Chatter Vibrations of Machine Tool or Work with Directional Stiffness Inequality*

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The cutting limit of stability can be increased by using a boring bar with a non-circular cross section. This fact has been reported by J. Thusty, L. K. Kuchma, and S. Kato-E. Marui. Among these analytical or experimental reports, there seems to be no coincidence in the most effective orientation angle between cutting edge and minimum direction of boring bar stiffness.

In this paper, many factors affecting the cutting limit of stability, that is, cutting conditions, time lag of cutting force to chip thickness fluctuation, shaft rigidity, asymmetry of stiffness, natural frequency, damping coefficient and angular velocity of shaft are investigated. Consequently, it is verified that three reports above mentioned coincide with the special cases of certain factors combined. The obtained results in this paper may be applied not only to the vibration-proof problems of boring bar, planer-tool, shaper-tool, lathe-tool, but also to those of workpiece set on lathe spindle.

1. Introduction

It is well known that in the cutting process using a boring bar of unequal bending stiffness chatter vibrations occur very easily with certain orientations between the cutting edge and the minimum stiffness of bar, but, on the contrary, they do not occur with other certain orientations (1)-(3). As a matter of fact the removal of vibrations may be expected from appropriate combination of orientation and stiffness inequality. The tool is set at an angle $\phi$ forward to the direction of rotation of the bar from the axis O-x', in which direction the bending stiffness of tool is minimum (Fig. 1). Figure 2 shows qualitatively the relation between $\phi$ and the critical values of stability of self-excited vibration by polar coordinates. The hatched areas in Fig. 2 show the dynamically unstable regions of a nonrotating machine tool which are analytical results given by J. Thusty (1). Solid shapes of the figure 8 are two examples of the critical value of cutting depth with a boring bar which are experimental results obtained by L. K. Kuchma (2). This curve takes the maximum value at $\phi = 45^\circ \sim 60^\circ$ where the bar is most stable in cutting. Broken lines with the shapes of starfish are two examples of the critical value of boring bar analyzed by S. Kato and E. Marui (3) and they take maximum values sharply at $\phi = 10^\circ \sim 20^\circ$, and $\phi = 100^\circ \sim 110^\circ$, respectively. From these three reports the chatter vibration of tool with unequal stiffness is known to occur very easily in the neighborhood of $\phi = 150^\circ$. But there does not yet exist a clear congruity among them regarding the most effective orientation for elimination of chatter vibration.

This paper states the effects of cutting conditions,

Fig. 1 Vibrations of a boring bar with directional stiffness (Rotating coordinate system O-x', y')

Fig. 2 Critical values of stability and orientation $\phi$ of a cutting edge

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time lag of the cutting force upon chip thickness fluctuation, mean value and asymmetry of bending stiffness, natural frequency of the bar, damping coefficient, rotating speed of the bar, etc. upon the critical value of the unstable region, and a scrutiny for the increase of the upper limit of the stable region. Moreover, the authors verified that these three reports can be completely elucidated by combining each of the factors above mentioned, though there exists a difference in them.

The conclusions regarding the removal of chatter vibration of a boring bar can be easily applied to vibration of the planer-tool, shaper-tool, lathe-tool, and also to that of the workpiece on a lathe.

2. Equations of motion and stability criterion of a boring bar

The vibratory system of a boring bar shown in Fig. 1 is regarded as a "two-degree-of-freedom" system having a lumped mass \( m \) where the vibration of work is omitted since the stiffness of the work may be assumed to be sufficiently larger than that of the boring bar. An analysis is made for the orthogonal cutting.

Take the gravitational center \( G \) in steady cutting as the origin \( O \). The \( x' \)-axis and \( y' \)-axis coincide with the directions of the minimum and maximum stiffness of the boring bar. The rotating coordinate system \( O-x', y' \) turns with the same angular velocity \( \omega \) as the boring bar. The spring constant, viscous damping coefficient, and natural frequency of the bar in the direction \( oz'(Oy') \) are \( k_s, c_s, p_s, (k_p, c_p, p_p) \), respectively. The thrust force and tangential cutting force are assumed to be as proportional to the depth of cut and to have time lags upon chip thickness fluctuation. \( K_N \) and \( K_T \) indicate two components of cutting force for a unit depth of cut, and \( H \) and \( h \) indicate the time lags of two components behind the instantaneous depth of cut. Mean value of the natural frequencies of the boring bar is \( p = \sqrt{(k_p^2 + p_p)^2} \). Assuming \( pH, ph \ll 1 \) rad and a small vibration \( x', y' \), the equations of motion about \( G \) in the coordinate system \( O-x', y' \) are as follows:

\[
\begin{align*}
mx'' + (c_s - KNH \cos \phi + Kh \sin \phi \cos \phi)x'' \\
+ (k_s - m\omega^2 + KNH \cos \phi - K_T \sin \phi \cos \phi)x' \\
+ (\omega^2 - m\omega^2 - KNH \sin \phi \cos \phi + Kh \sin \phi \cos \phi)y' \\
+ (KNH \sin \phi \cos \phi - K_T \sin^2 \phi)y'' = 0 \\
mv'' + (c_s - KNH \sin \phi \sin \phi + Kh \cos \phi \cos \phi)v'' \\
+ (k_s - m\omega^2 + KNH \sin \phi \sin \phi - K_T \sin \phi \sin \phi)v' \\
+ (\omega^2 - m\omega^2 - KNH \sin \phi \cos \phi + Kh \cos \phi \cos \phi)\phi' \\
+ (KNH \sin \phi \cos \phi + K_T \cos^2 \phi)\phi'' = 0
\end{align*}
\]

(1)

For the sake of convenience, the following dimensionless quantities are introduced:

\[
k = \frac{\Delta k}{k_s}, \quad k + \Delta k = k_s, \quad n = \frac{N}{c / m} \\
n + \Delta n = c / 2m, \quad \lambda = KN / m, \quad \varepsilon = \Delta H / k \\
\mu = KN / k = \lambda N / p, \quad \kappa = \lambda / K_T, \quad \eta = h / H
\]

Put the solution of Eq. (1) into Eq. (1), and the following characteristic equation (4) is derived.

\[
f(s) = s^4 + Ts^3 + Us^2 + Vs + W = 0
\]

where

\[
T = 4n - \lambda H \\
U = 2[\omega^2 + \omega^2 - 2(\Delta n)^2 + \lambda N(1 - 2nH) - 2\omega H - 2(\Delta n) \lambda H (\cos 2\phi - \kappa \sin 2\phi)] \\
V = (4n - \lambda H)(\omega^2 - \omega^2) - \varepsilon p^4(4\Delta n) \\
+ \lambda N (\cos 2\phi - \kappa \sin 2\phi) \\
+ 2\lambda N(\varepsilon p^2 + \omega^2 - 2\omega H) \lambda N \\
+ \varepsilon p^2 \lambda N (\cos 2\phi - \kappa \sin 2\phi)
\]

(5)

Provided that the four roots of Eq. (4) are all complex numbers, these roots become two pairs of a conjugate complex as follows:

\[
s_1, s_2 = \alpha_1 \pm i \beta_1, \quad s_3, s_4 = \alpha_2 \pm i \beta_2
\]

Expanding \( f(s) = (s - s_1)(s - s_2)(s - s_3)(s - s_4) = 0 \), the following relations are derived:

\[
T = -2(\alpha_1 + \alpha_2) \\
U = (\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2) + 4\alpha_1 \alpha_2 \\
V = -2(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2) - 2\beta_1 \alpha_2 \alpha_2 + \beta_1^2 + \beta_2^2 \\
W = (\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)
\]

(7)

In case that \( T < 0 \) or \( V < 0 \) holds, both or either of the two real parts \( \alpha_1, \alpha_2 \) of Eq. (6) must be positive and the system becomes dynamically unstable and the chatter vibration occurs.

