621.52

The Axial Flow Molecular Pump*

(4th Report, Performance of a Rotor with a Single Blade Row in the Transition Flow Regime)

By Tadashi SAWADA**

As a forevacuum pump of an axial flow molecular pump, a diffusion pump is commonly used. In this case, the pressure in the forevacuum side of the molecular pump is kept below 10^{-4} mmHg, and therefore a free molecule flow takes place in all the blade rows. Occasionally, only a rotary pump is used as the forevacuum pump. In this case, the pressure in the forevacuum side is $10^{-3} \sim 10^{-2}$ mmHg and collisions between gas molecules occur as frequently as between gas molecules and blade plates of some rows placed near the forevacuum side. This state is called the transition flow regime. The performance of a rotating blade row in this regime is studied theoretically and experimentally.

The theoretical performance agrees with the experimental results when the Knudsen number is larger than 5 to 1, but the difference between the theoretical and experimental results increases as the Knudsen number decreases.

1. Introduction

For the purpose of clarifying the pumping mechanism of an axial flow molecular pump, the performance of an axial blade row in the free molecule flow regime has been studied by the author and his colleagues.⁽¹⁾⁽²⁾

In the case where an oil diffusion pump is used as the forevacuum pump, a free molecule flow takes place in all the blade rows which constitute the axial flow molecular pump. On the other hand, occasionally, only a rotary pump is used as the fore pump. In such a case, the forevacuum is in the range from 10^{-3} to 10^{-2} mmHg. Consequently, in some blade rows which are placed near the outlet side of the molecular pump, collisions between molecules occur as frequently as between molecules and blade plates. This state is called the transition flow regime.

Several studies on the transition flow regime have been reported, but the method to deal with the transition flow problem has not yet been established.

Schaetzle has solved a Couette flow and the boundary layer at the leading edge of a flat plate in low speed incompressible and compressible flows using relaxation techniques⁽³⁾⁽⁴⁾. The technique divides the flow field into small cubical volume elements and assigns a molecular velocity distribution to the molecules with a collision in the volume element.

In this paper, it is tried to apply Schaetzle's method to the blade row problem and to approach the problem using the kinetic theory of gases. The performance of a single row with flat plate blades in the transition flow regime is analyzed theoretically and the theoretical results are compared and discussed with the results of experiments which are performed in the pressure range from 10^{-4} to 10^{-1} mmHg by using several rotors with various blade angles and spacing-chord ratios.

2. List of symbols

- a : most probable molecular velocity $= \sqrt{2RT/M}$
- b : blade chord
- c : blade velocity ratio = blade velocity/ a
- f_2 : molecular velocity distribution function projected on a plane
- f_3 : three dimensional molecular velocity distribution function
- $F_{\eta\lambda}(\eta, \lambda)$: fraction of molecules emitted from $d\eta$ which impinge directly on $d\lambda$ without collision on the way. Either η or λ expresses one of x, y, s, s' (Fig. 6)
- $G_{p\eta}(\mathbf{p}, \eta)$: ratio of number of molecules which reach $d\eta$ without collision on the way after colliding in element $d\mathbf{p}$ to number density n_p . Symbol η expresses one of x, y, s, s'
- $h_{\eta p}(\eta, \mathbf{p}, v)$: fraction of molecules with velocity v emitted from $d\eta$ which enter element $d\mathbf{p}$ around point P and stay there for unit time

* Received 15th July, 1971.

** Scientist, The Institute of Physical and Chemical Research, Wako.

Symbol η expresses one of x, y, s, s', q (q is a point except p)

$H_{\eta p}(\eta, p)$: fraction of molecules emitted from $d\eta$ which enter element $d\mathbf{p}$ around point P and stay there for unit time. Symbol η is identical with $h_{\eta p}(\eta, p, v)$

l_0 : mean free path of all molecules with velocities from 0 to ∞

M : molecular weight

n : molecular number density

$n_u du$: number density of molecules with velocity between u and $u+du$

n_η : molecular number density around η . Symbol η expresses one of s, s', x, y, p, q

R : universal gas constant

$s(s')$: coordinate along the inlet (outlet) section

s_0 : blade spacing

S_0 : spacing-chord ratio $= s_0/b$

T : gas temperature

u : molecular velocity in the three dimensional space

u' : molecular velocity projected on a plane

v : u/a or u'/a

x : coordinate along blade $\otimes$

y : coordinate along blade $\odot$

α : blade angle

σ : collision diameter

Fig. 1 Molecular fluxes coming to point A

Fig. 2 Blade row of infinite blade height

3. Analysis

How to deal with the problem will be explained at the beginning of this section.

In Fig. 1, the number of molecules coming into ΔV_A around point A per unit time, is the sum of the fluxes of molecules $N_{B \rightarrow A}, N_{C \rightarrow A}, N_{D \rightarrow A}, \dots$ which have a collision in small volume elements $\Delta V_B, \Delta V_C, \Delta V_D, \dots$ around points B, C, D... and then reach point A without another collision on their way from these elements. Therefore, it is necessary to know the number of molecules which have a collision in elements $\Delta V_B, \Delta V_C, \Delta V_D, \dots$, and the number of molecules which have a collision on their way from points B, C, D... to point A. Since the number of colliding molecules is related to the molecular velocity and number, after all the problem comes to obtain the molecular number density and the molecular velocity distribution function in each element.

