - \cos(sv) \sin^3(sv) \\ H &\equiv \sinh^4(su) - 6 \sinh^2(su) \sin^2(sv) + \sin^4(sv) \\ I &\equiv -4 \sinh^3(su) \sin(sv) + 4 \sinh(su) \sin^3(sv) \\ \coth^3(sw/2) &= (B+iC)/A^4, \quad \coth^2(sw/2) = (D+iE)/A^4, \quad \coth^3(sw/2) = (F+iG)/A^4, \\ \coth^4(sw/2) &= (H+iI)/A^4 \end{aligned}$$

Therefore, the following equations are obtained from Eq. (6).

$$g_1 = (s/2) \{ (D+iE) - A^4 \} / A^4 - \{ B+iC \} / A^4, \quad g_2 = s \{ (F-B) + i(G-C) \} / A^4 - \{ D+iE \} / A^4, \\ g_3 = (3s/2) \{ (H-D) + i(I-E) \} / A^4 - \{ F+iG \} / A^4$$

Moreover, by giving the following equations

$$\begin{aligned} V &\equiv A^4 K_0 - (s/2) C_1 (D - A^4) - s C_2 (F - B) - (3s/2) C_3 (H - D) + C_1 B + C_2 D + C_3 F \\ W &\equiv - (s/2) C_1 E - s C_2 (G - C) - (3s/2) C_3 (I - E) + C_1 C + C_2 E + C_3 G \\ X &\equiv A^4 - (s/2) a_1 (D - A^4) - s a_2 (F - B) - (3s/2) a_3 (H - D) + a_1 B + a_2 D + a_3 F \\ Y &\equiv - (s/2) a_1 E - s a_2 (G - C) - (3s/2) a_3 (I - E) + a_1 C + a_2 E + a_3 G \end{aligned}$$

Equation (14) is given as follows.

$$\sigma_{\text{Rand}} = 8R_e [\{ VX + WY + i(WX - VY) \} / (X^2 + Y^2)] = 8 [\{ VX + WY \} / (X^2 + Y^2)] \dots \quad (19)$$

From equations mentioned above, the stress at the root fillet of practical force-fitted gear tooth can be calculated theoretically.

2.3 Calculated results by theoretical equations

By substituting dimensions of the practical gear into theoretical equations mentioned above, theoretical calculation was carried out.

Dimensions and force-fitting conditions of gear chosen as the subject to calculate are shown in Table 1. Thickness of gear rim mentioned in Table 1 is a half of the difference (T in Fig. 4) between diameter of root circle d_r and inside diameter of gear rim D_i .

Calculated results are shown in Table 2.

The stress at point A or A' in Fig. 4 is shown in Table 1, here σ_R is stress in the place where Hofer's weakest section crosses trochoid curve at the root fillet of gear tooth and σ_{BA} is stress at the center of tooth root curves. These tables show representatively cases of pressure angle $\alpha=20^\circ$, thickness of gear rim $T=7.9(\text{mm})$, $20.4(\text{mm})$. As evident from these tables, σ_R and σ_{BA} values become larger as interference gets larger, that is, they are proportional to the interference. σ_R and σ_{BA} values in tooth profile of B type are larger by $0.3\sim 2.0 \text{ kg/mm}^2$ than those in tooth profile of A type. This is the case of number of teeth $z=30$, and the difference between these stresses becomes larger as the number of teeth gets smaller. It's natural that σ_R and σ_{BA} become

smaller as thickness of gear rim gets larger.

3. Measured results of stress at the root fillet of force-fitted gear tooth and discussions on them

3.1 Measurement of stress caused by force-fitting at the root fillet of the practical gear tooth

The test gears made of 0.45% carbon steel are produced in the same size and same force-fitting conditions as the gears chosen as the subjects of theoretical calculation, and the stresses at the root fillet of force-fitted gear were measured by means of strain gauge (F101 type made by Shinko Communication Industry Co., Ltd. $0.8 \text{ mm} \times 1.0 \text{ mm}$). Test gears were annealed in vacuum atmosphere for one hour at the temperature of 850°C after tooth hobbing to remove the strain occurring in the process of gear hobbing. Figure 5(a) shows the method of force-fitting the shaft into the gear rim.

