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On Some Basic Properties of Generalized Solution in Elastostatics*

By Rohzō FURUHASHI**

In the present paper, some basic properties of generalized solutions for the equation of 3-dimensional elastostatics were considered and the following results were obtained: (a) The elementary solution of elastostatics (Somigliana's tensor) satisfies the basic equation in the sense of distribution. (b) The basic formulas for the theory of elastic potentials are derived from the convolution of elementary solution and a distribution. (c) If body force is absent, any generalized solution for the bounded domain is infinitely differentiable. (d) The displacement solution for the problem of uniform extension of an infinite body is explained as a generalized solution in the space of slowly increasing distribution. (e) If a displacement solution of the equation with no body force is bounded in the whole domain R^3 , then it is a constant.

1. Introduction

To study the problems of elasticity in the classical sense, it is necessary to introduce the notion of regularity, which requires that the displacement vector is continuously twice differentiable in a domain and decreases to zero at the infinity in the order of inverse of the distance from the origin if the elastic body is unbounded. However, in many practical problems, it becomes necessary to use some typical solutions which are not regular. For example, the solution for the case of concentrated force at a point (where the displacement is not finite at that point): the solution for the uniform extension of an infinite body (where the displacement becomes infinite at the infinity). Therefore, it seems natural to inquire some generalizations of the solution in elasticity to include an important class of singular solutions. To do so, in this paper, we shall establish some basic properties of generalized solutions in the sense of distribution.

2. Some elementary properties of distribution

We shall quote some elementary properties of distribution without proof. The proofs will be found in the book of V. S. Vladimirov⁽¹⁾.

(A) Distribution. Let $\mathcal{D}$ be a set of all test functions, which means a set of all infinitely dif-

ferentiable functions with compact support in 3-dimensional Euclidean space R^3 . We say a sequence $\{\varphi_k\} \subset \mathcal{D}$ converges to φ in $\mathcal{D}$ if and only if the following two conditions are satisfied: (a) There exists a sphere U_R with radius R centered at the origin such that $\text{car}(\varphi_k) \subset U_R$. [By $\text{car}(\varphi_k)$ we denote the support of φ_k]. (b) For any multi-index α , the derivative $D^\alpha \varphi_k(x)$ converges uniformly in R^3 to $D^\alpha \varphi(x)$ as $k \rightarrow \infty$. Where x is a point in R^3 with coordinate (x_1, x_2, x_3) and D^α stands for the following differential operator:

$$D^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \dots (1)$$

The differential operator D^α maps $\mathcal{D}$ into $\mathcal{D}$.

A continuous linear functional f over $\mathcal{D}$ is called a distribution. This statement means that f satisfies the following three conditions: (a) For any $\varphi \in \mathcal{D}$, there is correspondingly a complex number $\langle f, \varphi \rangle$. (b) For any $\varphi, \psi \in \mathcal{D}$ and arbitrary complex numbers α and β , the next relation holds:

$$\langle f, \alpha\varphi + \beta\psi \rangle = \alpha\langle f, \varphi \rangle + \beta\langle f, \psi \rangle \dots (2)$$

(c) If $\varphi_k \rightarrow \varphi$ in $\mathcal{D}$ as $k \rightarrow \infty$, then $\langle f, \varphi_k \rangle \rightarrow \langle f, \varphi \rangle$ as $k \rightarrow \infty$. Let $\mathcal{D}'$ be a set of all distributions.

If $\langle f, \varphi \rangle = \langle g, \varphi \rangle$ for all $\varphi \in \mathcal{D}$, we say that the distributions f and g coincide, and they will be written simply by $f = g$. The differentiation of the distribution is defined by

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle, \quad \varphi \in \mathcal{D} \dots (3)$$

All the distributions are infinitely differentiable in the above sense. Moreover, if the support of

* Received 16th September, 1971.

** Assistant Professor, Faculty of Engineering, Meiji University, Kawasaki.

the distribution f is $\{0\}$, then f is represented uniquely in the following form:

$$f(x) = \sum_{|\alpha|=0}^m C_\alpha D^\alpha \delta(x), \quad m \geq 0, \quad \dots\dots\dots(4)$$

where C_α is a constant and $\delta(x)$ is Dirac's measure at 0.

(B) Convolution of the distribution. We assume $f(x)$ and $g(x)$ to be locally integrable in R^3 . By the convolution of f and g we mean the following:

$$(f * g)(x) = \int_{R^3} f(x-y)g(y)dy \quad \dots\dots\dots(5)$$

For any locally integrable function f and an arbitrary distribution g with compact support, there exists a convolution $f * g$ in $\mathcal{D}'$, and the following expression is permitted:

$$\langle f * g, \varphi \rangle = \langle f(x)g(y), \eta(y)\varphi(x+y) \rangle, \quad \varphi \in \mathcal{D} \quad \dots\dots\dots(6)$$

where η denotes a test function which equals to 1 in the neighborhood of $\text{car}(g)$. If the convolution $f * g$ exists, then $D^\alpha f * g$ and $f * D^\alpha g$ exist also, and in $\mathcal{D}'$ the formula

$$D^\alpha f * g = D^\alpha(f * g) = f * D^\alpha g \quad \dots\dots\dots(7)$$

is valid.