The passage of blade row is divided into a finite number of volume elements, and a uniform molecular number density and a uniform molecular velocity distribution are assigned to each element. The molecular number density varies in general as a function of location and direction. Therefore, taking note of the mean free path of molecules coming from point

Fig. 3 Projection of velocity on a plane

Fig. 4 Velocity distribution function

H to point A in Fig. 1, it is given as a function of location and direction, for instance, as $l_{H\theta}$ in element ΔV_H or $l_{D\theta}$ in element ΔV_D . But, since the problem is too complicated to deal with under these conditions, it is simplified by some assumptions. First, it is assumed that the number of colliding molecules and the mean free path are the functions of velocity and location only, and are independent of direction. Still more, it is assumed that the mean free path of molecules proceeding from point H to point A takes a constant value $l = (l_A + l_H)/2$ throughout the path, where l_A is the mean free path at ending point and l_H that at starting point. Some more assumptions will be made, but they will be described at the part where they are required. In order to simplify the problem still more, the blade row of infinite blade height shown in Fig. 2 is adopted as a model of blade rotor and the problem is transformed into a two dimensional one. The passage of blade row is divided into $6 \times 6 = 36$ volume elements of equal size. Since molecules move in a three dimensional space, it is necessary to transform the quantities related to molecular motion into two dimensional quantities before analysis is begun.

3.1 Preparation for analysis

3.1.1 Molecular velocity distribution

Let $\mathbf{u}$ be the velocity of molecules in the three dimensional space whose components relative to the rectangular coordinate system (x, y, z) are u_x, u_y, u_z (see Fig. 3). Then the molecular velocity distribution function is

$$f_3(u_x, u_y, u_z) du_x du_y du_z = \left(\frac{1}{\pi a^2} \right)^{3/2} \exp \{ -(u_x^2 + u_y^2 + u_z^2)/a^2 \} du_x du_y du_z \quad (1)$$

where $a = \sqrt{2RT/M}$, the most probable velocity.

If the projected velocity of $\mathbf{u}$ on a plane (x - y plane) is denoted by $\mathbf{u}'$, the molecular velocity distribution function is expressed as follows by integration of Eq. (1) with respect to u_z from $-\infty$ to ∞ :

$$f_2(u_x, u_y) du_x du_y = \frac{1}{\pi a^2} \exp \{ -(u_x^2 + u_y^2)/a^2 \} du_x du_y \quad (2)$$

Using polar coordinate expression, Eq. (2) is rewritten as follows:

$$f_2(u', \theta) du' d\theta = \frac{1}{\pi a^2} u' \exp(-u'^2/a^2) du' d\theta \quad (3)$$

Therefore, by integration of Eq. (3) with respect to θ from 0 to 2π , the molecular velocity distribution function taking note of magnitude only and neglecting direction is given by

$$f_2(u') du' = \frac{2}{a^2} u' \exp(-u'^2/a^2) du' \quad (4)$$

Using $v = u'/a$, Eq. (4) can be rewritten as follows:

$$f_2(v) dv = 2v \exp(-v^2) dv \quad (5)$$

Similarly, the three dimensional molecular velocity distribution function regardless of direction is

$$f_3(v) dv = \frac{4}{\sqrt{\pi}} v^2 \exp(-v^2) dv \quad (6)$$

Function f_2 is shown in Fig. 4 compared with f_3 . As shown in Fig. 4, molecules which have velocities larger than $V=3$ are very few in the total number of molecules. It is therefore sufficient that numerical integration is performed with respect to velocity V below 4 even if the effect of blade velocity is estimated somewhat large.

3.1.2 Mean free path

The term "mean free path" is used here in two ways. The first use is as the mean free path of all molecules with velocities from 0 to ∞ , and the second is as the mean free path of molecules with velocity v .

Hereafter, the former is called simply "mean free path" and in order to distinguish from the former the latter is called "mean free path of molecules with velocity v ". Since collisions are related to molecular velocities, the latter plays an important role. Mean collision frequency per unit time of molecules which have the projected velocity u' (Fig. 3) and the three dimensional velocity u is given by Chapman and Cowling⁽⁵⁾ as follows:

$$P(u') = n^2 \sigma \left(\frac{2\pi RT}{M} \right)^{1/2} \left\{ \exp(-x^2) + \left(2x + \frac{1}{x} \right) \operatorname{erf}(x) \right\} \quad (7)$$

where $x = u' \sqrt{M/2RT} \cos \eta$, $\operatorname{erf}(x) = \int_0^x \exp(-y^2) dy$ and $u' = u \cos \eta$.

