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Flexural Wave Propagation in a Thin Plate with Circular Holes*

By Hideo SAITO** and Kosuke NAGAYA***

This paper deals with the problems of transverse and torsional wave propagation in an infinite thin elastic plate with circular holes. The dynamical moment concentrations along holes are obtained and compared with those for a single hole.

When the wave length becomes infinity, it is ascertained that the dynamical solutions tend to statical ones.

1. Introduction

The problems of the wave propagation in an elastic medium is of great importance, and many papers have been published on the problems during the past years. Pao, Mow and Mente^{(1)~(3)} have treated many problems of the wave propagation in an infinite elastic medium with a circular cavity or inclusion. Recently, the flexural wave propagations in the plate with a circular hole or inclusion have also been examined by authors^{(4)~(5)}. However, the cases when a plate has two or more than two obstacles have not been investigated enough in the past, though such plates have been widely used in designing structures.

This paper deals with the problems of the transverse or torsional wave propagation in an infinite thin elastic plate with more than one hole. To solve the problem of this kind, the method which was developed in the elastic and acoustic wave problems by Cheng⁽⁶⁾ and Twersky⁽⁷⁾ will be available. In this report, this method is simplified and applied to the flexural wave propagation problem. As numerical examples, we consider two cases of thin plates with two and three circular holes of equal size, and the dynamical moment concentrations around holes are examined in detail. The results obtained are compared with those for a plate with a single hole. It is ascertained that, when the wave length of the incident wave becomes infinity, the dynamical solution tends to a statical one^{(8)~(9)}.

2. Transverse wave propagation

An infinite thin elastic plate with circular holes is shown in Fig. 1 in which three arbitrary holes 0, s, s' are chosen as the reference holes, and x, y axes are spaced on the middle plane of the plate and z axis normal to the plate surface. The radii of the holes are denoted by $a_0, a_s, a_{s'}$ respectively. Let us choose the center O of the hole 0 as the origin of Cartesian coordinates x, y and denote coordinates from holes 0, s and s' to any point P by vectors $r(r, \theta)$, $r_s(r_s, \theta_s)$ and $r_{s'}(r_{s'}, \theta_{s'})$ respectively.

If we denote the displacement of the plate in the z direction by w , the differential equation of motion for the thin elastic plate is

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \quad \dots\dots\dots (1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

and

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

Fig. 1 Infinite thin plate with circular holes

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** Professor, Faculty of Engineering, Tohoku University.

*** Lecturer, Faculty of Engineering, Yamagata University, Yonezawa.

is the flexural rigidity of the plate, E the modulus of elasticity, ρ the mass density, ν Poisson's ratio, h the thickness of the plate and t time.

Now let us consider the hole s . The hole is excited by the incident wave w_i and the scattered waves from the holes except hole s

$$\sum_{m=1}^{\infty} \sum_{\substack{s'=0 \\ (s' \neq s)}}^{N-1} {}^s w^m \dots\dots\dots (2)$$

where m is the order of scattering and N the total number of holes. The total displacement ${}^s \bar{w}$ of the plate can be given by superposing these waves and the scattered wave from the hole s

$$\sum_{m=1}^{\infty} {}^s w^m \dots\dots\dots (3)$$

as

$${}^s \bar{w} = w_i + {}^s w^1 + \sum_{m=2}^{\infty} \left(\sum_{\substack{s'=0 \\ (s' \neq s)}}^{N-1} {}^s w^{m-1} + {}^s w^m \right) \dots\dots\dots (4)$$

The incident wave which propagates in the direction making an angle γ with the x axis is denoted as

$$w_i = w_0 e^{i\alpha r - i\omega t} \dots\dots\dots (5)$$

$${}^s w^m = \sum_{n=-\infty}^{\infty} \{ {}^s C_{1n}^m H_n(\alpha r_s) + {}^s C_{2n}^m K_n(\alpha r_s) \} e^{in\theta_s} e^{-i\omega t}$$

$${}^{s'} w^m = \sum_{n=-\infty}^{\infty} \{ {}^{s'} C_{1n}^m H_n(\alpha r_{s'}) + {}^{s'} C_{2n}^m K_n(\alpha r_{s'}) \} e^{in\theta_{s'}} e^{-i\omega t}$$

in which $H_n(\alpha r_s)$ is a Hankel function of the first kind, $K_n(\alpha r_s)$ a modified Bessel function of the second kind of order n and ${}^s C_{1n}^m$, ${}^s C_{2n}^m$ are arbitrary constants to be determined from the boundary conditions.