(C) Fourier transform of distribution. Let $\mathcal{S}$ be a set of all functions which belong to $C^\infty(R^3)$ and decrease with all its derivatives more rapidly than any powers of $|x|^{-1}$ when $|x| \rightarrow \infty$. For any sequence $\{\varphi_k\} \subset \mathcal{S}$, the expression $\varphi_k \rightarrow \varphi$ ($k \rightarrow \infty$) in $\mathcal{S}$ means that $x^\beta D^\alpha \varphi_k(x)$ converges uniformly to $x^\beta D^\alpha \varphi(x)$ as $k \rightarrow \infty$ for an arbitrary multi-index α and β . The continuous linear functional over $\mathcal{S}$ is called the slowly increasing distribution, and the set of all them will be denoted by $\mathcal{S}'$. Fourier transform of $f \in \mathcal{S}'$ is defined by the following:

$$\mathcal{F}[f] = \langle f, \mathcal{F}[\varphi] \rangle, \quad f \in \mathcal{S}', \quad \varphi \in \mathcal{D} \quad \dots\dots\dots(8)$$

Here,

$$\mathcal{F}[\varphi] = \int_{R^3} f(x)e^{ix \cdot x} dx, \quad \varphi \in \mathcal{D} \quad \dots\dots\dots(9)$$

where $i = \sqrt{-1}$ is an imaginary number. The inverse transform is given by the formula,

$$\mathcal{F}[f] = \frac{1}{(2\pi)^3} \mathcal{F}[f(-x)], \quad f \in \mathcal{S}' \quad \dots\dots\dots(10)$$

Moreover, for any $f \in \mathcal{S}'$, the formula

$$D^\alpha \mathcal{F}[f] = \mathcal{F}[(ix)^\alpha f] \quad \dots\dots\dots(11)$$

is valid.

3. Definition of generalized solution

Let G be a bounded open domain in R^3 , $\bar{G}$ its closure and we suppose the boundary ∂G of $\bar{G}$ is piecewise smooth. As is well known, the displacement vector $u(x)$ must satisfy in G the following partial differential equation of second order:

$$Au(x) \equiv \mu \Delta u(x) + (\lambda + \mu) \nabla \nabla \cdot u(x) = -f(x)$$

$$\dots\dots\dots(12)$$

where λ and μ stand for Lamé's constants, f denotes a body force. The definition of generalized solution for Eq. (12) is simple enough. Since A is a self-adjoint operator, in view of the definition of differentiation of the distribution, we say that $u \in \mathcal{D}$ is a generalized solution of Eq. (12) if it satisfies the following equation for any $\varphi \in \mathcal{D}$ and $f \in \mathcal{D}'$:

$$\langle u, A\varphi \rangle = -\langle f, \varphi \rangle \quad \dots\dots\dots(13)$$

Therefore, when the body force is absent, the generalized solution is defined as a solution $u \in \mathcal{D}'$ of the equation such that,

$$\langle u, A\varphi \rangle = 0, \quad \varphi \in \mathcal{D} \quad \dots\dots\dots(14)$$

If $u(x)$ is a classical solution of Eq. (12), the displacement $u(x)$ has a continuous second order derivatives so that it clearly satisfies Eq. (13). Conversely, if the generalized solution $u(x)$ is twice continuously differentiable in G and f is continuous, it is a classical solution of Eq. (12). To prove this we will use the following proposition⁽²⁾.

(Proposition) Any locally integrable function $f(x)$ in G equals to zero in the sense of distribution if and only if it vanishes except in the set of Lebesgue measure zero.

Proof of our statement: Because of the condition, u belongs to the class $\mathcal{D}' \cap C^2(G)$, then the classical and the generalized second order derivatives coincide in G . (Here $C^k(G)$ means the class of continuously differentiable functions up to the order k in the domain G). Since $Au + f$ is continuous in G and u is a generalized solution of Eq. (13), we see that the equality

$$\langle Au + f, \varphi \rangle = 0, \quad \varphi \in \mathcal{D} \quad \dots\dots\dots(15)$$

is valid. Then, from property (A) of Chapter 2, $Au + f$ is continuous in G and equals to zero in the sense of distribution. Therefore, the above proposition states that there holds $Au + f = 0$ for all points in G . This concludes our statement.

In general, the converse does not hold. This result implies that the generalized solution, which is defined by Eq. (13), is an extension for the concept of solutions in the classical sense.

4. Reciprocal theorem and elementary solution

By the elementary solution for the operator A we mean the distribution $I \in \mathcal{S}'$ if it satisfies the following equation

$$AI(x) = -\delta(x)I \quad \dots\dots\dots(16)$$

where I stands for 3×3 unit matrix. After Fourier transform, we have

$$\mathfrak{F}[\Gamma](\xi) = \frac{1}{\mu(\lambda+2\mu)} \left((\lambda+3\mu) \frac{1}{|\xi|^2} \delta_{jk} - (\lambda+\mu) \frac{\xi_j \xi_k}{|\xi|^4}, \begin{matrix} j \downarrow 1.2.3 \\ k \rightarrow 1.2.3 \end{matrix} \right) \dots\dots\dots (17)$$