After $P(u')$ is multiplied by the ratio of the number of molecules with the projected velocity u' and the three dimensional velocity u to the number of all molecules with the projected velocity u' , integration is performed with respect to η . The mean collision frequency per unit time of molecules with velocity u' projected on plane ϕ is given by

$$p(u') = \frac{\int_{-\pi/2}^{\pi/2} \frac{4}{\sqrt{\pi}} \frac{u'^2}{a^2} \frac{1}{\cos^2 \eta} \exp\left(-\frac{u'^2}{a^2} \frac{1}{\cos^2 \eta}\right) P(u') d\eta}{\int_{-\pi/2}^{\pi/2} \frac{4}{\sqrt{\pi}} \frac{u'^2}{a^2} \frac{1}{\cos^2 \eta} \exp\left(-\frac{u'^2}{a^2} \frac{1}{\cos^2 \eta}\right) d\eta} = \frac{2u' \int_0^{\pi/2} \frac{1}{\cos^2 \eta} \exp\left(-\frac{u'^2}{a^2} \frac{1}{\cos^2 \eta}\right) P(u') d\eta}{\sqrt{\pi} a \exp\left(-\frac{u'^2}{a^2}\right)} \quad (8)$$

Therefore, using Eqs. (7) and (8), the mean free path l_u' of molecules with projected velocity u' is

$$l_u' = \frac{u'}{p(u')} = \frac{u' \exp(-u'^2/a^2)}{2n\sigma^2 u' \int_0^{\pi/2} \frac{1}{\cos^2 \eta} \exp(-x^2) \left\{ \exp(-x^2) + \left(2x + \frac{1}{x}\right) \operatorname{erf}(x) \right\} d\eta} \quad \dots\dots\dots (9)$$

Equation (9) is rearranged as follows after u' is replaced by $v=u'/a$:

$$l_v = \frac{v \exp(-v^2)}{n\sigma^2 \left[\sqrt{\frac{\pi}{2}} \exp(-2v^2) + 2v \int_0^{\pi/2} \frac{1}{\cos^2 \eta} \exp\left(-\frac{v^2}{\cos^2 \eta}\right) \left(2\frac{v}{\cos \eta} + \frac{\cos \eta}{v}\right) \operatorname{erf}\left(\frac{v}{\cos \eta}\right) d\eta \right]} \quad \dots\dots\dots (10)$$

Since the mean free path l_0 is expressed with $l_0 = 1/(\sqrt{2}\pi n\sigma^2)$, l_v/l_0 is given by

$$\frac{l_v}{l_0} = \frac{\sqrt{2}\pi}{\sqrt{\frac{\pi}{2}} \frac{1}{v} \exp(-v^2) + 2 \int_0^{\pi/2} \frac{1}{\cos^2 \eta} \exp\left(-\frac{\sin^2 \eta}{\cos^2 \eta} v^2\right) \left(2\frac{v}{\cos \eta} + \frac{\cos \eta}{v}\right) \operatorname{erf}\left(\frac{v}{\cos \eta}\right) d\eta} \quad \dots\dots\dots (11)$$

The calculated values of Eq. (11) are shown in Fig. 5 compared with the three dimensional mean free path of molecules with velocity v .

3.1.3 Probability that a molecule moving with velocity u' describes a free path at least equal to r

The probability that a molecule moving with velocity u' projected on a plane will have a collision while travelling a further distance dr is⁽⁵⁾

$$\frac{dr}{u'} p(u') = \frac{dr}{l_u'} \quad \dots\dots\dots (12)$$

Therefore, the probability $q(r+dr, u')$ that the molecule will describe a free path at least equal to $r+dr$ is given by

$$q(r+dr, u') = q(r, u') \left(1 - \frac{dr}{l_u'}\right) \quad \dots\dots\dots (13)$$

In Eq. (13),

$$q(r, u') \left(1 - \frac{dr}{l_u'}\right) = q(r, u') + dr \frac{\partial q(r, u')}{\partial r} \quad \dots\dots\dots (14)$$

Then,

$$\frac{\partial \ln q(r, u')}{\partial r} = -\frac{1}{l_u'} \quad \dots\dots\dots (15)$$

Integrating Eq. (15) and using the fact that $q(0, u') = 1$, the following equation is obtained.

$$q(r, u') = \exp\left(-\frac{r}{l_u'}\right) \quad \dots\dots\dots (16)$$

Fig. 5 Mean free path of molecules with velocity v

Substituting $v=u'/a$ into Eq. (16),

$$q(r, v) = \exp\left(-\frac{r}{l_v}\right) \quad \dots\dots\dots (17)$$

3.1.4 Number of molecules with velocity u' which have a collision in unit volume

In the case where l_u' is much larger than the representative length of a volume element ΔV , since the probability that a molecule has two or more collisions in element ΔV is considered very small, collision frequency is almost equal to the number of molecules which have one collision in the element. Therefore, when l_u' is much larger than unit length, the number $N_c du'$ of molecules with velocity u' which have a collision in unit volume is

$$N_c du' = n_u' p(u') du' = \frac{n u'}{l_u'} du' \int f^*(u', \theta) d\theta \quad \dots\dots\dots (18)$$

where $f^*(u', \theta) du' d\theta$ is the velocity distribution function considering the direction of u' , and it is written as follows if nondimensional velocity $v=u'/a$ is used:

$$f(v, \theta) dv d\theta = \frac{1}{\pi} v \exp[-\{v^2 - 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\}] dv d\theta$$

where V is the mass velocity, V_ξ and V_ζ are the parallel and normal components of V to the blade row respectively and θ is the angle between v and ζ -direction (Fig. 6).