If we substitute Eqs. (7) and (8) into Eq. (4) and transform $r_{s'}$, $\theta_{s'}$ coordinates into r_s , θ_s coordinates⁽¹⁰⁾, the total displacement of the plate can be represented in the form

$$\begin{aligned} {}^s \bar{w} = & \sum_{n=-\infty}^{\infty} \left[w_0 \epsilon_s J_n(\alpha r_s) e^{in\xi} + {}^s C_{1n}^1 H_n(\alpha r_s) + {}^s C_{2n}^1 K_n(\alpha r_s) + \sum_{m=2}^{\infty} \left[{}^s C_{1n}^m H_n(\alpha r_s) \right. \right. \\ & + {}^s C_{2n}^m K_n(\alpha r_s) + \sum_{\substack{s'=0 \\ (s' \neq s)}}^{N-1} \sum_{l=-\infty}^{\infty} \{ {}^{s'} C_{1l}^{m-1} H_{n-l}(\alpha r_{s'}) \\ & \times J_n(\alpha r_s) + (-1)^l {}^{s'} C_{2l}^{m-1} K_{n-l}(\alpha r_{s'}) I_n(\alpha r_s) \} e^{-i(n-l)\theta_{s'}} \left. \right] e^{in\theta_s} e^{-i\omega t} \dots\dots\dots (9) \end{aligned}$$

where $I_n(\alpha r_s)$ is a modified Bessel function of the first kind of order n .

The bending moment, twisting moment and shearing force are derived from Eq. (9)⁽¹¹⁾. The bending moment M_r and equivalent shearing force V_r must be zero at the edges $r_s = a_s$ of the holes. Thus the boundary conditions are

$$\left. \begin{aligned} (M_r)_{r_s=a_s} &= 0, \\ (V_r)_{r_s=a_s} &= 0, \end{aligned} \quad s=0, 1, 2, \dots, N-1 \right\} \dots\dots\dots (10)$$

Substituting Eq. (9) into Eqs. (10), we have simultaneous equations with the coefficients of the same order of m . Then the unknown constants are obtained as follows:

$${}^s C_{1n}^m = w_0 {}^s A_n^m, \quad {}^s C_{2n}^m = w_0 {}^s B_n^m \dots\dots\dots (11)$$

in which

$$\left. \begin{aligned} {}^s A_n^1 &= (f_{1n} E_{2n} - e_{1n} F_{2n}) e^{in\xi} \epsilon_s / \Delta_n, & {}^s B_n^1 &= (e_{1n} F_{1n} - f_{1n} E_{1n}) e^{in\xi} \epsilon_s / \Delta_n \\ {}^s A_n^m &= -(D_{1n} F_{2n} - D_{2n} E_{2n}) / \Delta_n, & {}^s B_n^m &= -(D_{2n} E_{1n} - D_{1n} F_{1n}) / \Delta_n \quad m \geq 2 \end{aligned} \right\} \dots\dots\dots (12)$$

and

$$\left. \begin{aligned} E_{1n} &= \{ n(n+1)(1-\nu) - \alpha^2 a_s^2 \} H_n(\alpha a_s) - \alpha a_s (1-\nu) H_{n-1}(\alpha a_s) \\ E_{2n} &= \{ n(n+1)(1-\nu) + \alpha^2 a_s^2 \} K_n(\alpha a_s) + \alpha a_s (1-\nu) K_{n-1}(\alpha a_s) \\ e_{1n} &= \{ n(n+1)(1-\nu) - \alpha^2 a_s^2 \} J_n(\alpha a_s) - \alpha a_s (1-\nu) J_{n-1}(\alpha a_s) \\ e_{2n} &= \{ n(n+1)(1-\nu) + \alpha^2 a_s^2 \} I_n(\alpha a_s) - \alpha a_s (1-\nu) I_{n-1}(\alpha a_s) \end{aligned} \right\}$$