Figure 5(b) shows the sticking position of strain gauge on the gear tooth.

In Fig. 6 the results of stress distribution at the root fillet of gear tooth measured practically are compared with those calculated theoretically, and it is found that both of them are almost the same at the weakest section and at the center of the gear tooth root curve.

It is also shown in Fig. 6 that the stress at the addendum part is nearly zero. Therefore, in the case of force-fitted gear, the stress at the root fillet

Table 1 Dimensions of gear used in theoretical calculation and force-fitting conditions (standard spur gear)

Tooth profile	A type (top clearance 0.157 m) B type (top clearance 0.25 m)
Pressure angle: α	20° , 14.5°
Module: m	4, 8, 10
Number of teeth: z	18, 25, 30, 40, 60
Rim thickness: $T \text{ mm}$	5.4, 7.9, 16.0, 20.4
Interference: $\delta \text{ mm}$	0.046, 0.09, 0.12

Table 2 Calculated results of fillet stress of tooth induced by force-fitting (pressure angle 20°)

Tooth profile Rim thickness Number of teeth	Module	A type (top clearance: 0.157 m)												B type (top clearance: 0.250 m)											
		7.9 (mm)						20.4 (mm)						7.9 (mm)						20.4 (mm)					
		Interference (mm)						Interference (mm)						Interference (mm)						Interference (mm)					
		0.046		0.09		0.12		0.046		0.09		0.12		0.046		0.09		0.12		0.046		0.09		0.12	
		σ_R	σ_{BA}	σ_R	σ_{BA}	σ_R	σ_{BA}	σ_R	σ_{BA}	σ_R	σ_{BA}	σ_R	σ_{BA}	σ_R	σ_{BA}	σ_R	σ_{BA}	σ_R	σ_{BA}	σ_R	σ_{BA}	σ_R	σ_{BA}	σ_R	σ_{BA}
18	4	16.6	15.6	32.5	30.6	43.3	40.8	7.8	7.3	15.2	14.3	20.2	19.1	18.5	18.8	36.2	36.7	48.3	49.0	8.5	8.6	16.6	16.9	22.1	22.5
	8	9.7	9.1	19.0	17.9	25.3	23.8	7.5	7.0	14.7	13.8	19.5	18.4	10.8	11.0	21.2	21.5	28.3	28.7	8.3	8.6	16.3	16.5	21.7	22.1
	10	8.0	7.6	15.7	14.8	20.9	19.7	6.6	6.2	12.9	12.2	17.2	16.2	8.6	9.0	17.5	17.7	23.3	23.6	7.3	7.4	14.3	14.5	19.1	19.4
25	4	14.7	14.4	28.7	28.1	38.3	37.5	9.8	9.6	19.1	18.7	25.5	25.0	15.2	15.5	29.7	30.3	39.5	40.4	10.1	10.3	19.7	20.1	26.2	26.8
	8	8.1	7.9	15.9	15.3	21.2	20.7	6.9	6.7	13.5	13.2	18.0	17.6	8.4	8.6	16.4	16.7	21.9	22.3	7.1	7.3	13.9	14.2	18.6	18.9
	10	6.5	6.2	12.7	12.1	16.9	16.1	5.7	5.4	11.2	10.7	14.9	14.2	6.8	7.0	13.4	13.7	17.8	18.2	6.0	6.1	11.8	12.0	15.7	16.0
30	4	13.0	12.8	25.5	25.1	34.0	33.5	9.6	9.5	18.8	18.5	25.1	24.7	13.6	14.1	26.6	27.6	35.4	36.7	10.0	10.3	19.5	20.2	26.0	27.0
	8	7.1	7.0	13.8	13.6	18.4	18.1	6.2	6.1	12.1	12.0	16.2	15.9	7.4	7.6	14.4	14.9	19.2	19.9	6.5	6.7	12.6	13.1	16.8	17.5
	10	5.7	5.6	11.2	11.1	15.0	14.7	5.2	5.1	10.2	10.0	13.5	13.3	6.0	6.2	11.7	12.1	15.6	16.2	5.4	5.6	10.6	11.0	14.1	14.6
40	4	10.4	10.3	20.3	20.2	27.1	26.9	8.5	8.4	16.6	16.4	22.1	21.9	10.8	11.1	21.1	21.8	28.2	29.0	8.8	9.1	17.2	17.7	22.9	23.6
	8	5.5	5.4	10.8	10.7	14.4	14.2	5.1	5.0	9.8	9.7	13.1	13.0	5.7	5.9	11.2	11.5	14.9	15.4	5.2	5.4	10.2	10.5	13.6	14.0
	10	4.5	4.4	8.7	8.6	11.6	11.5	4.1	4.1	8.1	8.0	10.8	10.7	4.6	4.8	9.0	9.3	12.1	12.4	4.3	4.4	8.4	8.7	11.2	11.6
60	4	7.1	7.0	13.9	13.7	18.6	18.3	6.3	6.2	12.3	12.1	16.4	16.2	7.8	7.9	15.2	15.5	20.2	20.7	6.8	7.0	13.4	13.9	17.9	18.3
	8	3.7	3.6	7.2	7.1	9.6	9.5	3.5	3.4	6.8	6.7	9.1	8.9	4.0	4.1	7.9	8.0	10.5	10.7	3.8	3.9	7.4	7.6	9.9	10.1
	10	3.0	2.9	5.8	5.7	7.8	7.6	2.8	2.8	5.6	5.5	7.4	7.3	3.2	3.3	6.3	6.5	8.4	8.6	3.1	3.2	6.1	6.2	8.1	8.2