3.2 Equations

On the basis of preparation for analysis described above, the density and the velocity distribution in the blade row, the incoming molecular number on blade plates $\textcircled{X}$ and $\textcircled{Y}$, and those on the inlet and outlet sections are discussed. In the blade row shown in Fig. 6, let $F_{zs}(x, s) dx ds$ be the fraction of molecules emitted from a small area dx on blade plate $\textcircled{X}$ which impinge directly on a small area ds on inlet section $\textcircled{S}$ without collision on the way and let F_{ys} and F_{zs}' be similarly defined. The number $N_s ds$ of molecules which reach a small area ds from the inside of blade row is

$$N_s ds = \int_{-0.5b}^{0.5b} N_z F_{zs}(x, s) dx ds + \int_{-0.5b}^{0.5b} N_y F_{ys}(y, s) dy ds + \int_{-0.5s_0}^{0.5s_0} \bar{N}_s' F_{s's}(s', s) ds' ds + \int_{\mathbf{p}} n_p G_{ps}(\mathbf{p}, s) d\mathbf{p} ds \quad (19)$$

where n_p is the molecular number density in a small volume element $d\mathbf{p} = \sin\alpha dp_\theta dp_\phi$ around point P, $G_{ps}(\mathbf{p}, s)$ is the ratio of the number of molecules which reach ds after colliding in element $d\mathbf{p}$ to the number density n_p , and $\mathbf{p}$ represents coordinates (p_θ, p_ϕ) of point P. Integration is performed over the total space between two blade plates. In the same manner, $\bar{N}_s' ds'$, $N_x dx$ and $N_y dy$ are given by

$$\left. \begin{aligned} N_s' ds' &= \int_{-0.5b}^{0.5b} N_z F_{zs'}(x, s') dx ds' + \int_{-0.5b}^{0.5b} N_y F_{ys'}(y, s') dy ds' \\ &\quad + \int_{-0.5s_0}^{0.5s_0} \bar{N}_s F_{ss'}(s, s') ds ds' + \int_{\mathbf{p}} n_p G_{ps'}(\mathbf{p}, s') d\mathbf{p} ds' \\ N_x dx &= \int_{-0.5b}^{0.5b} N_y F_{yx}(y, x) dy dx + \int_{-0.5s_0}^{0.5s_0} \bar{N}_s F_{sx}(s, x) ds dx \\ &\quad + \int_{-0.5s_0}^{0.5s_0} \bar{N}_s' F_{s'x}(s', x) ds' dx + \int_{\mathbf{p}} n_p G_{px}(\mathbf{p}, x) d\mathbf{p} dx \\ N_y dy &= \int_{-0.5b}^{0.5b} N_z F_{zy}(x, y) dx dy + \int_{-0.5s_0}^{0.5s_0} \bar{N}_s F_{sy}(s, y) ds dy \\ &\quad + \int_{-0.5s_0}^{0.5s_0} \bar{N}_s' F_{s'y}(s', y) ds' dy + \int_{\mathbf{p}} n_p G_{py}(\mathbf{p}, y) d\mathbf{p} dy \\ \bar{N}_s &= \frac{1}{s_0} \int_{-0.5s_0}^{0.5s_0} N_s ds, \quad \bar{N}_s' = \frac{1}{s_0} \int_{-0.5s_0}^{0.5s_0} N_s' ds' \end{aligned} \right\} \quad (20)$$

The number $n_p d\mathbf{p}$ in element $d\mathbf{p}$ is expressed by

$$\left. \begin{aligned} n_p d\mathbf{p} &= \int_{-0.5s_0}^{0.5s_0} \bar{N}_s H_{sp}(s, \mathbf{p}) ds d\mathbf{p} + \int_{-0.5s_0}^{0.5s_0} \bar{N}_s' H_{s'p}(s', \mathbf{p}) ds' d\mathbf{p} + \int_{-0.5b}^{0.5b} N_x H_{xp}(x, \mathbf{p}) dx d\mathbf{p} \\ &\quad + \int_{-0.5b}^{0.5b} N_y H_{yp}(y, \mathbf{p}) dy d\mathbf{p} + \int_{\mathbf{q}} n_q H_{pq}(\mathbf{q}, \mathbf{p}) d\mathbf{q} d\mathbf{p} \end{aligned} \right\} \quad (21)$$