where

$$\alpha = \alpha(i \cos \gamma - j \sin \gamma)$$

$$\alpha^4 = \frac{12\omega^2 \rho (1-\nu^2)}{E h^2}$$

and w_0 is the amplitude, α the propagation vector, α the wave number, ω the circular frequency, i , j the unit vectors with respect to x and y axes, respectively. There are the following relations between an arbitrary point P and the holes 0, s , s' :

$$r_{0s} = r_{0s'} - r_{ss'} = r - r_s \dots\dots\dots (6)$$

Substituting Eq. (6) into Eq. (5) and expanding this equation into Fourier Bessel series, we obtain

$$w_i = w_0 \epsilon_s \sum_{n=-\infty}^{\infty} J_n(\alpha r_s) e^{in(\theta_s + \xi)} e^{-i\omega t} \dots\dots\dots (7)$$

where

$$\epsilon_s = e^{i\alpha r_{0s} \cos(\theta_{0s} + \gamma)}$$

$$\xi = \gamma + \pi/2$$

and $J_n(\alpha r)$ is a Bessel function of the first kind of order n . The displacement of the plate excited by the scattered waves is given as

$$\left. \begin{aligned} {}^s w^m &= \sum_{n=-\infty}^{\infty} \{ {}^s C_{1n}^m H_n(\alpha r_s) + {}^s C_{2n}^m K_n(\alpha r_s) \} e^{in\theta_s} e^{-i\omega t} \\ {}^{s'} w^m &= \sum_{n=-\infty}^{\infty} \{ {}^{s'} C_{1n}^m H_n(\alpha r_{s'}) + {}^{s'} C_{2n}^m K_n(\alpha r_{s'}) \} e^{in\theta_{s'}} e^{-i\omega t} \end{aligned} \right\} \dots\dots\dots (8)$$

$$\left. \begin{aligned}
F_{1n} &= \{\alpha^2 a_s^2 n + n^2(n+1)(1-\nu)\} H_n(\alpha a_s) - \{\alpha^3 a_s^3 + \alpha a_s n^2(1-\nu)\} H_{n-1}(\alpha a_s) \\
F_{2n} &= \{-\alpha^2 a_s^2 n + n^2(n+1)(1-\nu)\} K_n(\alpha a_s) - \{\alpha^3 a_s^3 - \alpha a_s n^2(1-\nu)\} K_{n-1}(\alpha a_s) \\
f_{1n} &= \{\alpha^2 a_s^2 n + n^2(n+1)(1-\nu)\} J_n(\alpha a_s) - \{\alpha^3 a_s^3 + \alpha a_s n^2(1-\nu)\} J_{n-1}(\alpha a_s) \\
f_{2n} &= \{-\alpha^2 a_s^2 n + n^2(n+1)(1-\nu)\} I_n(\alpha a_s) + \{\alpha^3 a_s^3 - \alpha a_s n^2(1-\nu)\} I_{n-1}(\alpha a_s) \\
D_{1n} &= \sum_{s'=0}^{N-1} \sum_{l=-\infty}^{\infty} \left\{ {}^s A_l^{m-1} H_{n-l}(\alpha r_{ss'}) e_{1n} + (-1)^l {}^s B_l^{m-1} K_{n-l}(\alpha r_{ss'}) e_{2n} \right\} e^{-i(n-l)\theta_{ss'}} \\
D_{2n} &= \sum_{s'=0}^{N-1} \sum_{l=-\infty}^{\infty} \left\{ {}^s A_l^{m-1} H_{n-l}(\alpha r_{ss'}) f_{1n} + (-1)^l {}^s B_l^{m-1} K_{n-l}(\alpha r_{ss'}) f_{2n} \right\} e^{-i(n-l)\theta_{ss'}} \\
J_n &= E_{1n} F_{2n} - E_{2n} F_{1n}
\end{aligned} \right\} \dots\dots\dots (13)$$