Here δ_{jk} stands for Kronecker's delta. In view of the formulas

$$\frac{1}{(2\pi)^3} \int_{R^3} \frac{e^{i\xi x}}{|\xi|^2} d\xi = \frac{1}{4\pi|x|},$$

$$-\frac{1}{(2\pi)^3} \int_{R^3} \frac{e^{i\xi x}}{|\xi|^4} d\xi = \frac{|x|}{8\pi} \dots\dots\dots (18)$$

the inverse transformation, which is given by Eq. (10), reduces easily to the following expression:

$$\Gamma(x) = \frac{1}{8\pi\mu(\lambda+2\mu)} \left(\frac{\lambda+3\mu}{|x|} \delta_{jk} + \frac{\lambda+\mu}{|x|} \times \frac{\partial|x|}{\partial x_j} \frac{\partial|x|}{\partial x_k}, \begin{matrix} j \downarrow 1.2.3 \\ k \rightarrow 1.2.3 \end{matrix} \right) \dots\dots\dots (19)$$

Conversely, the matrix (19) satisfies Eq. (16) in the sense of distribution. Before doing proof of this statement we will establish a reciprocal theorem which is an extension of the classical Maxwell-Betti formula to the generalized solutions. To begin with, we introduce the following concept:

Let ζ be continuous on ∂G , and we define the functionals $\zeta \delta_s$ and $\frac{d}{dn}(\zeta \delta_s)$ as follows:

$$\langle \zeta \delta_s, \varphi \rangle = \int_{\partial G} \zeta(x) \varphi(x) ds, \quad \varphi \in \mathfrak{D} \dots\dots\dots (20)$$

$$\langle -\frac{d}{dn}(\zeta \delta_s), \varphi \rangle = \int_{\partial G} \zeta(x) \frac{d\varphi(x)}{dn} ds \dots\dots\dots (21)$$

where d/dn is the stress operator defined as follows.

$$\frac{d\varphi}{dn} \equiv \lambda n \nabla \cdot \varphi + 2\mu \frac{\partial \varphi}{\partial n} + \mu(n \times \nabla \times) \varphi \dots\dots\dots (22)$$

Here n stands for the unit outward normal on ∂G . The functionals $\zeta \delta_s$ and $-\frac{d}{dn}(\zeta \delta_s)$ will be called the generalized simple and double layers, respectively.

The next results are easily derived from the above definitions.

$$\left. \begin{aligned} \zeta \delta_s &\in \mathfrak{D}', \quad \text{car}(\zeta \delta_s) \subset \partial G \\ -\frac{d}{dn}(\zeta \delta_s) &\in \mathfrak{D}', \quad \text{car}\left[-\frac{d}{dn}(\zeta \delta_s)\right] \subset \partial G \end{aligned} \right\} \dots\dots\dots (23)$$

We prove the case of generalized simple layer $\zeta \delta_s$. To do so, it is sufficient to verify that the functional $\zeta \delta_s$ defined above satisfies the three conditions for distribution, which were explained in Chapter 2-(A). For any complex number α , β and φ , $\psi \in \mathfrak{D}$, the following equality is obvious.

$$\left. \begin{aligned} \langle \zeta \delta_s, \alpha\varphi + \beta\psi \rangle &= \int_{\partial G} \zeta(x) \{\alpha\varphi(x) + \beta\psi(x)\} ds \\ &= \alpha \int_{\partial G} \zeta(x) \varphi(x) ds + \beta \int_{\partial G} \zeta(x) \psi(x) ds \\ &= \alpha \langle \zeta \delta_s, \varphi \rangle + \beta \langle \zeta \delta_s, \psi \rangle \end{aligned} \right\}$$

$$\dots\dots\dots (24)$$

Hence the functional $\zeta \delta_s$ is linear. If $\varphi_k \rightarrow \varphi$ ($k \rightarrow \infty$) in $\mathfrak{D}$, in view of Lebesgue's theorem, there holds the relation such that,

$$\langle \zeta \delta_s, \varphi_k \rangle = \int_{\partial G} \zeta(x) \varphi_k(x) ds$$

$$\xrightarrow{k \rightarrow \infty} \int_{\partial G} \zeta(x) \varphi(x) ds = \langle \zeta \delta_s, \varphi \rangle \dots\dots\dots (25)$$

Hence the functional $\zeta \delta_s$ is continuous. Then we have the statement $\zeta \delta_s \in \mathfrak{D}'$. On the other hand, $\zeta \delta_s$ vanishes except on ∂G , then the statement $\text{car}(\zeta \delta_s) \subset \partial G$ is proved.

(Theorem 1) For any vector valued function $u \in C^2(\bar{G}) \cap C^2(\bar{G}_1)$, the following formula holds.

$$Au = \{Au\} + \frac{d}{dn}([u]_s \delta_s) + \left[\frac{du}{dn} \right]_s \delta_s \dots\dots\dots (26)$$

where $\{Au\}$ means the ordinary derivative, G_1 stands for the complement of $\bar{G}$, i. e., $G_1 = R^3 \setminus \bar{G}$, and $[u]_s$ denotes the jump of u when it passes through the surface ∂G , that is:

$$[u]_s(x) = \lim_{x' \in G_1 \rightarrow x \in \partial G} u(x') - \lim_{x' \in G \rightarrow x \in \partial G} u(x') \dots\dots\dots (27)$$