where, for instance, $H_{sp}(s, \mathbf{p})$ is the fraction of molecules which enter element $d\mathbf{p}$ around point P and stay there for unit time. These simultaneous equations (19)~(21) should be solved for N_s , N_s' , N_x and N_y . For this purpose, $F_{zs}(x, s)$, $F_{ys}(y, s)$, $G_{ps}(\mathbf{p}, s)$, $G_{ps'}(\mathbf{p}, s')$, $H_{xp}(x, \mathbf{p})$, $H_{yp}(y, \mathbf{p})$, $H_{sp}(s, \mathbf{p})$, $H_{s'p}(s', \mathbf{p})$, $H_{pq}(\mathbf{q}, \mathbf{p})$ must be known. But it becomes tedious to describe all of these functions and the basic point of view is almost the same for all, so derivations of $F_{zs}(x, s)$, $G_{ps}(\mathbf{p}, s)$ and $H_{xp}(x, \mathbf{p})$ are described as the typical examples.

Fig. 6 Blade row of infinite blade height (Illustration of symbols)

3.2.1 $F_{zs}(x, s)$

The number of molecules emitted into an angle $d\varphi$ from a small area dx is

$$\begin{aligned} N_x v(x, \varphi, v) dx d\varphi dv \\ = \frac{2}{\sqrt{\pi}} N_x v^2 \exp(-v^2) \cos \varphi dx d\varphi dv \quad (22) \end{aligned}$$

The angle φ is illustrated in Fig. 6.

The fraction of molecules which have no collision on their way to ds from dx is given by $\exp(-r/l_v)$ in Eq. (17). Since, l_v is a function of molecular number density, the mean free path changes as the molecules proceed from dx to ds . But it is impossible to obtain the molecular number density in a functional form, so l_v as mentioned at the beginning of this section is approximated with the arithmetic average of the mean free paths of the starting and ending points, and is denoted by $l_{zs}(v)$. Let n_1 be the molecular number density in region ①, let n_s be that around ds , and let $l_1(v)$ be the mean free path of molecules with velocity v in region ①. Then, the mean free path $l_{zs}(v)$ is expressed as follows:

$$l_{zs}(v) = \frac{2n_1}{n_x + n_s} l_1(v) \quad (23)$$

Therefore, the number of molecules which reach ds from dx without collision on the way is

$$\begin{aligned} N_{zs}(x, s) dx ds = - \int_{v=0}^{\infty} \frac{2}{\sqrt{\pi}} N_x v^2 \exp(-v^2) \\ \times \exp\left\{-\frac{r_{zs}}{l_{zs}(v)}\right\} \cos \varphi dv d\varphi dx \quad (24) \end{aligned}$$

where r_{zs} , $\cos \varphi$ and $d\varphi$ are (see Fig. 6)

$$\left. \begin{aligned} r_{zs} &= [(s+0.5s_0)^2 \sin^2 \alpha + (x+0.5b \\ &\quad + (s+0.5s_0) \cos \alpha)^2]^{1/2} \\ \cos \varphi &= \frac{(s+0.5s_0) \sin \alpha}{r_{zs}} \\ d\varphi &= -\frac{(x+0.5b) \sin \alpha}{r_{zs}^2} ds \end{aligned} \right\} \dots\dots\dots (25)$$

Dividing Eq. (24) by N_z and using Eq. (26), $F_{zs}(x, s)$ is given by

$$F_{zs}(x, s) = \frac{2}{\sqrt{\pi}} \frac{(x+0.5b)(s+0.5s_0)}{r_{zs}^3} \times \sin^2 \alpha \int_0^\infty v^2 \exp \left\{ -v^2 - \frac{r_{zs}}{l_{zs}(v)} \right\} dv \dots\dots\dots (26)$$

3.2.2 $G_{pz}(\mathbf{p}, x)$

Let n_p be the molecular number density in a volume element $d\mathbf{p}$ around point P, let $l_p(v)$ be the mean free path of molecules with velocity v in $d\mathbf{p}$ and let $f_p(v, \theta)dv d\theta$ be the velocity distribution function in $d\mathbf{p}$. Then the number of molecules which have a collision in $d\mathbf{p}$ is obtained from Eq. (18) as follows:

$$N_c dv d\mathbf{p} = \frac{n_p v}{l_p(v)} \int_{\theta=0}^{2\pi} f_p(v, \theta) dv d\theta d\mathbf{p} \dots\dots\dots (27)$$

where

$$l_p(v) = \frac{n_1}{n_p} l_1(v) \dots\dots\dots (28)$$

The velocity distribution function $f_p(v, \theta)dv d\theta$ around point P is assumed in the following form:

$$f_p(v, \theta) dv d\theta = \frac{1}{\pi} v \exp \left[-\{v^2 - 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\} \right] dv d\theta \dots\dots\dots (29)$$