In case that \( U < 0 \), the relation \( \alpha_1, \alpha_2 < 0 \) holds, i.e., either \( \alpha_1 \) or \( \alpha_2 \) becomes positive, and hence the system becomes dynamically unstable too. Next, in case that \( W < 0 \), contrary to \( W = (\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2) > 0 \) in Eq. (7), Eq. (6) is not satisfied. In this case the relations \( W = f(0) < 0 \), and \( f(\pm \infty) > 0 \) hold, and \( s \) has a positive root between \( s = 0 \) and \( s = \pm \infty \). The system is statically unstable in case that \( W < 0 \), i.e., the cutting operation can not be carried out.

In critical condition between stability and instability, the real part \( \alpha = 0 \) holds and also \( f(i \beta) = 0 \) of Eq. (4) holds. Therefore the real and imaginary parts must be equal to zero simultaneously, as follows:

\[
R[f(i \beta)] = \beta^4 - U \beta^2 + W = 0
\]

\[
I[f(i \beta)] = -\beta^4 - T \beta^2 - V = 0
\]

Frequency \( \beta \) of stationary vibration occurring in the critical condition is satisfied simultaneously by the following equations (10), (11) derived from Eqs. (8), (9), respectively.

\[
\beta = \sqrt{U \pm \sqrt{U^2 - 4W}} / 2
\]

\[
\beta = \sqrt{V/T
\]

(10)

(11)
Equating Eq. (10) to Eq. (11) Routh-Hurwitz' stability criterion is obtained as follows:
\[ R = TVU - V^2 - TVW = 4\alpha_1\alpha_2(\alpha_1 + \alpha_2)^2 + (\beta_1 - \beta_2)^2(\alpha_1 + \alpha_2)^2 + (\beta_1 + \beta_2)^2(\alpha_1 \alpha_2 - \beta_1 \beta_2) = 0 \]  
(12)
In the stable condition that \(\alpha_1 < 0, \alpha_2 < 0, i.e., R > 0\)  
\((R > 0 \text{ contains } U > 0)\), therefore the statically and dynamically stable condition is that all coefficients \(T, U, V, W\) of the characteristic equation (4) are positive, and also \(R\) of Eq. (12) is positive.

In Fig. 3 (\(\lambda N\)) vs. (\(k_m\)) or \(\mu_s = (k_n)\) vs. \(\phi = 0^\circ - 180^\circ\) relation of the curves \(R = 0, T = 0, V = 0\) are shown by full, thin, and broken lines, respectively with parameters of Experiment (B-3) achieved by Kato and Maruji. The parameters in this figure are \(K = K/N = 3, H = 0.5, PH = 0.6135, H = 0.384, n = 0.4827, \)  
\(p = 0.0317, \)  
\(p = 1.227 \text{ rad/sec, and } \omega_p = 0.01\). The critical values (\(\lambda N\)) for \(0^\circ - 360^\circ\) are the same as those for \(0^\circ - 180^\circ\) as shown in Fig. 2 or Eq. (5).

In the region \(\lambda N < 10^\circ(\text{rad/sec})^2\) of Fig. 3, \(U > 0, V > 0\), and \(R > 0\) always hold, but moreover \(T > 0, V > 0, R > 0\) must be satisfied for stability of the bar. In the region of \(\lambda N\) lower than the critical value \(\lambda N_{cr}\) (i.e., \(R = 0\)) and \(\lambda N_{opt}\) (i.e., \(T = 0\)) in Fig. 3, the relations \(T > 0, V > 0, R > 0\) are satisfied simultaneously. On the peaks of \(\lambda N\) vs. \(\phi\) curves, i.e., \(\phi = 0^\circ - 20^\circ\) and \(100^\circ - 110^\circ\), two curves \(T = 0\) and \(V = 0\) cross each other at the symbol \(O\). The condition \(\lambda N\) \(\phi\) which satisfies \(T = 0, V = 0\) simultaneously always yields \(R = 0\). \(T = V = 0\) (\(R = 0\)) is a sole condition that the real parts of four roots equal to zero simultaneously (\(\alpha_1 = \alpha_2 = 0\)). And in this case \(U = \beta_1^2 + \beta_2^2 > 0\), and \(W = \beta_1^2 \beta_2^2 > 0\) hold. Without inequality in stiffness \(\varepsilon, R = 0\) holds for Eq. (12) in the region of \(\lambda N\) smaller than either the curve \(T = 0\) or \(V = 0\), whereas existence \(\varepsilon\) makes two curves \(T = 0\) and \(V = 0\) cross each other, and enhances the critical value (\(\lambda N\)) (\(R = 0\)) up to the optimum value (\(T = 0\)), i.e.,
\[ \lambda N_{opt} = 4n/H \]  
(13)
Even though the relation \(R = 0\) is satisfied at the cross point of the curves \(T = 0\) and \(V = 0\) in case where \(\omega_0 = 0.04, \omega_0, R < 0\) holds in the vicinity of this point (mark \(O\)). When the cutting conditions and dimensions of the vibratory system are given, the most optimum orientation \(\phi_{opt}\) for stability is determined. Inserting \(\lambda N = 4n/H\) derived from \(T = 0\) into \(V = 0\) of Eq. (5) the equation for \(\phi_{opt}\) is obtained as follows:
\[ E \sin 2(\phi_{opt}) + F \cos 2(\phi_{opt}) = G \]  
(14)
where
\[ E = -\varepsilon(\cos^2 H - 2(\Delta n)) \]  
\[ F = \varepsilon^2 H - 2(\Delta n) \]  
\[ G = 2n + 2\omega_0 - \varepsilon(\Delta n/n)^2H \]  
\[ J = G/\sqrt{E + F} \]  
(14-a)
From Eqs. (14) and (14-a), \(\phi_{opt}\) is obtained easily as follows:
\[ \phi_{opt} = (1/2) \sin^{-1} [1 - (1/2) \tan^{-1}(F/E)] \]  
(15)
Provided that \(J > 1\) is satisfied, \(\phi_{opt}\) has two real roots. Selecting the orientation \(\phi = \phi_{opt}\), the system may be set in the most stable condition (\(\lambda N_{max} = \phi_{opt}\)).