(Proof) We will express the differential operator A and d/dn defined by Eq. (12) and (22) in matrix form. The (i, j) element A^{ij} and d^{ij}/dn are represented as follows⁽³⁾:

$$\left. \begin{aligned} A^{ij} &\equiv \mu \delta_{ij} + (\lambda + \mu) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \\ \frac{d^{ij}}{dn} &\equiv \lambda n_i \frac{\partial}{\partial x_j} + \mu n_j \frac{\partial}{\partial x_i} + \mu \delta_{ij} \frac{\partial}{\partial n} \end{aligned} \right\} \dots\dots\dots (28)$$

It is well known that the next formula⁽⁴⁾ holds for any scalar function $f \in C^2(\bar{G}) \cap C^2(\bar{G}_1)$:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \left\{ \frac{\partial^2 f}{\partial x_i \partial x_j} \right\} + \frac{\partial}{\partial x_j} ([f]_s \cos(n, x_i) \delta_s)$$

$$+ \left[\left\{ \frac{\partial f}{\partial x_i} \right\} \right]_s \cos(n, x_j) \delta_s \dots\dots\dots (29)$$

Accordingly, for any scalar function $f \in C^2(\bar{G}) \cap C^2(\bar{G}_1)$, we have

$$A^{ij} f = \{A^{ij} f\} + \lambda \frac{\partial}{\partial x_j} ([f]_s n_i \delta_s) + \mu \frac{\partial}{\partial x_i}$$

$$\times ([f]_s n_j \delta_s) + \mu \delta_{ij} \frac{\partial}{\partial n} ([f]_s \delta_s) + \lambda n_i \left[\frac{\partial f}{\partial x_j} \right]_s$$

$$\times \delta_s + \mu n_j \left[\frac{\partial f}{\partial x_i} \right]_s \delta_s + \mu \delta_{ij} \left[\frac{\partial f}{\partial n} \right]_s \delta_s \dots\dots\dots (30)$$

Then, because of the definition of the operator d^{ij}/dn , we have

$$A^{ij} f = \{A^{ij} f\} + \frac{d^{ij}}{dn} ([f]_s \delta_s) + \left[\frac{d^{ij} f}{dn} \right]_s \delta_s \dots\dots\dots (31)$$

This equality implies the formula (26) for any vector valued function $u \in C^2(\bar{G}) \cap C^2(\bar{G}_1)$. (q.e.d.)

The equality (26) is nothing but Maxwell-Betti reciprocal theorem written in the words of distribution. In particular, if we put $u=0$ in G_1 , Eq. (26) will be reduced to the following:

$$Au = \{Au\} - \frac{du}{dn} \delta_s - \frac{d}{dn} (u \delta_s) \dots\dots\dots (32)$$

Hence, for any $\varphi \in \mathcal{D}$, the relation

$$\begin{aligned} \langle u, A\varphi \rangle &= \langle Au, \varphi \rangle \\ &= \langle \{Au\}, \varphi \rangle - \langle \frac{du}{dn} \delta_s, \varphi \rangle - \langle \frac{d}{dn} (u \delta_s), \varphi \rangle \end{aligned} \dots\dots\dots (33)$$

is valid. Then, if we rewrite this in the integral form, we get easily the ordinary Maxwell-Betti reciprocal formula as follows:

$$\int_G (\varphi Au - u A\varphi) dx = \int_{\partial G} (\varphi \frac{du}{dn} - u \frac{d\varphi}{dn}) ds \dots\dots\dots (34)$$

By turning to account of the above result we can explain that the elementary solution defined by Eq. (19) satisfies Eq. (16) in the sense of distribution. From Eq. (13), we get

$$\langle A\Gamma, \varphi \rangle = \int_{R^3} \Gamma(x) A\varphi(x) dx, \quad \varphi \in \mathcal{D} \dots\dots\dots (35)$$

The integral on the right hand side is understood to be of Cauchy's principal value. Then it is sufficient to evaluate the integral

$$\int_{R^3} \Gamma(x) A\varphi(x) dx = \lim_{\varepsilon \rightarrow 0} \int_{|x| > \varepsilon} \Gamma(x) A\varphi(x) dx, \quad \varphi \in \mathcal{D} \dots\dots\dots (36)$$

Since $\text{car}(\varphi)$ is finite, the domain of integration on the right hand side is bounded. In view of the reciprocal formula (33), we get the following equality if we consider the fact that the elementary solution (19) belongs to the class C^∞ except at the origin:

$$\begin{aligned} \int_{|x| > \varepsilon} \Gamma(x) A\varphi(x) dx &= \int_{|x| > \varepsilon} A\Gamma(x) \varphi(x) dx \\ &- \int_{|x| = \varepsilon} \frac{d\Gamma(x)}{dn} \varphi(x) ds + \int_{|x| = \varepsilon} \Gamma(x) \\ &\times \frac{d\varphi(x)}{dn} ds \dots\dots\dots (37) \end{aligned}$$

Since the matrix (19) satisfies the following equation in the domain $|x| > 0$,

$$A\Gamma(x) = 0 \dots\dots\dots (38)$$

from the formula⁽⁵⁾

$$\int_{|x| = \varepsilon} \frac{d\Gamma(x)}{dn} ds = I \dots\dots\dots (39)$$

we can easily obtain the following results:

$$\left. \begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{|x| = \varepsilon} \frac{d\Gamma(x)}{dn} \varphi(x) ds &= \varphi(0) \\ \lim_{\varepsilon \rightarrow 0} \int_{|x| = \varepsilon} \Gamma(x) \frac{d\varphi(x)}{dn} ds &= 0 \end{aligned} \right\} \dots\dots\dots (40)$$

Then, Eq. (34) is deformed into the following:

$$\langle A\Gamma, \varphi \rangle = -\varphi(0) \dots\dots\dots (41)$$

Whence, we have

$$\langle A\Gamma, \varphi \rangle = -\langle \delta I, \varphi \rangle, \quad \varphi \in \mathcal{D} \dots\dots\dots (42)$$

This equality implies that the matrix (19) is a generalized solution of Eq. (16).