In Eqs. (11) ~ (13), if we change s from 0 to $N-1$, the expressions of all unknown constants ${}^0C_{1n}^m$, ${}^0C_{2n}^m$, ${}^1C_{1n}^m$, ${}^1C_{2n}^m$, ... for all holes can be obtained. The constants ${}^sC_{1n}^m$, ${}^sC_{2n}^m$ or ${}^sA_n^m$, ${}^sB_n^m$ are determined from ${}^sA_l^{m-1}$, ${}^sB_l^{m-1}$. This implies that, if ${}^sA_n^1$, ${}^sB_n^1$, ${}^sA_l^1$, ${}^sB_l^1$ are determined first, ${}^sC_{1n}^2$, ${}^sC_{2n}^2$, ..., ${}^sC_{1n}^m$, ${}^sC_{2n}^m$, ${}^sC_{1l}^2$, ${}^sC_{2l}^2$, ..., ${}^sC_{1l}^m$, ${}^sC_{2l}^m$ can also be obtained. It is also observed that, if $r_{ss'}$ ($s'=0, 1, 2, \dots, N-1$, $s' \neq s$) becomes infinity, $H_{n-l}(\alpha r_{ss'})$, $K_{n-l}(\alpha r_{ss'})$ become zero and the coefficients ${}^sA_n^m$, ${}^sB_n^m$ become zero except those of $m=1$. The problem of flexural wave propagation in a plate with one hole has been already discussed in reference (4). The expressions of ${}^sA_n^1$, ${}^sB_n^1$ in Eqs. (11) ~ (13) are reduced to those of reference (4). Accordingly, the unknown constants can be obtained from those of the single hole case.

The maximum bending moment in the plate due to the incident wave in one cycle of vibrations can be represented in the form⁽⁴⁾

$$M_0 = Dw_0 \alpha^2 \dots\dots\dots (14)$$

The bending moment and shearing force quantities are derived from Eqs. (9), (11) ~ (13) in a dimensionless form using Eq. (14). If we calculate these with respect to r , θ coordinates, we obtain

$$\left. \begin{aligned}
\frac{M_\theta}{M_0} &= - \sum_{n=-\infty}^{\infty} \left[m_{1n} e^{in\theta} + \sum_{m=1}^{\infty} \left\{ {}^0A_n^m M_{1n} + {}^0B_n^m M_{2n} \right\} + \sum_{m=2}^{N-1} \sum_{s'=1}^{\infty} \sum_{l=-\infty}^{\infty} \left\{ {}^sA_l^{m-1} H_{n-l}(\alpha r_{0s'}) m_{1n} \right. \right. \\
&\quad \left. \left. + (-1)^l {}^sB_l^{m-1} K_{n-l}(\alpha r_{0s'}) m_{2n} \right\} e^{-i(n-l)\theta_{0s'}} \right] e^{in\theta} e^{-i\omega t} \\
\frac{a_0 Q_\theta}{M_0} &= \frac{a_0}{r} i \sum_{n=-\infty}^{\infty} n \left[J_n(\alpha r) e^{in\theta} + \sum_{m=1}^{\infty} \left\{ {}^0A_n^m H_n(\alpha r) - {}^0B_n^m K_n(\alpha r) \right\} \right. \\
&\quad \left. + \sum_{m=2}^{N-1} \sum_{s'=1}^{\infty} \sum_{l=-\infty}^{\infty} \left\{ {}^sA_l^{m-1} H_{n-l}(\alpha r_{0s'}) J_n(\alpha r) - (-1)^l {}^sB_l^{m-1} K_{n-l}(\alpha r_{0s'}) \right. \right. \\
&\quad \left. \left. \times I_n(\alpha r) \right\} e^{-i(n-l)\theta_{0s'}} \right] e^{in\theta} e^{-i\omega t}
\end{aligned} \right\} \dots\dots\dots (15)$$

where

$$\left. \begin{aligned}
m_{1n} &= \left\{ \frac{-n(n+1)(1-\nu)}{\alpha^2 r^2} - \nu \right\} J_n(\alpha r) + \frac{1-\nu}{\alpha r} J_{n-1}(\alpha r) \\
m_{2n} &= \left\{ \frac{-n(n+1)(1-\nu)}{\alpha^2 r^2} + \nu \right\} I_n(\alpha r) - \frac{1-\nu}{\alpha r} I_{n-1}(\alpha r) \\
M_{1n} &= \left\{ \frac{-n(n+1)(1-\nu)}{\alpha^2 r^2} - \nu \right\} H_n(\alpha r) + \frac{1-\nu}{\alpha r} H_{n-1}(\alpha r) \\
M_{2n} &= \left\{ \frac{-n(n+1)(1-\nu)}{\alpha^2 r^2} + \nu \right\} K_n(\alpha r) - \frac{1-\nu}{\alpha r} K_{n-1}(\alpha r)
\end{aligned} \right\} \dots\dots\dots (16)$$

In Eqs. (15), if we consider one complete cycle, the real parts R yield the values at $t=0$ and the imaginary parts I yield the values at $t=T/4$, where $T=(2\pi/\omega)$ is the period of the incident wave. The absolute values $(R^2+I^2)^{1/2}$ give the maximum moment and shearing force concentration factors which occur at an instant between $t=0$ and $t=T/4$.