of gear tooth is not so much influenced by shape of tooth top, but is dependent on the shape of dedendum and root of gear tooth. This shows that in the theoretical calculation it was correct to determine profile coefficient " a_m " so as to make the theoretical tooth profile as close as possible to the shape of dedendum and root of gear tooth, even though the whole depth of tooth is smaller than the theoretical tooth profile. That's to say, this calculating method is very effective to obtain the stress at the root fillet of force-fitted gear tooth. If examined more in detail, values measured by strain gauge are a little lower than the values calculated theoretically.

Main reason for these results seems to be that curvature at the root fillet of gear tooth used in theoretical calculation didn't agree perfectly with practical ones, in particular at root of tooth, and the other reason is because $K_t (=K_0)$ was determined approximately as a disk. As for residual tensile stress

caused by force-fitting, either fillet stress σ_R at A or fillet stress σ_{BA} at the center of tooth root curve in Fig. 4 will be larger than the other for a certain value of curvature radius and the shape of tooth root curve, but the difference between σ_R and σ_{BA} is small, and stresses along the fillet and root curve distribute with almost the same values from the root to dedendum part of gear tooth.

3.2 Results of photo-elastic experiment

Gear rims of epoxide resin with the same dimensions as the gear rims whose stress measurement had been done by strain gauges were annealed for 5 hours at the temperature of 120°C after gear hobbing.

Also, force-fitted shafts were made by the same process and their interference was 0.09 mm. The inside diameter of gear rim and outside diameter of force-fitted shaft were measured up to 1/1 000 mm by universal measuring microscope. Figure 7 shows the stress distribution by photo-elastic experiments. As is obvious from Fig. 7 the maximum fillet stress in this case occurs at the center of tooth root curve. This is the case of tooth profile of B type and fairly agrees with the calculated results mentioned above. Also as is clear from Fig. 7, stress values at the outside part over pitch circle of tooth are very small. These results agree with those of photo-elastic experiments carried out by G. Lechner⁽⁴⁾ and the results mentioned above measured by means of strain gauge, too.