5. Convolution in elasticity

In this chapter, we define the convolutions which appear in elastostatics, and establish their basic properties. First of all, we would examine the convolution of the elementary solution Γ and the distribution f , that is

$$V = \Gamma * f \dots\dots\dots (43)$$

When the support of f is bounded, from Property (B) of Chap. 2, there exists the convolution (43). Because of Eq. (7) and (16), we have

$$AV = A(\Gamma * f) = A\Gamma * f = -\delta I * f = -f \dots\dots\dots (44)$$

Then, we see that the convolution (43) is a generalized solution of Eq. (12). Clearly, when f is locally integrable, we can represent Eq. (43) in the integral form as follows:

$$V(x) = \int_G \Gamma(x-y) f(y) dy \dots\dots\dots (45)$$

where G denotes $\text{car}(f)$. The expression (45) has also an important meaning when we make research of Eq. (12) in the classical sense, and we can prove⁽⁶⁾ that if f belongs to the class $C^{k+\alpha}$, $1 > \alpha > 0$, the vector (45) belongs to the class C^{k+2} and it satisfies Eq. (12), then if f is Hölder continuous, the convolution (45) represents a particular solution of Eq. (12) in the classical sense. However, the convolution (43) is not always a classical solution when f is a distribution. We get the following result:

(Theorem 2) If the support of the distribution f is bounded, the convolution $u = \Gamma * f$ is a generalized solution of Eq. (12).

(Corollary) Suppose $u \in \mathcal{D}'$. If $\Gamma * u$ exists, the formula

$$u = -\Gamma * Au \dots\dots\dots (46)$$

is valid in the sense of distribution.

As an application of this theorem, we extend Neuber-Papkovich formula to the class of generalized solutions.

(Proposition) If the support of the distribution f is bounded, the generalized solution of Eq. (12) is expressed as follows:

$$u = \mathcal{P}(xB + B_0) - 4(1-\nu)B \dots\dots\dots (47)$$

where, ν stands for Poisson's ratio, and B and B_0 are assumed to be the generalized solution of the following equations:

$$\Delta B = \frac{f}{4\mu(1-\nu)}, \quad \Delta B_0 = -\frac{xf}{4\mu(1-\nu)} \dots\dots\dots (48)$$

Indeed, by transforming the elementary solution (19) such that,

$$\begin{aligned} \Gamma(x) &= -\frac{1}{16\pi\mu(1-\nu)} \mathcal{P}\left(\frac{x}{|x|}\right) + \frac{4(1-\nu)}{16\pi\mu(1-\nu)} \\ &\times \frac{1}{|x|} I \dots\dots\dots (49) \end{aligned}$$

we can easily rewrite the convolution $u = \Gamma * f$ as follows:

$$u = \frac{1}{16\pi\mu(1-\nu)} \left\{ \nabla \left[\frac{1}{|x|} * yf \right] - \nabla \left[x \left(\frac{1}{|x|} * f \right) \right] + 4(1-\nu) \left[\frac{1}{|x|} * f \right] \right\} \dots\dots\dots (50)$$

In view of Theorem 2, Eq. (50) is a generalized solution of Eq. (12). Let us introduce the notations such that

$$B = -\frac{1}{16\pi\mu(1-\nu)} \frac{1}{|x|} * f$$

$$B_0 = \frac{1}{16\pi\mu(1-\nu)} \frac{1}{|x|} * yf \dots\dots\dots (51)$$

Then Eq. (50) takes the form (47). Further, it is well known that B , B_0 defined by Eq. (51) are the generalized solutions of Poisson's equations (48), then our statement was proved.

Now, we introduce the convolutions of Γ and the distributions $\zeta\delta_s$, $-d(\zeta\delta_s)/dn$ defined by Eq. (20), (21), such that

$$V^0 = \Gamma * \zeta\delta_s, V^1 = -\Gamma * \frac{d}{dn}(\zeta\delta_s) \dots\dots\dots (52)$$

Since the supports of distributions $\zeta\delta_s$ and $-d(\zeta\delta_s)/dn$ are bounded, the above convolutions exist. Because of Eq. (44), we have

$$AV^0 = -\zeta\delta_s, AV^1 = \frac{d}{dn}(\zeta\delta_s) \dots\dots\dots (53)$$