3. Numerical examples of transverse wave propagation

As numerical examples, let us consider two cases of plates having two and three circular holes of equal size, centers of which are spaced on a straight line as in Fig. 2. Then we have

$$r_{01}=Z, \theta_{01}=\pi/2, \theta_{10}=-\pi/2$$

for the case of two holes, and

$$r_{01}=Z, r_{02}=2Z, r_{12}=Z$$

$$\theta_{01}=\theta_{02}=\theta_{12}=\pi/2$$

$\theta_{10} = \theta_{20} = \theta_{21} = -\pi/2$
for the case of three holes. By taking $\gamma=0$, $\nu=0.3$ and $Z/a=10/3$, the variation of the bending moment concentration factors $|M_\theta/M_0|_{r=a}$ at the edge of the hole 0 with wave number αa for the

Fig. 2

Fig. 3 Bending moment concentration factors at $r=a$, $\theta=\pi/2, 3\pi/2$ of the hole 0 in the case of two holes of equal size ($\gamma=0$, $\nu=0.3$, $Z/a=10/3$)

Fig. 4 Bending moment concentration factors at $r=a$, $\theta=\pi/2$ of the hole 0 in the case of two holes of equal size ($\gamma=0$, $\nu=0.3$)

case of two holes of equal size is shown in Fig. 3. Figures 4 and 5 illustrate the effects of separation between two holes on the bending moment concentration factors for $\gamma=0$, $\nu=0.3$. In these figures, the solid line shows the values for the

Fig. 5 Bending moment concentration factors at $r=a$, $\theta=3\pi/2$ of the hole 0 in the case of two holes of equal size ($\gamma=0$, $\nu=0.3$)

Fig. 6 Bending moment concentration factors at $r=a$, $\theta=0, 3\pi/2$ of the hole 0 in the case of two holes of equal size ($\gamma=-\pi/2$, $\nu=0.3$, $Z/a=10/3$)

Fig. 7 Bending moment concentration factors at $r=a$, $\theta=0$ of the hole 0 in the case of two holes of equal size ($\gamma=-\pi/2$, $\nu=0.3$)

case of two holes and dashed line shows those for the single hole case which was given by authors⁽⁴⁾. From these figures, it is noted that the bending moment concentration factor is very sensitive to the wave number and the separation between two holes. From Figs. 4 and 5, we can also see that the curve of bending moment concentration reduces rapidly to that for the single hole case at low frequencies ($\alpha a = 0.1$), while at high frequencies ($\alpha a = 2.0$), this curve has a complicated shape and converges to one with oscillations. Figures 6 and 7 show the cases for $\gamma = -\pi/2$. Above discussions are also applicable to these cases. If the wave length of the incident transverse wave approaches infinity ($\alpha \rightarrow 0$), the results obtained should reduce to statical ones. In Figs. 3 and 6, the small circle $\bigcirc$ denotes the value of statical bending for the plate with two circular holes of equal size which was given by Tamate and Ishii⁽³⁾. We can see, of course, from these figures that, as the incident wave number decreases, the values of the bending moment concentration factors tend to those of the statical case. Figure 8 shows the variation of $|M_\theta/M_0|_{r=a}$ at the edge of the hole 0 with wave number αa for the case of three circular holes of equal size with $\nu = 0.3$, $\gamma = 0$, $Z/a = 10/3$.