3. Self-excited vibrations of nonrotating tool without damping and time lag of cutting force (comparison with J. Trusty's analytical result)

3.1 Unstable regions

In case of nonrotating tool system (\(\omega = 0\)) like planer-tool or shaper-tool and without time lag of cutting force (\(H = k = 0\)) and damping (\(n = 0\)) in the system, equations of motion of tool are given by omitting 2nd and 4th terms in Eq. (1). And a mode coupling between \(x'\) and \(y'\) exists only by the 5th terms in Eq. (1). As for coefficients of characteristic equation (4) the odd order coefficients \(T = 0, V = 0\) hold, and the even order \(U, W\) are
\[ U_0 = 2p^2 + \lambda N > 0 \]  
\[ W_0 = p^2(1 - \varepsilon^2) \]  
\[ + p^2 \lambda N (1 + \varepsilon (\cos 2 \phi - \kappa \sin 2 \phi)) \]  
(5-a)
A quadratic equation in \(s^2\)
\[ s^2 + U_0 s + W_0 = 0 \]  
(16)
gives the following roots,
\[ s^2 = (U_0 + \sqrt{U_0^2 - 4W_0})/2, \]  
\[ D_0 = U_0^2 - 4W_0 \]  
(17)
In case that \(s^2\) is a complex number, i.e., \(D_0 < 0\), \(s = \alpha + \beta\), where
\[ \alpha = \pm \sqrt{2} \sqrt{W_0 - U_b/2}, \quad \beta = \pm \sqrt{2} \sqrt{W_0 + U_b/2} \]

The real part \( \alpha \) is positive, and then the system becomes dynamically unstable. Critical value of stability \( \mu_c \) is derived from \( D_0 = 0 \) in Eq. (17) as follows:

\[ \mu_c = \frac{(\lambda_N)_c}{\beta^2} = 2\varepsilon_0 \left( \cos 2\phi - \kappa \sin 2\phi \right) \]

\[ \pm \sqrt{(\cos 2\phi - \kappa \sin 2\phi)^2 - 1} \] .................................(19)

The frequency \( \beta \) of steady vibration is given,

\[ \beta^2 = U_b/2 = \beta^2 + (\lambda_N)_c/2 \] .................................(20)

This value \( \beta^2 \) coincides with the arithmetic mean value of the natural frequencies of \( \beta = \beta^2 \), and \( \beta^2 + \lambda_N \) which are derived without inequality in stiffness (\( \varepsilon = 0 \)).

The dynamically unstable regions (\( D_0 \leq 0 \)) for \( \varepsilon = 0.5 \) are shown by broken lines with parameter \( \kappa = 1 \sim 4 \) in Fig. 4. Generally in case where \( \varepsilon \approx 0.5 \) an ordinate \( \mu_c \) may be multiplied by \( 2\varepsilon_0 \) [cf. Eq. (19)].

The larger \( \kappa = K_T / K_N \) becomes, the wider the unstable region becomes. And the smaller \( \varepsilon \) becomes, the thinner the does unstable region. Dynamically unstable region is restricted by \( \phi = 90^\circ + \tan^{-1} (1/\kappa) \sim 180^\circ \) because \( (\cos 2\phi - \kappa \sin 2\phi)^2 \geq 1 \), i.e., \( \sin \phi (\sin \phi + \kappa \cos \phi) \leq 0 \). From the minimum of \( \mu_c \), i.e., \( \phi = 0 \), the system becomes unstable most easily at \( \phi = 180^\circ - (1/2) \tan^{-1} \kappa \). Boundaries of statically unstable region \( W_b \leq 0 \) are shown by full line curves in Fig. 4. The statically unstable region \( W_b \geq 0 \) and the dynamically unstable region \( D_0 \leq 0 \) do not overlap each other, because \( W_b > 0 \) holds in case \( D_0 \leq 0 \), and \( W_b \leq 0 \) holds in case \( D_0 > 0 \) [cf. Eq. (17)].

Relations \( \alpha, \beta, \phi \) are derived from Eq. (16) and shown in Fig. 5 in case \( \kappa = 3, \varepsilon = 0.5 \) and as parameter \( \mu = 0 \sim 5 \). In the dynamically unstable region \( D_0 < 0 \), \( \alpha, \beta \) are derived from Eq. (18) instead of (16) and shown by broken lines.

The results obtained from section 3.1 agree perfectly with Tlusty's results[1] by putting \( K_N \sqrt{1 + \kappa^2} \)

\[ = k_n, \quad k_\alpha = \lambda_\alpha, \quad k_\beta = \lambda_\beta, \quad \kappa = \tan \theta, \quad 180^\circ - \phi = \theta \]. Lower critical value shown by the minus sign ahead of a square root in Eq. (19) corresponds to Tlusty's critical value shown by a single dotted chain line in Fig. 2.

3-2 Orbit and energy source of self-excited vibration

In case the system is statically and dynamically stable, i.e., \( W_b > 0, \quad D_0 > 0 \) purely imaginary root \( s = \pm i\beta \) in Eq. (16), and the vibration of gravity center \( G \) becomes a reciprocal line motion passing an equilibrium point \( 0 \) as follows:

\[ x = A \cos \beta t, \quad y = B \cos \beta t \] .................................(21)

In this case there is no energy variation in the system over one cycle of vibration. Tangent \( B/A \) of straight orbit of vibration is as follows:

\[ \frac{y'}{x'} = \frac{B}{A} = \frac{-\lambda_N \cos \phi}{(1 + \varepsilon) - \beta^2 + \lambda_N \sin \phi (\sin \phi + \kappa \cos \phi)} \]

.................................(22)

Tangent \( B/A \) depends upon frequencies \( \beta = \beta_1, \beta_2 \), and these two straight orbits do not make a right angle with each other because \( (B/A)_{\beta_1}(B/A)_{\beta_2} = -1 + \kappa \cot \phi / (1 - \kappa \tan \phi) = -1 \).

Next, in the dynamically unstable region \( W_b > 0 \) and \( D_0 < 0 \) \( x^* \) and \( y^* \) with real part \( \alpha \) shown by Eq. (18) are expressed as follows:

\[ x^* = e^{\alpha t} (A \cos \beta t - A^* \sin \beta t) \]

\[ y^* = e^{\alpha t} (B \cos \beta t) \] .................................(23)

Amplitude ratio in Eq. (23) is \( A^* : B = p(1 + \varepsilon) + (\alpha^2 - \beta^2) + \lambda_N \sin \phi (\sin \phi + \kappa \cos \phi): 2 \alpha \beta = -\lambda_N \cos \phi \times (\sin \phi + \kappa \cos \phi) \). The vibrating orbit of \( G \) is an ellipse from Eq. (23). The direction of the elliptic orbit is determined only by the sign of \( A^*/B \) and independent of the sign of \( B/A \). When the elliptic orbit has the direction in which the cutting force is smaller for the approach of tool to the workpiece than for the separation from the workpiece, the tool system gains the
energy, and self-excited vibration is induced \((\alpha>0)\). When the direction is reversed, the tool loses energy and the vibration is damped out \((\alpha<0)\). Influence of \(H, n, \omega_t\), etc. is discussed in Chapters 4 and 5.

4. Influences of cutting conditions and vibrational properties on the self-excited vibration of a boring bar

4.1 Influence of stiffness inequality \(\varepsilon\)

Although it is very effective against chatter vibration of a boring bar to make the diameter larger and/or the length shorter, the dimensions of bar are decided before cutting. Section 4.1 shows that a boring bar may be stabilized by milling two flats on the boring bar.