However, in view of the property (23), V^0 and V^1 satisfy the basic equations of elasticity for an arbitrary point $x \in G$, i. e.,

$$AV^0(x) = 0, AV^1(x) = 0, x \in G \dots\dots\dots (54)$$

Therefore we can conclude that the convolutions V^0 and V^1 are, for any continuous vector ζ on ∂G , the generalized solutions for the equation of elasticity with no body force. On the other hand, because of Eq. (6), the equality

$$\langle \Gamma * \zeta\delta_s, \varphi \rangle = \langle \Gamma(\xi)\zeta\delta_s, \eta(y)\varphi(y+\xi) \rangle$$

$$= \int_{R^3} \varphi(x) \left(\int_{\partial G} \Gamma(x-y)\zeta(y)dS_y \right) dx \dots\dots (55)$$

holds for any test function η which is equal to the unit matrix in a neighborhood of $\text{car}(\zeta\delta_s)$. Then the convolution V^0 can be represented in the integral form as follows:

$$V^0(x) = \int_{\partial G} \Gamma(x-y)\zeta(y)ds \dots\dots\dots (56)$$

Similarly, we have the following representation.

$$V^1(x) = \int_{\partial G} \left(\frac{d\Gamma(x-y)}{dn_y} \right)' \zeta(y)ds \dots\dots\dots (57)$$

where $(\cdot)'$ denotes the transpose of matrix $(\cdot)$. Obviously $\Gamma(x-y)$ belongs to the class $C^\infty(\bar{G}-y)$, which follows from Eq. (19). Thus, the convolutions $V^0(x)$ and $V^1(x)$ are infinitely differentiable in the open domain G .

The convolutions (56), (57) play an essential

role in the classical theory of elastic potentials. It is well known that the existence theorems of classical solutions for the displacement and traction boundary value problems of elastostatics can be proved on the basis of the properties of $V^0(x)$ and $V^1(x)$ under the assumption that ζ is Hölder continuous⁽⁷⁾. At this stage, the convolutions V^0 and V^1 are said to be the generalized potentials of simple and double layers, respectively.

(Proposition) If the vector $u \in C^2(\bar{G})$ vanishes in the exterior closed domain $\bar{G}_1$, the following formula is valid.

$$u = -\Gamma * \{Au\} + \Gamma * \left(\frac{du}{dn} \zeta_s \right) + \Gamma * \frac{d}{dn}(u\delta_s) \dots\dots\dots (58)$$

Indeed, when the above assumption of u is satisfied, the distribution Au defined by Eq. (32) has a compact support, and there exists a convolution $\Gamma * Au$. Hence Eq. (46) concludes easily our statement.

In particular, from Eqs. (45), (56) and (57), we get

$$u(x) = - \int_G \Gamma(x-y)Au(y)dy$$

$$+ \int_{\partial G} \left\{ \Gamma(x-y) \frac{du(y)}{dn} - \left(\frac{d\Gamma(x-y)}{dn_y} \right)' u(y) \right\} \times ds \dots\dots\dots (59)$$

This expression is nothing but the classical Somigliana formula.

6. Local property of generalized solution

We would establish a fundamental result which is stated as follows:

(Theorem 3) If a body force is absent, any generalized solution is infinitely differentiable in the open domain G . Hence, the operator A defined by Eq. (12) is hypo-elliptic.

(Proof) Let G' be an arbitrary proper subset of G , i. e., $G' \subset G$. By definition of the generalized solution u , this vector belongs to the class $C(G)$. Then we have $u \in C(\bar{G}')$. Suppose $\bar{u}$ is an extension of u to the exterior of $\bar{G}'$ with the value zero. Since $\text{car}(\bar{u})$ is bounded, there exists the convolution $\Gamma * \bar{u}$, therefore, in view of Eq. (46), the following relation holds in the domain G'

$$\bar{u} = -\Gamma * A\bar{u} \dots\dots\dots (60)$$

From Eq. (6) and the symmetry of matrix Γ , we get

$$\langle \Gamma * A\bar{u}, \varphi \rangle = \langle A\bar{u}(y), \eta(y) \int_{R^3} \Gamma(x-y)\varphi(x)dx \rangle \dots\dots\dots (61)$$

for an arbitrary $\varphi \in \mathcal{D}$. Suppose $G'' \subset G'$. Since we can select the support of the test function η which is defined similarly as for Eq. (55), we

assume such that

$$\text{car}(\eta) \cap \bar{G}'' = \emptyset \quad (62)$$

where $\emptyset$ stands for the empty set. At this stage, $\eta(y)\Gamma(x-y)\varphi(x)$ belongs to $\mathcal{D}(R^{3+3})$, since $\Gamma(x-y) \in C^\infty$ if $x \neq y$. From this, we can easily conclude the following:

$$u(x) = \langle A\bar{u}(y), -\eta(y)\Gamma(x-y) \rangle, \quad x \in G'' \quad (63)$$

Then, in view of the linearity of functional, we have

$$\frac{u(x + \Delta x_i e^i) - u(x)}{\Delta x_i} = \langle A\bar{u}(y), -\eta(y) \frac{\Gamma(x + \Delta x_i e^i - y) - \Gamma(x - y)}{\Delta x_i} \rangle, \quad (64)$$

where e^i stands for the unit vector in R^3 . Since $\eta(y)\Gamma(x-y) \in C^\infty$, which follows immediately from Eq. (62), we get the following result as a limit of $\Delta x_i \rightarrow 0$:

$$\frac{\partial u(x)}{\partial x_i} = \langle A\bar{u}(y), -\eta(y) \frac{\partial \Gamma(x-y)}{\partial x_i} \rangle \quad (65)$$

In general, repeating this process, we have

$$D^\alpha u(x) = \langle A\bar{u}(y), -\eta(y) D_x^\alpha \Gamma(x-y) \rangle, \quad x \in G'' \quad (66)$$

Then, we get $u \in C^\alpha(G'')$. Since α , G' , G'' are arbitrary, this result implies that $u \in C^\infty(G)$. (q. e. d.)