4. Torsional wave propagation

The differential equation of a thin plate excited by the torsional wave may also be written in the same form as Eq. (1). The displacement of the plate subjected to a torsional wave which propagates along SQ axis making an angle γ with the x axis as in Fig. 9 can be obtained by superposing two transverse waves which have the same

amplitude and wave length and propagate along S_1Q and S_2Q axes making angles $+45^\circ$ and -45° with SQ axis respectively. The displacement of the plate is antisymmetric with respect to SQ axis. Thus putting $\gamma = \gamma_k'$ ($k=1, 2$) in Eq. (7), we obtain

$$g_k(\gamma_k') = w_0 \epsilon_s' \sum_{n=-\infty}^{\infty} J_n(\alpha r_s) e^{in(\theta_s + \xi_k)} e^{-i\omega t} \dots (17)$$

in which

$$\epsilon_s' = e^{i\alpha r_{0s} \cos(\theta_{0s} + \gamma)}$$

$$\xi_k = \gamma_k' + \pi/2$$

and

$$\gamma_1' = \gamma + \pi/4, \quad \gamma_2' = \gamma - \pi/4$$

represent the angles between S_1Q , S_2Q and SQ axes respectively. Then the displacement of a plate subjected to the incident wave is shown as

$$w_i = g_1(\gamma_1') - g_2(\gamma_2') \dots (18)$$

Substituting Eq. (17) into Eq. (18), the displacement of the plate due to the incident torsional wave is obtained as follows:

$$\begin{aligned} w_i &= 2w_0 i e^{i\alpha r_{0s}(1/\sqrt{2})\cos(\theta_{0s} + \gamma)} \sum_{n=-\infty}^{\infty} J_n(\alpha r_s) \\ &\quad \times e^{in(\theta_s + \xi)} \sin \left\{ \frac{n\pi}{4} - \frac{\alpha r_{0s}}{\sqrt{2}} \sin(\theta_{0s} + \gamma) \right\} e^{-i\omega t} \\ &= 2w_0 i \sum_{n=-\infty}^{\infty} J_n(\alpha r) e^{in(\theta + \xi)} \sin \left(\frac{n\pi}{4} \right) e^{-i\omega t} \dots (19) \end{aligned}$$

Figure 10 shows the shape of the incident torsional wave. Putting $\gamma = 0$ into Eq. (19), the displacement of the plate subjected to the incident torsional wave propagating along x axis can be obtained as

$$w_i = -2w_0 \sum_{n=1}^{\infty} \eta_n i^n J_n(\alpha r) \sin \left(\frac{n\pi}{4} \right) \sin n\theta e^{-i\omega t} \dots (20)$$

where

$$\eta_n = \begin{cases} 1 & n=0 \\ 2 & n \geq 1 \end{cases}$$

Fig. 9 Infinite thin plate with circular holes subjected to torsional waves

Fig. 10 Shapes of incident torsional waves

Fig. 8 Bending moment concentration factors at $r=a$, $\theta = \pi/2, 3\pi/2$ of the hole 0 in the case of three holes of equal size ($\gamma = 0$, $\nu = 0.3$, $Z/a = 10/3$)

If we shift the origin of time in Eq. (20) a half period, Eq. (20) corresponds to Eq. (15) of reference (4).

The displacement of the plate caused by the scattered waves is the same as in Eqs. (8). The total displacement is obtained in a similar way to the case of the transverse wave propagation, and is given by

$$\begin{aligned} {}^s\bar{w} = & 2i \sum_{n=-\infty}^{\infty} \left[w_0 \varepsilon_s^* J_n(\alpha r_s) e^{in\zeta} + {}^sC_{1n} {}^1H_n(\alpha r_s) + {}^sC_{2n} {}^1K_n(\alpha r_s) + \sum_{m=2}^{\infty} \left[{}^sC_{1n} {}^mH_n(\alpha r_s) \right. \right. \\ & + {}^sC_{2n} {}^mK_n(\alpha r_s) + \sum_{\substack{s'=0 \\ (s' \neq s)}}^{N-1} \sum_{l=-\infty}^{\infty} \left\{ {}^{s'}C_{1l} {}^{m-1}H_{n-l}(\alpha r_{ss'}) J_n(\alpha r_s) + (-1)^l {}^{s'}C_{2l} {}^{m-1} \right. \\ & \left. \left. \times K_{n-l}(\alpha r_{ss'}) I_n(\alpha r_s) \right\} e^{-i(n-l)\theta_{ss'}} \right] e^{in\theta_s} e^{-i\omega t} \dots \dots \dots (21) \end{aligned}$$

where

$$\varepsilon_s^* = e^{iar_{0s}(1/\sqrt{2})\cos(\theta_{0s}+\gamma)} \sin \left\{ \frac{n\pi}{4} - \frac{\alpha r_{0s}}{\sqrt{2}} \sin(\theta_{0s}+\gamma) \right\}$$

The boundary conditions for this case are also the same as in Eqs. (10). The unknown constants can be found by substituting Eq. (21) into Eqs. (10). In this case, if we put ε_s^* in place of ε_s in Eqs. (12), we have the same forms for the unknown constants ${}^sC_{1n}^m$, ${}^sC_{2n}^m$ as in Eqs. (11) and (12). E_{1n} , E_{2n} , ... are the same as in Eq. (13).