A boring bar with circular cross section has diameter \(d\), distributed bar mass \(m_{dt}\), concentrated mass of tool holder \(m_0\), spring constant at tool center \(k_0\), and natural frequency \(p_0\). Two parallel flats were machined along the boring bar so that the width between these flats was \(t\) as shown in Fig. 1. This noncircular boring bar has equivalent concentrated mass \(m\), spring constant \(k\equiv \Delta k\), and mean natural frequency \(p\), and they are expressed as follows:

\[
\begin{align*}
    m &= m_t + 0.23 m_0 \left( \frac{1}{\cos^{-1} \left( \frac{t}{d} \right)} - \frac{2}{\pi} \cos^{-1} \left( \frac{t}{d} \right) \right) + \frac{2}{\pi} \left( \frac{t}{d} \right) \sqrt{1 - \left( \frac{t}{d} \right)^2} \\
    k &= k_0 \left( \frac{2}{\pi} \cos^{-1} \left( \frac{t}{d} \right) + \frac{4}{\pi} \left( \frac{t}{d} \right) \right) \times \sqrt{1 - \left( \frac{t}{d} \right)^2} \left[ \frac{1}{2} + \left( \frac{t}{d} \right)^2 \right] \\
    \varepsilon &= \frac{\Delta k}{k} = \frac{8}{3\pi} \left( \frac{t}{d} \right) \sqrt{1 - \left( \frac{t}{d} \right)^2} \left( \frac{m_t + 0.23 m_0}{m} \right) \left( \frac{k_0}{k} \right) \quad \text{(26)}
\end{align*}
\]

In case that damping coefficients \(n \neq n_0\) are proportional to natural frequencies \(p, p_0\), respectively, the equation

\[
\begin{align*}
    \Delta n/n = \sqrt{1 + \varepsilon - 1} = \varepsilon/2 \\
    \Delta n/n = \varepsilon
\end{align*}
\]

holds. In case \(n = n_0\) are proportional to \(p, p_0\), Eq. (29) holds.

\[
\begin{align*}
    \Delta n/n = \varepsilon
\end{align*}
\]

Relations \(\varepsilon, k/k_0, p/p_0, p_0/p_0\), and \(k_0/p_0\) vs. \(t/d\) are shown in Fig. 6 \((c)\) derived from Eq. (24)\((~27)\) of a noncircular mild steel boring bar with \(d=5.90\ cm, l=31.3\ cm, m_0=1.936\ kg, m_{dt}/m_t=3.47, p_0=1.855\ rad/sec\). Circular marks in Fig. 6 \((c)\) indicate the experimental results\((~24)\) of \(\varepsilon, p/p_0, p_0/p_0\). Experimental results agree well with analytical ones in the vicinity of \(t/d=1\), but they do not coincide near \(t/d=0.5\) because the boring bar used in this experiment is not milled at both bar ends.

Figure 7 shows the relations of \((\lambda, t), \beta, \varphi\) with parameter \(t/d=0.5\sim 1.0, n/p=0.05\). On the mark \(\bigcirc\) which is the cross point of \(T=0\) and \(V=0\), the values of \(\beta\) in Eq. (11) do not exist, and these values of \(\beta\) (mark \(\bigcirc\)) are derived from Eq. (10).
Frequency $\beta$ during $BCD$ is higher ($\approx p_B$) and that during $DEAB$ is lower ($\approx p_A$) and $\beta$ suddenly changes at the point $B$, $D$ corresponding to $\phi=\phi_{\text{opt}}$. In order to maintain $T=0(\lambda_N=4n/H)$ independently of $t/d$, value $(k_B/k_A)(K_N/k_0)$ is used instead of $\lambda_N$ in Fig. 7. With a decrease of $t/d$, $(k_B/k_0)$ becomes smaller as shown in Fig. 6(c). Though $T=0$ is constant as $(k_B/k_A)(K_N/k_0)_{\text{opt}}=(4n/p)/(p_BH)$ holds, $(K_N/k_0)_{\text{opt}}$ becomes smaller in proportion to $(k_B/k_0)$ with a decrease of $t/d$, so $t/d$ must not be fairly small.

As $n/p$ is sufficiently smaller compared with $p_H$ in practical cutting, Eq. (15a) is derived from Eq. (14a),

$$J=\left[\frac{-\Delta n}{n}+2\left(\frac{n}{p}+\frac{\kappa \omega}{p}\right)/\varepsilon(p_H)\right]/\sqrt{1+(\kappa \gamma)^2}$$

(15-a)

In case $|J|\geq 1$, number of $(\phi)_{\text{opt}}$ is either one or zero, and $(\lambda_N)$ curve has one convex and one concave during $\phi=0^9-180^9$. On the other hand in case $|J|<1$, two roots of $(\phi)_{\text{opt}}$ exist in existence, and $(\lambda_N)$ curve has two peaks $B, D$ and two valleys $C, D$ during $\phi=0^9-180^9$. Thus, Kato's analytical results are obtained for the case $|J|\geq 1$, and Kuchma's experimental results(2) are obtained for the case $|J|<1$. Eq. (15-a) shows that the case $|J|\geq 1$, i.e., Kuchma's result is easily derived for larger values of $n/p$, $k_B/k_A$ and also for smaller values of $\varepsilon$, $p_H$, $k_B/k_A$, whereas the case $|J|<1$, i.e., Kato's result is contrary to the above tendency of parameters.

Curves $(\lambda_N)_n$ are convex at $\phi=20^9-60^9$, and concave at $\phi=150^9-180^9$ as in Kuchma's result in case of small $\varepsilon$ (i.e., $t/d\leq 0.091$) as shown in Fig. 7. But $(\lambda_N)_n$ has two peaks $B, D$ in case of large $\varepsilon$ (i.e., $t/d<0.91$) and maximum points $B, D$ (symbol $\bigcirc$) just equal to $T=0$ in the case that $t/d<0.7$.

According to Kato's analytical results**, regarding the influences of $\varepsilon=2-4$, $\gamma=0.4-0.8$ on $(\phi)_{\text{opt}}$, the orientation $(\phi)_{\text{opt}}$ becomes smaller with an increase of $k_B/k_A$. These results depend partially on the decrease of $|J|$ in Eq. (15-a) with increase of $k_B/k_A$, but mainly on the decrease of the second term $-1/2 \times \tan^{-1}(F/E)$ in $1/2 \times \tan^{-1}(1/(k_B/k_A))$ of Eq. (15).

Peak value $(K_N/k_0)_{\text{max}}$ at the points $B, D$ of critical value $(K_N/k_0)_C$ with parameter $n/p=0.05, 0.10, 0.15$ and its orientations $\phi_B, \phi_D$ are marked by circle in Fig. 6(a), (b). $T=0$, i.e., $(K_N/k_0)_{\text{opt}}=(4n/p)/(p_BH)$ and $(\phi)_{\text{opt}}$ derived from Eq. (15) are shown by thin lines. With a decrease of $t/d$, $(K_N/k_0)_{\text{max}}$ comes close to the curve $T=0$ and moves to the left along $T=0$. The optimum value of $t/d$ almost exists in the range $0.6-0.8$, although it decreases with an increase of $n/p$.

Putting $\varepsilon=\Delta n/n=\omega/p=0$ in Eq. (12)

$$R=2\pi(2n-\lambda_NH)(p_H(4n-\lambda_NH)^2$$

$$+8n^2\lambda_NH(1-2nH)\lambda_N)^2$$

(12-a)

The critical condition

$$(\lambda_N)_{\text{opt}}^2=(K_N/k_0)_C^2=(4n/p)/(p_BH)$$

(12-b)

is marked by $\bigcirc$ on the line $t/d=1$ in Fig. 6(a), and this value is just a half of the value marked $\bigcirc$ satisfying $T=0$, i.e., $(K_N/k_0)_{\text{opt}}=(4n/p)/(p_BH)$. After all using a flat shaft milled from a circular one, the value of $(K_N/k_0)_{\text{max}}$ may be nearly doubled.

4. Influence of damping coefficient $n$ and time lag of thrust force $H$

Thus, the optimum combination of inequality $\varepsilon$ and orientation $(\phi)_{\text{opt}}$ can make the critical value $(\lambda_N)_{\text{opt}}^2$ up to $(\lambda_N)_{\text{opt}}^2=4n/H$. Increasing $n$, and decreasing $H$ is most desirable for preventing self-excited vibrations of the boring bar. In comparison with the influence of $n, H$, that of $k, \gamma$ and $\Delta n/n$ may be considered small and secondary.