From the above statement it follows that, if the body force is absent, any generalized solution is also a classical solution.

(Corollary) If a sequence of classical solutions $\{u_n\}$ converges weakly to $u \in C(G)$, i. e., if the relation

$$\lim_{n \rightarrow \infty} \int_G u_n(x) \varphi(x) dx = \int_G u(x) \varphi(x) dx, \quad \varphi \in \mathcal{D}(G) \quad (67)$$

holds, a weak limit u is also a classical solution in G .

(Proof) Since $\{u_n\}$ is a sequence of classical solutions, it satisfies the following equation for any $\varphi \in \mathcal{D}(G)$

$$\int_G u_k(x) A\varphi(x) dx = 0 \quad (68)$$

Then, using Lebesgue's theorem, we get as a limit of $k \rightarrow \infty$,

$$\int_G u(x) A\varphi(x) dx = 0 \quad (69)$$

Namely $u(x)$ is a generalized solution. Hence we have immediately our statement from Theorem 3.

7. Generalized solution in the space $\mathcal{S}'$

Finally we consider some properties of a generalized solution in the space of slowly increasing distribution $\mathcal{S}'$.

(Proposition) If $u \in \mathcal{S}'$ satisfies the equation

$$Au(x) = 0 \quad (70)$$

in the whole domain R^3 , then $u(x)$ is a polynomial.

(Proof) Doing Fourier transform of Eq. (70) we have

$$A(i\xi)\mathfrak{F}[u] = -(\mu|\xi|^2\delta_{jk} + (\lambda + \mu)\xi_j\xi_k) \mathfrak{F}[u](\xi) = 0 \quad (71)$$

In view of the property

$$-det A(i\xi) = (\lambda + 2\mu)\mu^2|\xi|^6 \geq 0 \quad (72)$$

we see $\mathfrak{F}[u](\xi) = 0$ or $\text{car}(\mathfrak{F}[u]) = \{0\}$. If $\mathfrak{F}[u](\xi) = 0$, the above statement is obvious. When $\text{car}(\mathfrak{F}[u]) = \{0\}$, from Eq. (4), we have

$$\mathfrak{F}[u](\xi) = \sum_{|\alpha|=0}^m C_\alpha D^\alpha \delta(\xi) \quad (73)$$

where C_α denotes some constant vector. Therefore,

$$u(x) = \sum_{|\alpha|=0}^m C_\alpha \mathfrak{F}[D^\alpha \delta(\xi)](x) = \frac{1}{(2\pi)^3} \sum_{|\alpha|=0}^m (-1)^\alpha C_\alpha x^\alpha \quad (74)$$

This equality concludes our statement. (q. e. d.)

(Corollary) Let $u \in \mathcal{S}'$ be a generalized solution of Eq. (70). If $u(x)$ satisfies the inequality:

$$|u(x)| \leq c(1+|x|)^m, \quad x \in R^3, \quad m \geq 0 \quad (75)$$

then $u(x)$ is a polynomial with degree less than m .

This result means that the displacement solution for the problem of uniform extension of infinite body, which has a singularity of $O(|x|)$ at the infinity, can be explained as a generalized solution in the space of slowly increasing distribution. (Since this solution is not regular as was mentioned in Chapter 1, we can not explain it in the sense of the classical theory.)

(Corollary) If a solution of Eq. (70) is bounded in R^3 , it is an identical constant throughout the whole domain R^3 . In particular, any solution of Eq. (70) which vanishes at the infinity must be identically zero in R^3 .

This corollary is an extension of the classical Liouville theorem for harmonic functions to the case of elastostatics. Because of this property any solution of Eq. (12) is uniquely determined in the class of distribution if it tends to zero at the infinity. Indeed, suppose u_1 and u_2 are such solutions, and the difference $u = u_1 - u_2$ satisfies Eq. (70) and vanishes at the infinity. Hence, the above corollary implies $u(x) \equiv 0$ in R^3 .

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Discussion

H. ABÉ (Tohoku University):

(1) It is interesting to use the slowly increasing distribution when we consider an irregular solution of problem for infinite domain. I ask you about the properties of generalized solutions when the domain is bounded. I suppose that Eq. (13) has its meaning in the sense of distribution when f is not Hölder continuous. What physical phenomena do you consider about such f ? If f stands for a body force, the temperature or the gravitational fields may be illustrated. How do you consider about the discontinuity of stresses?

(2) In proposition in Chapter 3 there is a phrase such that " $f(x)=0$ except on a set of Lebesgue measure zero". This phrase may imply that we can accept the case $f(x)\neq 0$ on the set of measure zero. Is it true? I guess that this phrase has no meaning if f is assumed to be a stress (or its derivative) or a body force. Or is this proposition the general situation of mathematics?

H. HASEGAWA (Hiroshima Denki Institute of Technology):

(3) To study the theory of elasticity from the theoretical point of view, it is important to make clear the mathematical foundations of the situation. It may be desirable to explain your opinion by using a certain model schema, if possible.