3.3 The fillet stress at the weakest section of gear tooth and at the center of tooth root curve in case when gear dimensions and force-fitted conditions are varied

Figure 8(a), (b) show the relation between the interference σ/d_r and the strain σ_R/E in case of tooth profile of A type (top clearance 0.15 m), B type (top clearance 0.25 m), where d_r , D_i and δ are respectively diameter of root circle, inside diameter

(a) Apparatus for force-fitting

(b) Measurement of stress by strain gauge
Fig. 5 Force-fitting method

Tooth profile: B type, Pressure angle: $\alpha = 20^\circ$,
Module: $m = 4$

Fig. 6 Distribution of fillet stress

Tooth profile: B type, Pressure angle: $\alpha = 20^\circ$
Module: $m = 4$, Number of teeth: $z = 30$, Width of tooth: $b = 10$ mm, Fringe stress: $\sigma_f = 0.106$ kg/mm²/mm, Modulus of longitudinal elasticity: $E_s = 306$ kg/mm²

Fig. 7 Fringe photograph of force-fitted gear

of gear rim and interference. As seen from in this figure, σ_R/E is proportional to δ/d_r and experimental values by strain gauge and by photo-elasticity shown in the same figure agree nearly with theoretically calculated values. In the case when σ_R/E is over 1.33×10^{-5} the experimental results for ratio of increase of δ/d_r is a little lower in comparison with the theoretical ones, because the stress σ_R attains the yield point of material of test gear. Figure 9(a), (b) show the relation between D_i/d_r and σ_R/E in both tooth profiles of A type and B type. Here, D_i/d_r is parameter (ratio of inside diameter of gear rim to diameter of root circle) indicating thickness of gear rim. D_i/d_r is approximately proportional to σ_R/E and the experimental results also coincide with the theoretical ones.

In addition, the dimensions of the test gear used in experiments of Figs. 8, 9 are pressure angle $\alpha = 20^\circ$, module $m = 4$ and number of teeth $z = 30$.

However, when D_i/d_r becomes larger than the order of 0.856, that is, thickness of gear rim becomes thin relatively in comparison with the inside diameter of gear rim, the experimental results don't coincide with theoretical ones.

This is because inside diameter of gear rim was approximately regarded as a circle passing through half height of circular projection around inside circumference in the calculation of theoretical value as mentioned above.

It is difficult to decide exactly the limit of the influence of the projection in inside diameter of gear rim upon σ_R/E . But the height of circular projection

inside diameter of gear rim is 2.1 mm in case of thickness of gear rim 7.9 mm, and 2.2 mm in case of thickness of gear rim 5.4 mm.

Judging from the fact that in the former cases the stresses measured at the root fillet of tooth nearly coincide, and in the latter case do not coincide, with the theoretical ones, the limit of theoretical height of circular projection over inside diameter of gear rim seems to be about a quarter of thickness of gear rim.

From these facts, the practical formula mentioned in next chapter is to be studied except in the latter case. The above discussion was made about the strain σ_R/E at the weakest section of tooth profile, but the same thing can be said in respect to the strain σ_{BA}/E at the center of tooth root curve.

4. Practical formula

4.1 On the practical formula for the fillet stress at the weakest section of the gear tooth

The stress σ_s at outside circumference of a force-fitting disk having an outside diameter equal to root diameter of test gear is expressed as Eq. (20).

$$\sigma_s = E(\delta/d_r)(D_i/d_r) \quad (20)$$

So the fillet stress σ_R at the weakest section of gear tooth is assumed as follows.

$$\sigma_R = \varphi_{Ri} \sigma_s \quad (21)$$

Then, from Eq. (20) and (21), a practical formula is given as follows.

$$\sigma_R/E = \varphi_{Ri}(\delta/d_r)(D_i/d_r) \quad (22)$$

$$\text{Where } d_r = m\{z - 2(c/m + 1)\}$$

(a) In the case of A type profile

(b) In the case of B type profile

Fig. 8 Relation between σ_R/E and δ/d_r at the weakest section of tooth

(a) In the case of A type profile

(b) In the case of B type profile

Fig. 9 Relation between σ_R/E and D_i/d_r at the weakest section of tooth

In above Eq. (22), (δ/d_r) , (D_i/d_r) and φ_{Ri} are parameters concerning respectively the force-fitted condition, thickness of gear rim and tooth profile. On the other hand, when the practical formula is considered in terms of the ratio of interference δ to inside diameter D_i of gear rim, Eq. (22) is expressed as follows.