Relations between $(K_N/k_0)_C$ and $\phi$ in case of $n/p=0.05, 0.15$ are shown in Figs. 7, 8, respectively. In this figure, the curves like Kuchma's experimental results(5) are given for the range $1/t/d\geq 0.77$ because $|J|\geq 1$. Kato's experimental results(5) resemble those for the range $t/d<0.77$. Even in case that peaks $B, D$ of $(K_N/k_0)_C$ do not reach the curve $T=0$, increase of $n/p$ by three times enhances the values $B, C$ nearly by three times as shown in Fig. 7 and 8. Increasing the damping of the bar is most effective in the improvement of stability.

In the lower figure of Fig. 9 the critical values $R=0$ ($B, 3, \omega/p=0$ in Fig. 3), $D=0$ of Eq. (17), $D=U^2-4W=0$ are shown by full, broken, and

* Values of $J$ in Experiment B-1, 2, 3 are 0.174(0.413), $-0.016(0.185)$, $-0.129(0.025)$ for $\omega/p=0(\omega/p=0.012)$.

** See Figs. 5, 6, 9, 10 in reference (3).
dotted lines, respectively. Real part $\alpha > 0$ and imaginary part $\beta$ of complex number $s$ derived from Eq. (4) are shown in the upper and middle figures of Fig. 9 for four cutting conditions $\mu = 0.0266$ (no cross point with $R = 0$, i.e., $R > 0$), 0.0724 (two points), 0.1461 (four points), 0.2179 (zero, $R < 0$). $\beta$ derived from Eq. (10), (11) is shown by full and one dotted chain line in the middle figure. In Tuy's unstable region ($D \leq 0$), $\alpha$ derived from Eq. (18) is shown by broken lines. In the region $D \leq 0$ where $\beta$ does not exist, $\alpha$ and $\beta$ shown by dotted lines are derived from $W$, $U$ instead of $W_0$, $U_0$ in Eq. (18). The difference between $\beta$ shown in Fig. 9 and $\beta$ shown in Fig. 5 is a small value of the order $(\omega/p)^2$, $(n/p)^2$, $(pH)^3$. But $\beta = \sqrt{V/T}$ of Eq. (11) shows that one dotted chain line is influenced largely by the small quantity because both $V$ and $T$ are the first order. Chain line and full line may cross each other as $V$ of Eq. (5) varies sinusoidally with $\phi$ by amplitude of the order $\epsilon(pH)$. These cross points are shown by $\bigcirc$ in Fig. 9. At these points Eq. (8) and (9) hold (i.e., $R = 0$), then Eq. (4) has a purely imaginary root, $s = i\beta$. At the angle $\phi$ between two marks $\bigcirc$ $\alpha \leq 0$ always holds. In case of $\mu = 0.2179$ two frequencies $\beta > p$, $\beta < p$ shown by the mark $\bigcirc$ occur simultaneously in the vicinity of $\phi = 20^\circ$, 102°.

Since Eqs. (8) and (9) have a common real root at the cross point $\bigcirc$ ($R = 0$) in Fig. 9, the region $R > 0$ is contained in the range of $D > 0$ and $V/T > 0$. Therefore the range $D \leq 0$ where Eq. (8) takes a complex root is involved in the unstable region $R \leq 0$.

As the difference between $D_0 = 0$ and $D = 0$ is in the second order of small quantity, the unstable region $R \leq 0$ is wider than the unstable region $D_0 \leq 0$ given by Tuy's (cf. Fig. 2). Real part $\alpha$ of $s$ (marks $\bigcirc$, $\bullet$ in Fig. 9) is always larger than the values of $\alpha$ obtained from Eq. (18) shown by broken lines; this fact means the "negative damping" effect of the time lag $H$.

Figure 10 shows the case of $pH = 0$. Full line curves $\beta$ of Eq. (10) are nearly equal to $\beta$ in Fig. 9. A numerator $V$ of chain line curve $\beta = \sqrt{V/T}$ varies sinusoidally with $\phi$ by the order $\epsilon(n/p)$ smaller than $\epsilon(pH)$. Cross point $\bigcirc$ of a full line and a chain line deviates from the point of equal roots ($D = 0$) by the order $\epsilon(n/p)$, $\kappa \omega/m$. As the cross points $\bigcirc$ exist on the real root $\beta$, $R \geq 0$ exists in the range $D > 0$ and $V/T > 0$, and $R = 0$ coincides nearly with $D = 0$ as $pH = 0$ in Fig. 10. Because of damping the maximum values of real part $\alpha$ (marks $\bullet$, $\bigcirc$) of $s$ are smaller than $\alpha$ of no damping (broken line) and the unstable region is slightly wider than that with no damping.

The unstable region like Fig. 10 cannot be observed in case of large $pH$, as in cutting mild steel, but may be observed in case of small $pH$ as in cutting cast iron.

4.3 Influence of angular velocity $\omega$

Coriolis' forces $-2m\omega y$, $2m\omega x$ and centrifugal forces $-m\omega^2 x^2$, $-m\omega^2 y$ necessarily exist in Eq. (1) because of rotation of boring bar. In boring operations the influence of $\omega$ has been usually neglected**.

---

** For example, $\omega/p = 0.012$ in reference (3), and $\omega/p$ is neglected in their numerical calculation.
The centrifugal force is in proportion to \((\omega/p)^2\) and it is actually smaller enough than the restoring force of the bar, and then the centrifugal force may be neglected in the quite stable region \((W' > 0)\) in boring operation. But Coriolis' forces in proportion to \(\omega/p\) make a dynamical coupling between \(x', y'\) and \(y', z'\), and tend to make an elliptical orbit of motion \((x', y')\). And they have a great influence on the occurrence of self-excited vibrations. Rotating speed \(\omega/p\) does not affect the maximum value of stability \((\lambda_x)_{\text{opt}} = 4n/H\) but fairly affects the optimum orientation \((\phi)_{\text{opt}}\). In a numerator of Eq. (15-a) \(\omega/p\) affects in the same order of \(n/p\).

Critical value of stability \((\lambda_x)_{\text{opt}}\) and frequency of steady vibration \(\beta\) vs. \(\phi\) are shown in Fig. 11 with the same parameters as Fig. 3. With an increase of \(\omega/p\), \(\phi)_{\text{opt}}\) approaches 150°. Peaks B, D become lower than \((\lambda_x)_{\text{opt}}\), in case where \(\omega > 0.04\). \((\phi)_{\text{opt}}\) of the cross point \((T' = 0, V = 0)\) derived from Eq. (15) vs. \(\omega/p = 0.125\) are shown in Fig. 12 by thin lines, and the values \(\mu_B, \mu_D, \mu_C, \mu_E\) orientation \(\phi_B, \phi_D, \phi_C, \phi_E\) are shown by circular marks. Equal roots \((J = 1)\) of \((\phi)_{\text{opt}}\) are obtained at \(\omega/p = 0.075\) and in the region \(\omega/p > 0.075\) cross points between \(T' = 0\) and \(V = 0\) (mark \(\circ\)) vanish, and then \((\lambda_x)\) curve has one peak in the vicinity of \(\phi = 135^\circ + (1/2)\tan^{-1}(1/Kx) \approx 150^\circ\).