Namely, it is assumed that $f_p(v, \theta)dv d\theta$ is a Maxwellian velocity distribution superposed on the mass Velocity V which makes an angle γ with the normal direction to the blade row as is illustrated in Fig. 6. This mass velocity V and the angle γ or ξ - and ζ -direction components of V , V_ξ and V_ζ are decided from the momentum of molecules entering the small volume element around point P. Since the molecules having small energy such as treated here neither undergo fission nor join together and their internal energy never changes in the process of collision between molecules, the total number, momentum and kinetic energy of molecules do not alter before and after collision between molecules. It is assumed that the molecules reaching $d\mathbf{p}$ are moving with a Maxwellian velocity distribution as expressed by Eq. (29), and the mass velocity V is decided with the momentum of molecules reaching $d\mathbf{p}$, so the conservation of momentum holds, but the conservation of kinetic energy does not. Besides, since the problem has been transformed into a plane problem, it is difficult to obtain the kinetic energy. But, the above assumption for the velocity distribution function is considered roughly

reasonable and the kinetic energy is supposed to be nearly conserved. The velocity components V_ξ and V_ζ in Eq. (29) are given by

$$\left. \begin{aligned} n_p V_\xi &= \int_{v=0}^\infty \int_{s=-0.5s_0}^{0.5s_0} \bar{N}_s h_{sp}(s, \mathbf{p}, v) v \sin \theta_s dv ds \\ &\quad + \int_{v=0}^\infty \int_{s'=-0.5s_0}^{0.5s_0} \bar{N}_{s'} h_{s'p}(s', \mathbf{p}, v) v \sin \theta_{s'} dv ds' \\ &\quad + \int_{v=0}^\infty \int_{x=-0.5b}^{0.5b} N_x h_{xp}(x, \mathbf{p}, v) v \sin \theta_x dv dx \\ &\quad + \int_{v=0}^\infty \int_{y=-0.5b}^{0.5b} N_y h_{yp}(y, \mathbf{p}, v) v \sin \theta_y dv dy \\ &\quad + \int_{v=0}^\infty \int_{\mathbf{q}} n_q h_{qp}(\mathbf{q}, \mathbf{p}, v) v \sin \theta_q dv d\mathbf{q} \\ n_p V_\zeta &= \int_{v=0}^\infty \int_{s=-0.5s_0}^{0.5s_0} \bar{N}_s h_{sp}(s, \mathbf{p}, v) v \cos \theta_s dv ds \\ &\quad + \int_{v=0}^\infty \int_{s'=-0.5s_0}^{0.5s_0} \bar{N}_{s'} h_{s'p}(s', \mathbf{p}, v) v \cos \theta_{s'} dv ds' \\ &\quad + \int_{v=0}^\infty \int_{x=-0.5b}^{0.5b} N_x h_{xp}(x, \mathbf{p}, v) v \cos \theta_x dv dx \\ &\quad + \int_{v=0}^\infty \int_{y=-0.5b}^{0.5b} N_y h_{yp}(y, \mathbf{p}, v) v \cos \theta_y dv dy \\ &\quad + \int_{v=0}^\infty \int_{\mathbf{q}} n_q h_{qp}(\mathbf{q}, \mathbf{p}, v) v \cos \theta_q dv d\mathbf{q} \end{aligned} \right\} \dots\dots\dots (30)$$

where $h_{sp}(s, \mathbf{p}, v)$, $h_{s'p}(s', \mathbf{p}, v) \dots$ are the ratios of number of molecules with velocity v which have entered element $d\mathbf{p}$ and stay there for unit time to $\bar{N}_s, \bar{N}_{s'} \dots$. These functions will be described in detail later. In Fig. 6, $\theta_s, \theta_{s'} \dots$ are illustrated. From Eq. (30), V_ξ and V_ζ are obtained and $f_p(v, \theta)dv d\theta$ is decided. The number of molecules with velocity u' which come within a small angle $d\theta$ and have a collision in unit volume is obtained by removal of the integral sign from Eq. (18). Since the steady state is discussed here, the number of molecules which go into $d\theta$ after a collision in the unit volume element is equal to the number of molecules which go through the reversed process. Therefore, the number of molecules with velocity v which go into the small angle $d\theta$ after a collision in a small volume element $d\mathbf{p}$ is given by

$$\{n_p d\mathbf{p} / l_p(v)\} (av^2 / \pi) \exp \left[-\{v^2 - 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\} \right] dv d\theta$$

The molecules which have a collision with other molecules on the way to ds from $d\mathbf{p}$ are taken into account in this value, so the flux of molecules which have a collision in a small volume element $d\mathbf{p}$ and reach ds without collision after that is expressed by

$$N_{ps} d\mathbf{p} ds = \frac{an_p}{\pi} \int_{v=0}^\infty \frac{v^2}{l_p(v)} \exp \left[-\{v^2 - 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\} \right] \exp \left\{ -\frac{r_{ps}}{l_{ps}(v)} \right\} dv d\mathbf{p} d\theta \dots (31)$$

where

$$\left. \begin{aligned} l_{ps}(v) &= \frac{l_p(v) + l_s(v)}{2}, & \cos \theta &= -\frac{(p_b + 0.5b)}{r_{ps}} \sin \alpha, & \sin \alpha &= \frac{(-p_s + s) + (p_b + 0.5b) \cos \alpha}{r_{ps}} \\ d\theta &= -\frac{ds \cos \theta}{r_{ps}}, & r_{ps} &= [(p_b + 0.5b)^2 \sin^2 \alpha + \{(p_b + 0.5b) \cos \alpha + (s - p_s)\}^2]^{1/2} \end{aligned} \right\} \dots\dots (32)$$