(4) Can we explain all contradictory points by changing the notion of regular solutions to the notion of generalized solutions? Can we consider that the definitions described in this paper are necessary and sufficient form to define the case of inhomogeneous materials or dynamical problems, too?

(5) What relation exists between Eq. (47) and Eq. (50)? Please explain to me about the case $f=0$ in Eq. (50).

(6) Can we apply the method of convolution to dynamical problems, too?

M. SHIBAHARA (Kanagawa University):

(7) Please show me your opinion about the actual application of generalized solution to the problems of elasticity when a body force exists.

M. КАТАОКА (Meiji University):

(8) Many results of the generalized solutions for the equation of elasticity were obtained in this paper. I expect an evolution of these results in future. I ask you about the role of generalized solutions in the case of concentrated force and uniform extension of infinite body. I suppose that these problems will suggest the necessity of a new idea for the equations of elasticity, which were constructed under the assumption that Lamé's coefficients are constant throughout the whole body. I suppose these problems may be settled by considering that Lamé's coefficients decrease in the case of concentrated force, while for the infinite body they vanish in the exterior of a certain bounded domain.

Y. ŌHASHI (Nagoya University):

(9) In the analysis of practical problems it seems unsuitable to accept the classical approximation of perfect elasticity under the assumption that the force affects instantaneously throughout the whole domain, since the theory of rheology has been developed in the recent years. The material should be understood as a visco-elasto-plastic body. When the finite force is affected instantaneously, the elastic part of the solution of visco-elasticity has been defined by a super function concerning the time^(*). The force which is applied instantaneously does not result in a momentary uniform deformation throughout the whole domain, but propagates in waves, and the viscosity is expected to influence the phenomena. Therefore, it seems reasonable to introduce the viscosity by considering a super function concerning the time variable rather than the position, as in general.

Author's closure

(1) We mentioned that Eq. (13) is a classical solution of Eq. (12) if f is Hölder continuous, in order to indicate a sufficient condition of f for which the convolution (43) should be a classical solution. The following are examples that the convolution $\Gamma * f$ is not a classical but a generalized solution. (a) If f denotes a concentrated force, the convolution $\Gamma * f$ exists since the support of f consists of a set of single points. This solution is not a classical but a generalized

solution. (b) Let $p(x_1)$ be a function which is equal to the distance from the origin to the nearest integer to $x_1 \in R$ and define the series $q(x_1)$ such that

$$q(x_1) = \sum_{n=0}^{\infty} 10^{-n} p(10^n x_1)$$

Then $q(x_1)$ is continuous everywhere but it has no derivatives^(*2). Therefore if we construct the convolution (43) by assuming $f = (q, 0, 0)$, it represents a generalized solution since f is continuous.

For the case of temperature field in your question, the solution of the problem with point source may be a generalized solution. We explained in this paper about the results concerning the statical problem, but the most interesting results will appear in dynamical cases, particularly, in impact problems.

Moreover, Theorem 3 implies that, if a body force is absent, the stresses in the bounded domain G are continuous, since any generalized solution in G is a classical solution. If a body force is continuous, the stresses are continuous also.

(2) This proposition is a general situation of mathematics, which is known as Du Bois Reymond's theorem.

(3) I can not understand clearly the meaning of model schema. However the difference of classical and generalized solutions may be roughly described as follows: The classical solution requires determination of the values at every point. But it seems unnatural, since the values of stress or displacement can be regarded as an average in the neighborhood of a point. On other hand, in view of Definitions (14) and (15), the generalized solutions require that they satisfy the equation as an average of the domain. Then the concept of generalized solutions may be more natural than the classical one.

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(4) It is not certain whether we can explain all contradictory points or not. The situation for inhomogeneous materials or dynamical problems will be similar essentially.

(5) Equation (47) is equivalent exactly to Eq. (50) because Eq. (50) implies obviously Eq. (47) when using the notations B_0 and B defined by Eq. (51). When $f=0$ the functions $B_0, B \in C(G)$ should denote generalized solution of Laplace equation. However, any generalized solution of Laplace equation belongs to the class $C^\infty(G)$. Therefore the result in this case coincides with the classical Neuber-Papkovich formula.

(6) We can also apply the method of convolution to the dynamical problems. At this stage we must use the convolution defined in the 4-dimensional Cartesian product $G \otimes (0, T)$, i. e., the Cartesian product of $G \subset R^3$ and a certain time interval $(0, T)$. In particular, the method of convolution will be useful when we consider the Cauchy problems of elastic wave equation.

(7) Though we discussed mainly in this article the local properties of generalized solution, we are interested in the problems with certain boundary conditions in practice. However it is well known that the boundary value problems of elasticity without body force under certain non-homogeneous boundary condition q , for instance, such as

$$Au(x) = 0, x \in G; u(x) = q, x \in \partial G,$$

are equivalent to a problem with body force f under homogeneous boundary condition as follows:

$$Au(x) = -f, x \in G; u(x) = 0, x \in \partial G.$$

Then it will be useful to consider the basic properties of elasticity equation with body force for the purpose of studying the problems of practical interest.

(8) Thank you for your interesting opinion.

(9) Thanks for your comments. Now, we are going to study the problems with time variable in connection with elastic wave propagation.