Generally the relation \(\mu_B, \mu_E\) holds as regards peaks B, D. Though two values \((\phi)_{\text{opt}}\) exist when \(\omega/p < 0.075\), peaks B, D may not reach the optimum value \((\lambda_x)_{\text{opt}}\) \((T' = 0)\) as shown in Figs. 11 and 12.

With an increase of \(\omega/p\), peaks \(\mu_B, \mu_D\) become smaller in case of a flat shaft just as a circular shaft \((\varepsilon = \Delta n/n = 0)\) as shown in Fig. 12. The stability criterion (12) is derived by using \(\lambda_x = 2n/H\) of Eq. (12-b) as follows:

\[
R = -16\pi(n/H)^2((\pi^2(4n^2 + 1) + 2n^2H))\omega^2
-16\pi(n/H)^2(1 + \pi^2(4n^2 + 1) + 2n^2H))\omega^2
\]

As the bracket \((-\cdots)\) in Eq. (12-c) is positive, \(R < 0\) holds with an increase of \(\omega/p\) in considering only the terms \(\omega, \phi\). Thus the existence of \(\omega\) enhances instability of the bar, and the critical value of stability in a circular shaft \((\varepsilon = \Delta n/n = 0)\) becomes always \((\lambda_x)_{\text{crit}} < 2n/H = (\lambda_x)_{\text{opt}}/2\). Value \((\lambda_x)_{\text{crit}}\) for \(\omega/p = 0.01\) becomes nearly a half of that for \(\omega/p = 0\).

5. Chatter vibration of a workpiece set on a lathe

The vibration of a boring bar is considered in Chapter 4, and the two-degree-of-freedom system of planer-tool, shaper-tool and lathe-tool can be treated as the particular case \((\omega/p = 0)\) of a boring bar. The vibration of a workpiece with inequality in headstock rigidity or in tailstock rigidity is considered in Chapter 5.

5.1 Equations of motion of workpiece having inequality in stationary stiffness

The tool is rigid enough in comparison with the workpiece and therefore the vibration of the tool is here neglected. The system consisting of a spindle and a workpiece is considered a two-degree-of-freedom system having a lumped mass \(m\). Let deflection of workpiece center \(G\) be \(x, y\), and let spring constant of shaft connecting the rigidity of headstock and tailstock, and damping coefficient be \(k_x, c_x\) in \(x\)-direction, and \(k_y, c_y\) in \(y\)-direction, respectively. The equations of motion about \(G\) are as follows:

\[
\begin{bmatrix}
mk + (c_x - K_x \cos^2 \phi + K_T \sin \phi \cos \phi) & + (k_x + K_x \cos^2 \phi - K_T \sin \phi \cos \phi) x \\
(k_x + K_x \cos^2 \phi - K_T \sin \phi \cos \phi) & + (k_T \sin \phi \cos \phi - K_T \sin^2 \phi) y
\end{bmatrix}
\begin{bmatrix}
x' \\
y'
\end{bmatrix}
= \begin{bmatrix}
x'' \\
y''
\end{bmatrix} = 0
\]

\[
my' + (c_y - K_y \sin^2 \phi + K_T \sin \phi \cos \phi) y' + (k_y + K_y \sin^2 \phi + K_T \sin \phi \cos \phi) y
\]

\[
- (K_x \sin \phi \cos \phi - K_T \sin^2 \phi) y + (k_x + K_x \cos^2 \phi + K_T \sin \phi \cos \phi) x
\]

\[
+ (K_T \sin \phi \cos \phi + K_T \cos^2 \phi) x = 0
\]

By putting \(x, y\) in place of \(x', y'\) and \(\omega = 0\) Eq. (1) coincides with Eq. (30). Therefore the vibration of
the workpiece can be eliminated by giving a stationary inequality \( \varepsilon \) in rigidity of spindle bearing support, and the results obtained in Chapter 4 are just applied to workpiece of lathe.

### 5.2 Particular case when the direction of inequality in bearing support coincides with lathe-tool direction (\( \phi=0^\circ \))

Stiffness of spindle bearing support is generally higher in the vertical direction than in the horizontal one. Thus the particular case of \( \phi=0^\circ \) is to be discussed. Putting \( \phi=0^\circ \) into Eq. (30), Eq. (30-a) is derived.

\[
\begin{align*}
&x = (2n-2\Delta n-\lambda N)\varepsilon + p^2(1-\varepsilon + \lambda N/p^2)x = 0 \\
y &= 2(n+\Delta n)\varepsilon + p^2(1+\varepsilon)y = \varepsilon \eta H H / - \varepsilon N N\n\end{align*}
\]

The horizontal vibration \( x \) is independent of the vertical vibration \( y \) and using Eq. (3).

\[
\beta = \sqrt{p^2(1-\varepsilon) + \lambda N - \alpha^2}
\]

The dynamically unstable vibration \( x \) occurs when \( \alpha \geq 0 \), i.e., \( \lambda N \geq 2n(1-\Delta n/n)/H \). Vertical vibration \( y \) is a forced one excited by \( x \) and \( \dot{x} \). Steady vibration \( x, y \) (\( \alpha = 0 \)) is expressed as follows:

\[
\begin{align*}
x &= A \cos \beta t - A' \sin \beta t \\
y &= B \cos \beta t - B' \sin \beta t
\end{align*}
\]

Where

\[
\begin{align*}
A &= -\varepsilon N + 2\varepsilon p^2 \\
A' &= 2(n + \Delta n)\sqrt{p^2(1-\varepsilon) + \lambda N} \\
B &= \varepsilon N \\
B' &= 2(n - \Delta n)\varepsilon \eta H H / - \varepsilon N N \\
Q &= A' - A - B' = 2\varepsilon n N N / \eta H H (1-\varepsilon) + \lambda N
\end{align*}
\]

According to whether \( Q > 0 \) or \( Q < 0 \) in Eq. (32) the loop turns anti-clockwise or clockwise, respectively as shown in Fig. 13. In the practical lathe with a little inequality in stiffness (\( \varepsilon = 0, \Delta n/n = 0 \)) the sign \( Q \approx 0 \) (i.e., \( \gamma \approx 1 \)) gives the same loop direction as shaft rotation, or the reverse one, severally.

### Conclusions

Conclusions obtained in this paper are summarized as follows:

1. The limit of stability of a boring bar is greatly affected by the following factors: Inequality in stiffness of the boring bar \( \varepsilon \), time lag of thrust force \( H \), mean value of damping coefficient \( n \), rotating speed of shaft \( \omega \), orientation between cutting edge and stiffness inequality \( \phi \), ratio of cutting force to thrust force \( \varepsilon = K T/K N \), and ratio of time lags \( \eta = h/H \).

2. Adoption of optimum orientation of a flat bar can raise the upper limit of stability up to twice the limit of a circular bar. Maximum value of stability is proportional to \( n \) and inversely proportional to \( H \).

3. Three papers have already been published with regard to the limit of stability of the tool with stiffness inequality, but the discrepancies in these papers may be satisfactorily explained, and they can be considered special cases of vibrational characteristics and cutting conditions.

4. Flucty's criterion of stability is derived as a particular case (\( H = n - \omega = 0 \)) of this paper. Influences of \( H, n, \omega \) are extensive and the unstable regions are greatly affected by their existence.

5. The larger \( n, \kappa n, 1/n, 1/H, 1/k \) become, the closer the unstable regions agree with Kuchma's experimental results which have a maximum and a minimum value in \( (\lambda N) \), for orientation \( \phi = 0^\circ \sim 180^\circ \). On the contrary, the smaller \( n, \kappa n, 1/n, 1/H, 1/k \) become, the unstable regions are closer to Kato's results having two peaks and two concaves.