$$\sigma_R/E = \varphi_{Ri}(\delta/D_i)(D_i/d_r)^2 \quad (23)$$

φ_{Ri} in Eqs. (22), (23) is to be determined from type of tooth profile as mentioned above, and it is expressed by the following equation:

$$\varphi_{Ri} = \varphi(z, m, \alpha, c) \quad (24)$$

The number of teeth z becomes equal to $f(m)$ under the condition that D_i/d_r is constant. So Eq. (24) is given by

$$\varphi_{Ri} = \varphi(z, \alpha, c) \quad (25)$$

Now, in the case when pressure angle α and top clearance c are considered to be parameters, φ_{Ri} is expressed for the variation of z as Fig. 10 (a), (b). In these figures, φ_{R1} is for the case of pressure angle $\alpha=20^\circ$, φ_{R2} is of pressure angle $\alpha=14.5^\circ$. The strain at the root fillet of force-fitted gear can be obtained simply from Fig. 10 (a), (b) and Eq. (22); therefore, stress σ_R is calculated from the relation of $\sigma_R = E\varepsilon$. As evident from these figures, φ_1, φ_2 get smaller as the number of teeth z becomes smaller, and vice versa, but the ratio of increase of these coefficients to z becomes gradually less. Especially, in the case of tooth profile A type, these coefficients approach a nearly constant value in case of

(a) In the case of 20° pressure angle

(b) In the case of 14.5° pressure angle

Fig. 10 Relation between number of teeth z and coefficient φ_R

the number of teeth $z=60$.

4.2 On the practical formula for the stress σ_{BA} at the center of tooth root curve

Practical formula for the stress σ_{BA} at the center of tooth root curve is also obtained by the same means as in the case of σ_R ,

$$\sigma_{BA}/E = \varphi_{Bi}(\delta/d_r)(D_i/d_r) \quad (26)$$

or

$$\sigma_{BA}/E = \varphi_{Bi}(\delta/D_i)(D_i/d_r)^2 \quad (27)$$

Figure 11(a), (b) show the relation between the number of teeth z and φ_{Bi} in various dimensions of gear. φ_{B1} is the value for the case of pressure angle $\alpha=20^\circ$ and φ_{B2} is of pressure angle 14.5° .

Moreover, $\varphi_{B1}, \varphi_{B2}$ have respectively the same tendency as $\varphi_{R1}, \varphi_{R2}$ as above mentioned.

5. Conclusions

(1) The stress at the root fillet of gear tooth when shaft was force-fitted into gear rim, could be obtained by applying the method of H. Neuber for gear, that is, the stress is decided through the theoretical analysis based on the transform function expressing the shape of gear and the theory of stress using complex variables.

(2) In case when the tooth profile is expressed by transform function expressed in Eq. (3), it is necessary to make the shape of the root fillet of theoretical tooth profile, that is, the curvature coincide

(a) In the case of 20° pressure angle

(b) In the case of 14.5° pressure angle

Fig. 11 Relation between number of teeth z and coefficient φ_B

well with that of the practical tooth profile. But, it seems that the noncoincidence in the addendum part of tooth has almost no influence on the calculated results of fillet stress by force-fitting, because of very small stress distributions by force-fitting in these areas. Therefore, this method of calculation is very effective even in case when $m=1, 2, 3$ is adopted in coefficient of " a_m " deciding the shape of the theoretical tooth profile.

(3) The practical formula to decide the fillet stress at the weakest section and at the center of tooth root curve of the force-fitted gear could be introduced by means of theoretical analysis in this paper, and these stresses can be easily calculated if diameter of gear and force-fitted conditions are given.

(4) The practical formulas to decide fillet stress σ_R at the weakest section and fillet stress σ_{BA} at the

center of tooth root curve of gear are expressed in a similar form except changing only the profile coefficient φ_i as shown in Eq. (22), (23), (26), (27).

This profile coefficient φ_i (φ_{R1} , φ_{R2} , φ_{B1} , φ_{B2}) is a function of number of teeth, tooth profile and pressure angle.

The magnitudes σ_R and φ_{BA} depend upon the radius of curvature at the root fillet of gear tooth and the shape of gear root.

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