Therefore, $G_{ps}(\mathbf{p}, s)$ is

$$G_{ps}(\mathbf{p}, s) = \frac{a}{\pi} \int_0^\infty \frac{v^2}{l_p(v)} \exp[-\{v^2 - 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\}] \exp\left[-\frac{r_{ps}}{l_{ps}(v)}\right] \frac{\cos \theta}{r_{ps}} dv \dots\dots\dots (33)$$

3.2.3 $h_{sp}(s, \mathbf{p}, v)$, $H_{sp}(s, \mathbf{p})$

The number of molecules $N_{s\theta s}(s, \theta_s, v) d\theta_s dv ds$ with velocity v emitted from a small area element ds within a small angle $d\theta_s$ is given by the following equation.

$$N_{s\theta s}(s, \theta_s, v) d\theta_s dv ds = \frac{n_s a}{\pi} v^2 \exp[-\{v^2 - 2v(V_\xi \sin \theta_s + V_\zeta \cos \theta_s) + V^2\}] \cos \theta_s d\theta_s dv ds \dots\dots\dots (34)$$

where n_s is the molecular number density around ds . Putting $rd\theta_s = dA$ and using θ given by Eq. (32) instead of θ_s , the number of molecules with velocity v impinging on the small area dA in element $d\mathbf{p}$ from ds is

$$\begin{aligned} N_{sp}(s, \mathbf{p}, v) dv ds d\theta_s &= -\frac{n_s a}{\pi} v^2 \exp[-\{v^2 + 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\}] \\ &\times \exp\left[-\frac{r_{sp}}{l_{sp}(v)}\right] \cos \theta \frac{dA}{r_{sp}} dv ds \dots\dots\dots (35) \end{aligned}$$

where the term $\exp\{-r_{sp}/l_{sp}(v)\}$ is the probability that a molecule will reach dA from ds without collision on the way, and $l_{sp}(v)$ is the mean free path for the average density of ds and $d\mathbf{p}$. Since these molecules travel the distance $u' = av$ in unit time, molecules $N_{sp} dv ds d\theta_s$ lie in volume $u' dA$. Therefore, the number of molecules with velocity u' which lie in unit volume is given by $N_{sp} dv ds d\theta_s / av dA$. Meanwhile, n_s is expressed as follows by using $\bar{N}_s$:

$$n_s = \frac{\bar{N}_s}{\frac{a}{\pi} \int_{v=0}^\infty \int_{\theta=-\pi/2}^{\pi/2} v^2 \exp[-\{v^2 + 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\}] \cos \theta d\theta dv} \dots\dots\dots (36)$$

Consequently, $h_{sp}(s, \mathbf{p}, v)$ is

$$h_{sp}(s, \mathbf{p}, v) = \frac{\exp[-\{v^2 + 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\}] \exp\left\{-\frac{r_{sp}}{l_{sp}(v)}\right\} v \cos \theta \frac{1}{r_{sp}} \frac{1}{a}}{\int_{v=0}^\infty \int_{\theta=-\pi/2}^{\pi/2} v^2 \exp[-\{v^2 + 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\}] \cos \theta d\theta dv} \dots\dots\dots (37)$$

Function $H_{sp}(s, \mathbf{p})$ is obtained by integration of $h_{sp}(s, \mathbf{p}, v)$ with respect to v .

$$H_{sp}(s, \mathbf{p}) = \frac{\int_{v=0}^\infty \exp[-\{v^2 + 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\}] \exp\left\{-\frac{r_{sp}}{l_{sp}(v)}\right\} v \cos \theta \frac{1}{r_{sp}} \frac{1}{a} dv}{\int_{v=0}^\infty \int_{\theta=-\pi/2}^{\pi/2} v^2 \exp[-\{v^2 + 2v(V_\xi \sin \theta + V_\zeta \cos \theta) + V^2\}] \cos \theta d\theta dv} \dots\dots\dots (38)$$

Since functions $F_{xy}(x, y) \dots$, $G_{px}(\mathbf{p}, x) \dots$, $H_{xp}(x, \mathbf{p}) \dots$ and $h_{xp}(x, \mathbf{p}, v) \dots$ are derived almost in the same manner, Eqs. (19)~(21) can be solved. But it is difficult to obtain the analytical solutions of them, so they are solved numerically by transforming the integrations in Eqs. (19)~(21) into the arithmetic sums and by combining them with Eqs. (30), (33), (37) and (38). Using the values of $\bar{N}_s$, $\bar{N}_s'$, N_x , N_y in the limit of the free molecule flow regime as the initial values and using the average value of the densities in the inlet and outlet sides as the initial value of molecular number density n_p in the passage of blade row, the equations are solved by the iteration method. An example of the molecular concentrations on the blades ⑧ and ⑨, those on the inlet and outlet sections and the molecular number densities in the passage of blade row obtained in the above manner is shown in Figs. 7~9. The symbol K_n in the figures indicates the Knudsen number (i.e., the mean free path in the

outlet side/blade spacing). The curves designated by $K_n = 45.5$ show the results in the case of the free molecule flow, and the curves designated by $K_n = 1.09$, the results in the case of the transition flow.