6. Vibrations of planer-tool, shaper-tool, lathe-tool, and workpiece of lathe can be treated as a particular case (\( \omega = 0 \)) of boring bar. Self-excited vibration may be prevented by giving a stationary stiffness inequality in tool, headstock, or tailstock.

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### References


Discussion

S. Ōno (University of Tokyo):

(1) Conclusions (1), (2), (4) and (6) indicate the influences of various parameters on the critical value of stability, but how are these conclusions supported by the experimental results so far?

(2) Conclusion (5) states that analytical results of this paper can be made nearly to coincide with experimental results [either reference (2) or reference (3)] by varying the parameters. Do the parameters in the cited experiments (2), (3) support a parameter tendency as indicated by conclusion (5)?

(3) Though Eq. (1) is derived by assuming \( p_H, \psi \ll 1 \), the values of \( p_H, \psi \) are not so small as shown in Figs. 3, 6~9, and 11~13. Are these facts independent of the conclusions?

(4) A theory of time lag in cutting force was introduced by Doi and Kato\(^{*1}\). Let me know any other reports regarding time lag.

(5) This paper assumes a difference between time lag \( H \) in thrust force and time lag \( h \) in main cutting force. How does one explain this fact \( (H \neq h) \) using the cutting mechanism since these two forces are only two components of a cutting force divided into two directions for convenience?

T. Hoshi (Kyoto University):

(6) The authors assume that \( p_H, \psi \ll 1 \) in the fundamental equation (1). This contradicts their numerical calculation, for example, a rather large value of \( p_H = 0.615 \) used in Fig. 3. If \( H = 0.005 \) sec is used as in reference (3), \( p = 12727 \) rad/sec = 12277/π (sec) = 195.3 (sec), and \( p_H = 195.5 \) (sec) × 0.005 (sec) = 0.976; hence, this is in conformity to the initial assumption that \( p_H \ll 1 \). What value of \( H \) is used in your calculation?

(7) The wave cutting at the previous cut necessarily causes a fluctuation of depth of cut, i.e., regenerative effect. However, this effect is not introduced in Eq. (1) and (30). What is the reason why this method starting from the equation of primary chatter is applicable to the chatter of the boring bar which essentially involves the regenerative effect?

(8) According to the theories of regenerative effect given by Tuaty, Tobias and Merritt, the variation of cutting force \( F \) is as follows:

\[
F = rb\{\mu X(t-T) - X(t)\}
\]

where \( r \) : dynamical specific cutting force, \( b \) : width of cut, \( \mu \) : overlap factor, \( -X(t-T) \) : depth of cut at the previous cut (time \( T \) before \( t)\), \( -X(t) \) : depth of cut at the instant (time \( t)\). In the case of the stationary vibration \( |X(t-T)| = |X(t)| \) where \( \mu = 1 \) (e.g., for the boring or turning operation) above is shown to be

\[
F = 2rb\left\{\sin \frac{\psi}{2} \left[-X \left\{t - \frac{1}{p} \left(\pi - \frac{\psi}{2} \right)\right\} \right]\right\}
\]

where \( p \) : frequency (rad/sec).

The author's answer (7) is interpreted to point out that the time lag \( H \) is equivalent to \( (\pi/2 - \psi/2)/p \), or \( p_H = (\pi/2 - \psi/2) \). But \( \psi \) is the phase difference between the previous cut \( X(t-T) \) and the present cut \( X(t) \), and it depends on \( \epsilon \) derived as the fraction part of the number of chatter marks per one revolution = \( (\psi/2\pi)T \) = \( N \) (an integer) \( + \epsilon \) (a fraction). Namely \( \psi \) is given by \( \psi = 2\pi(1 - \epsilon) \) and it can take any value between 0 and 2\( \pi \). Therefore the phase \( (\pi/2 - \psi/2) \) can hold a value between \( \pm \pi/2 \). When a general machine structure is in resonance, the decrease of the cutting thickness lags behind the increase of the cutting force. Thus, a self-excited chatter occurs when \( 0 < (\pi/2 - \psi/2) < \pi/2 \), the value being sharply variable according to the time \( T \) taken for one revolution.

If this \( (\pi/2 - \psi/2) \) is represented by \( p_H \) as interpreted by the authors, since it is shown in Appendix, Fig. 1(b) that \( (\mu)_{\text{max}} \) varies greatly with \( p_H \), the authors must be very careful in applying the results of this paper derived from a supposedly constant \( p_H \), to the self-excited chatter associated with the regenerative effect.

E. Marui (Mie University):

(9) In case that \( W_s > 0 \) and \( D_0 < 0 \), a root \( \epsilon \) of Eq. (18) has a negative and a positive real part, and the system becomes dynamically unstable. In which direction the elliptical orbit of \( G \) goes around depends upon the sign of the real part \( \alpha \) as shown in Eq. (23). Correspondence of the rotational direction of the orbit to the sign of \( \alpha \) is not so clear. The discussuer wishes to have this point explained concretely.

Is the discussuer justified in considering that there is a decrease in the limit of stability with an increase of \( \omega \) in section 4.3 as the existence of Coriolis' force makes the elliptical orbit, and that the energy goes into the system with a similar mechanism to section 3.2?

(10) Though the preceding discussion (9) concerns only the case that \( \omega = 0, H_s = 0, n = 0 \), how becomes the vibrational orbit of an actual boring bar for any orientation \( \phi \) in case that \( \omega \neq 0, H_s \neq 0, n \neq 0 \)?

Authors' closure

(1) The equations of motion (1) and the
characteristic equation (4) which give conclusions (1), (2), (4) and (6) are the same as Eq. (11) and Eq. (13) in reference (3). Analytical results derived from Eq. (1) and (4) coincide, at least qualitatively, with the experimental results in case of one blade cutting(4). Further, similar applications would be in good agreement with the experimental ones in case of two blades cutting(4).

(2) In reference (2) the cutting conditions, material of workpiece and vibrational properties except dimensions of the boring bar \((l/d=0.6\) and 0.8) are unknown, but probably the condition that \(p_{\phi} < 1\) is satisfied as seen in cutting of cast iron, and then \(|J|>1\) in Eq. (15-a) (cf. Fig. 8 and 10).

All experiments B-1, 2 and 3 in reference (3) show the case of \(|J|<1\) (as seen in the footnote* of p. 947), and the critical values of stability have two peaks and two concaves. Because the value \(|J|\) of B-1 is largest, i.e., \(\varepsilon\) is smallest among the three experiments, the critical limit of stability (B-1) resembles the experiment of reference (2). It seems to be the result of using a little larger value \((H=0.0005\) sec) than the actual one that the experimental results (B-1, 2 and 3) are qualitatively closer to Fig. 8 than Fig. 3 and 7.

(3) Equation (1) is derived from the approximation that a cutting force with time lag \(H\) is in proportion to \(x'(t-H)=x_0 \sin pt(t-H)=x'(t)\cos p(t-H)-Hx'(t)\sin p(t-H/p)\approx x'-Hx'\) when the vibration of the boring bar is expressed as \(x'(t)=x_0 \sin pt\). When \(pH\) is not so small, the assumptions that \(\cos pH\approx 1\) and \(\sin pH/(pH)\approx 1\) are not satisfied, and the error of approximation exists, but its effect is negligibly small, qualitatively, on the limit of stability if only \(pH > -\pi\). Append.-Fig. 1 (a) shows the relation \(\mu_0, \beta/p\) vs. \(\phi\) with a rather large \(pH=0.6135, 1\) and 2. Full lines are the exact solution(6) by another procedure considering \(pH\) not small, and broken lines are approximate solution of Eq. (12). Approximate solutions of \(\mu_0\) are slightly smaller than exact ones. Relation of maximum values of stability \((\mu_{0,max})\) vs. \(\phi\) is shown in Append.-Fig. 1 (b). Exact solutions of \((\mu_{0,max})\) are soon derived by multiplying approximate ones by \(pH/\sin pH\).