The maximum twisting moment H_0 in the plate subjected to the incident wave is ⁽⁴⁾

$$H_0 = D\omega_0 \alpha^2 (1-\nu) \dots \dots \dots (22)$$

The bending moment and shearing force quantities are calculated in a similar way to the case of transverse wave propagation, and those are shown as

$$\begin{aligned} \frac{M_\theta}{H_0} = & -\frac{2i}{1-\nu} \sum_{n=-\infty}^{\infty} \left[m_{1n} e^{in\zeta} + \sum_{m=1}^{\infty} ({}^0\bar{A}_n {}^mM_{1n} + {}^0\bar{B}_n {}^mM_{2n}) + \sum_{m=2}^{\infty} \sum_{s'=1}^{N-1} \sum_{l=-\infty}^{\infty} \left\{ {}^{s'}\bar{A}_l {}^{m-1} \right. \right. \\ & \left. \left. \times H_{n-l}(\alpha r_{0s'}) m_{1n} + (-1)^l {}^{s'}\bar{B}_l {}^{m-1} K_{n-l}(\alpha r_{0s'}) m_{2n} \right\} e^{-i(n-l)\theta_{ss'}} \right] e^{in\theta} e^{-i\omega t} \\ \frac{a_0 Q_\theta}{H_0} = & -\frac{2}{1-\nu} \frac{a_0}{r} \sum_{n=-\infty}^{\infty} n \left[J_n(\alpha r) e^{in\zeta} + \sum_{m=1}^{\infty} \{ {}^0\bar{A}_n {}^mH_n(\alpha r) - {}^0\bar{B}_n {}^mK_n(\alpha r) \} \right. \\ & + \sum_{m=2}^{\infty} \sum_{s'=1}^{N-1} \sum_{l=-\infty}^{\infty} \left\{ {}^{s'}\bar{A}_l {}^{m-1} H_{n-l}(\alpha r_{0s'}) J_n(\alpha r) - (-1)^l {}^{s'}\bar{B}_l {}^{m-1} \right. \\ & \left. \left. \times K_{n-l}(\alpha r_{0s'}) I_n(\alpha r) \right\} e^{-i(n-l)\theta_{ss'}} \right] e^{in\theta} e^{-i\omega t} \dots \dots \dots (23) \end{aligned}$$

where ${}^0\bar{A}_n^m$, ${}^{s'}\bar{A}_l^{m-1}$, ${}^0\bar{B}_n^m$ and ${}^{s'}\bar{B}_l^{m-1}$ represent the constants in Eqs. (12) in which ε_s is replaced by ε_s^* .

As numerical examples, let us consider the same arrangement of the holes for the case of the transverse wave propagation as in Fig. 2. By taking $\gamma=0$ and $-\pi/2$, $\nu=0.3$, $Z/a=10/3$, the variations of bending

Fig. 11 Bending moment concentration factors at $r=a$, $\theta=\pi/2, 3\pi/4$ of the hole 0 in the case of two holes of equal size ($\gamma=0$, $\nu=0.3$, $Z/a=10/3$)

Fig. 12 Bending moment concentration factors at $r=a$, $\theta=\pi/4, 7\pi/4$ of the hole 0 in the case of two holes of equal size ($\gamma=-\pi/2$, $\nu=0.3$, $Z/a=10/3$)

Fig. 13 Bending moment concentration factors at $r=a$, $\theta=3\pi/4$ of the hole 0 in the case of two holes of equal size ($\gamma=0$, $\nu=0.3$)