4. Comparisons of theoretical and experimental results

The experimental apparatus (Fig. 10) and procedure are almost similar to those in the first report⁽¹⁾. But the experiments of this paper were performed in the pressure range of 5×10^{-3} to 10^{-1} mmHg. The pressures below 10^{-3} mmHg were measured with an ionization gauge and the pressures 10^{-3} to 10^{-1} mmHg with a McLeod gauge. The experimental results are shown in Figs. 11~14 against the theoretical results. The theoretical results in these figures were obtained from the integrated values of the molecular concentrations as shown in Fig. 8, and the results were corrected in the equal proportion with the blade tip correction

in the first report. The theoretical results approximately agree with the experimental results for the case where the Knudsen number is larger than 5 to 1; but as the Knudsen number decreases, the difference between the theoretical and experimental results increases. This difference is considered to be mainly caused by the following differences between the actual conditions and the assumptions used in this theory. The number of divisions of the passage is set at 36 as shown in Fig. 2, so the size of the volume element relative to the mean free path becomes larger as the Knudsen number decreases. For instance, as for the number of molecules coming to point A (Fig. 2), the molecules coming from within element ΔV_A increase as the Knudsen number decreases, but these molecules are neglected in this theory. In this theory, the molecules entering the passage from the outside of blade row are assumed to have the mass velocity equal in magnitude and opposite in direction to the blade velocity. But actually, these molecules include some molecules which have experienced a collision in the vicinity of the inlet or outlet section with molecules coming from the inside of blade row. For a large Knudsen number, the proportion of such molecules to all the molecules entering the passage from the outside of blade row is very small and can be neglected; but as the Knudsen number decreases, their proportion increases. On account of this influence, the mass velocity of molecules relative to the blade row is likely smaller than the blade velocity. Therefore the theoretical values of maximum pressure ratios of this theory become larger than experimental values as the Knudsen number decreases as shown in Figs. 11~14.

Fig. 7 N_x, N_y

5. Summary

The theoretical values of this paper agree with the experimental data for Knudsen numbers larger than 5 to 1. Therefore, it is verified that the performance of the blade row can be analyzed theoretically by using the assumption described above. The maximum pressure ratio has been considered to be constant for Knudsen numbers larger than 10, but in the strict sense, the maximum pressure ratio decreases gradually with a decrease in the Knudsen number from a very large value and at Knudsen number 10, it takes a value about 5% less than the value of the free molecule flow limit. If the volume elements are divided more finely and the assumptions are partly altered, this theory can be fundamentally applied to a case of smaller Knudsen numbers, but since the procedure of calculation becomes more complicated with a decrease in the Knudsen number, it is actually limited to Knudsen numbers above about

(a)

(b)

Fig. 8 N_s, N_s'

Fig. 9 Molecular number density
($\alpha=20^\circ$, $s_0=0.5$, $c=0.364$)

- ① Vacuum vessel
- ② Rotor
- ③ High frequency induction motor
- ④ Leak valve of inlet side
- ⑤ Leak valve of outlet side
- ⑥ Ionization gauges
- ⑦ Cooling water inlet
- ⑧ Cooling water outlet
- ⑨ Rotation counter

Fig. 10 Experimental apparatus

Fig. 11 Maximum pressure ratio

1 that the problem can be approached from the free molecule flow side.

The author wishes to thank Dr. Osamu Taniguchi,

Fig. 12 Maximum pressure ratio

Fig. 13 Maximum pressure ratio

Fig. 14 Maximum pressure ratio

Chief Scientist of the Institute of Physical and Chemical Research, and Dr. Makoto Suzuki, Professor of Saitama University for their valuable advice and encouragement.

A part of this research has been supported by Scientific Research Fund from the Ministry of Education.

References

- (1) Sawada, T., et al., *Sci. Papers of the Institute of Physical and Chemical Research*, Vol. 62, No. 2 (1968), p. 49.
- (2) Sawada, T., et al., *Bull. JSME*, Vol. 14, No. 67 (1971-1), p. 48.
- (3) Schaetzle, W. J., *Rarefied Gas Dynamics, Supplement 2* (edited by Laurmann, J.A.), Vol. 1 (1963), p. 136, Academic Press.
- (4) Schaetzle, W. J., *Rarefied Gas Dynamics, Supplement 3*, (edited by Leeuw, J.H.), Vol. 1 (1965), p. 112, Academic Press.
- (5) Chapman, S. and Cowling, T.G., *The Mathematical Theory of Non-uniform Gases*, (1958), p. 95, Cambridge University Press.