(4) Some papers concerning the time lag in cutting forces have been presented by S. Doi and S. Kato(6), combining a

thrust force and a main cutting force on a synchroscope does not show a straight line \((H=0)\), but a loop \((H=\phi H)\).

In the cutting mechanism in chip formation of flow type, a resultant cutting force mainly consists of a shearing resistance of chip generated in the shearing plane and a frictional resistance of chip sliding on the tool face. Although these two resistances relate to one another, the latter is considered to lag in time behind the former upon a sudden change of cutting depth. The thrust force consists mainly of the frictional resistance, and partly of the shearing

---

resistance in the cutting mechanism, then the time lag \( H \) in thrust force is larger than time lag \( h \) in main cutting force \(^{(23)}^{(43)}^{(46)}\).

(6) Time lag of thrust force \( H = 0.0005 \) sec was used in our calculation. Circular measure \( p = 0.615 \) rad as \( p = 0.822 \) rad/sec in Figs. 2, 3, 9, 11 and 12, and also \( p = 0.975 \) rad as \( p = 1.85 \) rad/sec in Figs. 6, 7 and 8. Regarding the assumption \( pH < 1 \) please refer to the answer (3) for Mr. S. Ono.

(7) Primary and regenerative chatters are quite different in their mechanism of occurrence. As regards the relation between critical limit of stability and vibrational characteristics, it is permissible to consider that a time lag of cutting force exists both in these two chatters as mentioned in reference \(^{(43)}\). So this paper is also applicable to regenerative chatters considering an equivalent time lag.

(8) It should be firmly borne in mind that we must be careful in dealing with regenerative chatters. How to estimate an equivalent time lag in place of regenerative effect will be explained in the following paper \(^{(43)}\).

(9) Relation of vibrational orbit vs. \( \phi \) is shown in Appendix-Fig. 2 (a) where \( \kappa = 3, \epsilon = 0.5, \mu = 0.1, 0.2, 0.3 \) and 0.5 by inserting the root \( s \) of Eq. (16) into Eqs. (22) and (23). The major principal axis of tool orbit is ahead by angle \( \phi \) from \( x' \) axis in the anticlockwise direction. Two straight lines are shown by full line curves in Appendix-Fig. 2 (a) in that \( s = i \beta \), \( i \beta \) (\( \beta_1 > \beta_0 \)). Angle \( \phi \) of major principal axis of elliptical orbit and amplitude ratio \( -A'/B \) are shown by broken lines when \( s = \alpha \pm \beta \) (\( \alpha, \beta > 0 \)).

As \( \cos \phi (\sin \phi + \kappa \cos \phi) \) is always positive in the possible range of dynamic instability (\( \phi = 90^\circ + \tan^{-1} (1/2) \)) to \( \phi = 180^\circ \), amplitude ratio \( A'/B = -2A(\alpha - \beta) \times (\beta/|\mu \cos \phi (\sin \phi + \kappa \cos \phi)|) \leq 0 \) according to \( \alpha \geq 0 \), i.e., \( \alpha A'/B < 0 \). Independent of the sign of \( B/A \), four points \( 1, 2, 3, 4 \) on the elliptical orbit correspond to \( \beta = 0^\circ, 90^\circ, 180^\circ, 270^\circ \) when \( A'/B < 0 \), and inversely correspond to four points \( 1, 2, 3, 4 \) when \( A'/B > 0 \) as shown in Appendix-Fig. 2 (b). Thus, the vibration of boring bar increases more and more in amplitude describing a clockwise loop in case \( \alpha > 0 \), whereas it damps out describing an anti-clockwise loop in case \( \alpha < 0 \).

Lastly your opinion regarding the influence of \( \omega \) is the same as ours.

(10) In general, free vibrations of the boring bar \( x', y' \) in case that \( \omega = 0, H, h = 0, n = 0 \) are expressed in the similar equation to Eq. (23) as follows:

\[
\begin{align*}
x' &= e^{st} (A \cos \beta t - A' \sin \beta t) \\
y' &= e^{st} (B \cos \beta t - B' \sin \beta t)
\end{align*}
\]

where

\[
\begin{align*}
A &= \rho^2(1 + \epsilon) + \alpha^2 - \beta^2 - \omega^2 + \lambda_N \sin \phi \\
&\times (\sin \phi + \kappa \cos \phi) + \alpha(2(n + \lambda N)) \\
&- \lambda N H \sin \phi (\sin \phi + \kappa \cos \phi) \\
A' &= \beta(2 \alpha + 2(n + \lambda N)) \\
&- \lambda N H \sin \phi (\sin \phi + \kappa \cos \phi) \\
B &= -\lambda N \cos \phi (\sin \phi + \kappa \cos \phi) \\
&+ \alpha(-2\omega + \lambda N \cos \phi (\sin \phi + \kappa \cos \phi)) \\
B' &= \beta(-2\omega + \lambda N \cos \phi (\sin \phi + \kappa \cos \phi))
\end{align*}
\]

Consider the orbit \( (x', y') \) on the complex plane \( z = x' + iy' \). \( z(t) \) is shown by the sum of two vectors rotating with angular velocities \( \pm \beta \).

Major and minor principal diameters of elliptic orbit \( (x(t)) \) are:

\[
\sqrt{(A-B')^2 + (B+ A')^2} \geq \sqrt{(A+B')^2 + (B-A')^2}
\]

and angle \( \phi \) between the major principal axis and \( x' \) axis is as follows:

\[
\phi = (1/2) \tan^{-1} ((B + A')/(A - B')) + \tan^{-1} ((B-A')/(A+B'))
\]

\[
= (1/2) \times \tan^{-1} 2(AB + A'B')/(A^2 + A'^2 - B^2 - B'^2)
\]

Rotating direction of the vibrational orbit becomes

\[\text{(a)}\]

\[\text{(b)}\]

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identical, or opposite to the shaft rotation according to whether \((A'B-AB')>0\), or \((A'B-AB')<0\), respectively, and the orbit becomes a straight line when \((A'B-AB')=0\). The relation of the sign of \((A'B-AB')\) vs. orientation \(\phi\) is shown in Append.-Fig. 3 with the same dimensions as Fig. 9. Under cutting conditions \(\mu = 0.2179, 0.1461, 0.0724\) and \(0.0266\), signs of \((A'B-AB')\) are calculated for two pairs of conjugate complex number \(s_1, s_2 = a_1 \pm i\beta_1, s_2 = a_2 \pm i\beta_2\), and are shown by the circular symbols \((a \geq 0)\) and by triangular symbols \((a < 0)\). The sign of \((A'B-AB')\) along the critical limit of stability is shown by symbols \(\bigcirc, \bullet\). The turning direction of vibrational orbit is identical to shaft rotation (symbol \(\bigcirc\) in the region over the limit BCD), but it is opposite to shaft rotation (symbol \(\bullet\) in the region over the limit DEAB). These results coincide well with the experimental ones\(^{(3)}\).