Fig. 14 Bending moment concentration factors at $r=a$, $\theta=\pi/4$ of the hole 0 in the case of two holes of equal size ($\gamma=-\pi/2$, $\nu=0.3$)

moment concentration factors $|M_\theta/H_0|_{r=a}$ at the edge of the hole 0 with wave number αa for the case of two holes of equal size are shown in Fig. 11 and Fig. 12. The solid line shows the result of this analysis and the dashed line does the value of single hole case⁽⁴⁾. The small circle $\bigcirc$ denotes the statical result given by Tamate and Ishii^{(8) (9)}. From these figures, it is seen that the dynamical results agree with statical ones for $\alpha=0$. Figures 13 and 14 illustrate the effects of Z/a on the bending moment concentration factors. We can find that the bending moment concentration factor is very sensitive to the wave number and the separation between two holes. Figure 15 shows the results of the case for three circular holes of equal size with $\nu=0.3$, $\gamma=0$, $Z/a=10/3$.

Fig. 15 Bending moment concentration factors at $r=a$, $\theta=\pi/2$, $3\pi/4$ of the hole 0 in the case of three holes of equal size ($\gamma=0$, $\nu=0.3$, $Z/a=10/3$)

5. Conclusions

In this paper, the problems of transverse or torsional wave propagation in an infinite thin elastic plate having many holes were investigated. The dynamical moment concentrations around the holes were examined in detail. The results obtained were compared with those of the single hole case, and it was shown that the bending moment concentration factors are very sensitive to the wave numbers and the separations between holes. When the incident wave length becomes infinity, the dynamical solution given by this analysis tends to a statical one.

The numerical results were obtained by using NEAC-2200 Model 500 Computer of the Tohoku University Computing Center.

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Discussion

S. SUZUKI (Nagoya University) :

It is interesting to see that the dynamical moment concentration factor changes the value with the values of parameters, differing from the statical one.

We raise some questions as follows.

(1) The authors explain that the convergency of the solution expanded into Bessel Fourier series is not so good at high frequencies. How many terms of series should be taken in order to obtain sufficient accuracy for the engineering practices?

(2) The relations between moment concentration factors at $\theta=\pi/2$ and αa or Z/a for $\gamma=-\pi/2$ are not shown in the figures for the case of two holes. Are there any interesting aspects of this case from an engineering point of view?

(3) In Figs. 5 and 7, the moment concentration factors for small values of αa converge simply to those of the case of a single hole, but, as the value of αa increases, these factors converge with oscillations. What value does αa have on this change point?

(4) Figures 5 and 7 show the cases for $2.5 \leq Z/a \leq 12.5$. What aspects do the concentration factors show for $Z/a \rightarrow 2$?

(5) Pao reported that the stress concentration factors are dependent largely upon Poisson's ratio. How about this research?

(6) The authors state that this problem is important in practical use. What kind of practical problems is this study applied to?

Authors' closure

(1) The results for $n=\pm 14$, $m=6$ agree almost perfectly with those for $n=\pm 20$, $m=13$ in the case of $Z/a=2.5$. Therefore, $n=\pm 14$ and $m=3\sim 6$ were used in numerical calculations, where

m was selected from 3 to 6 according to the values of Z/a and wave length.

(2) The maximum bending moment in the plate subjected to the transverse wave occurs at about $\theta=0, \pi$. Since those values are important in practical problems, we examined those at $\theta=0, \pi$. The calculations of the moments at the points such as $\theta=\pi/2$ where the values of moments are small are omitted in this paper.

(3) The curve of the moment versus Z/a converges simply to the single hole case under statical bendings ($\alpha a=0$), but, under dynamical bendings ($\alpha a>0$), the curve converges with oscillations to the case of single hole. Since the period of the curve is very large for small values of αa like 0.1, these aspects can not be observed within the figure. However, when αa becomes large, those aspects will appear clearly in the figure.

(4) Since both holes contact each other when $Z/a=2.0$, the convergency of the series solution becomes quite poor. As terms of the series increase, the errors of calculation become large and it is difficult to get accurate values. Thus, in this report, the numerical calculations were limited to the case of $Z/a \geq 2.5$ where we can get enough accuracy for the engineering practices.

(5) Poisson's ratio has an effect on the bending moment concentration factors. A detailed discussion about this is reported in reference (4).

(6) It is usually observed that a plate with holes in the machines and structures is subjected to flexural waves. In practical problems, there are the effects of the outer edges of the plate. However, when the distance between the edge and hole is large, these effects can be neglected and the result of an infinite plate with holes is applicable to actual cases.