A Study on Anomalous Turbulent Flows of Non-Newtonian Fluids*
(3rd Report, Experiment and Analysis in Transitional Region)

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Transitional region of pipe flows of dilute polymer solutions is investigated. Experiment is made with HEC water solutions and the following results are obtained. Anomaly of the resistance coefficient is very small in the flow region before transition, though it is comparatively large for PEO solutions as reported in the previous paper. In this flow region a kind of fluctuations of static pressure is recognized and once the fluctuations occur, the appearance of issuing jets shows some anomalous winding effects.

An approximate analysis is made on the anomaly of the resistance coefficient in the flow region before transition. The results show that the resistance coefficient becomes larger than that of Poiseuille flow and that if Reynolds numbers and the fluctuations are constant, the rate of increase of the resistance coefficient enlarges as Weissenberg number increases. Comparison between analytical results and experimental data shows an approximate agreement.

1. Introduction

Drag reduction effect of dilute polymer solutions in turbulent flow regions has been studied by many investigators experimentally and theoretically. On the other hand, the transition region before the fully turbulent flow has not been examined widely(). Recently the authors made an experiment on the transition region of pipe flows with dilute solutions of polyethylene oxide (PEO) and clarified the following points in the previous paper(): (1) Before transition there is a flow region where the resistance coefficient λ departs gradually upward from the line λ\(=64/R_e\), and this region begins at a lower Reynolds number than the critical one of Newtonian fluids(\(=2300\)); (2) there appear wavy surfaces or windings on the jets in this flow region, and changes of the jet appearance correspond to those of the resistance coefficient approximately; (3) it is made clear that the wavy surfaces or windings occur by fluctuations in a pipe flow and these fluctuations are different from those in transition of water flow.

In this paper the authors will report the experimental results obtained using the hydroxyethyl cellulose (HEC) water solutions whose properties rather differ from those of PEO solutions stated in the previous paper, and furthermore make a theoretical analysis of the flow of transition region. Transition region (which contains both the preparing region and the transition zone, the term of transition zone is used here to mean the zone where the resistance coefficient increases with the Reynolds number. See Fig.1.) has not been studied precisely and as far as the authors know, a theoretical analysis has not been made so far. Hence, an experimental or theoretical work concerning it may be important and meaningful in the present stage.

Fig.1 Flow regions

2. Experimental apparatus and polymer solutions

Since the experimental apparatus is the same one that was mentioned in the previous paper, its details are omitted here, but the used pipe is a straight round one made of stainless steel with
inner diameter of 0.256 cm. In the experiment, pressure gradients and flow rates are measured, the pressure fluctuations at the pipe wall being recorded, and the issuing jets are sometimes photographed.

Fluids are 0.05-0.2% HEC(hydroxyethyl cellulose made by Union Carbide Company) water solutions. The city water is used as solvent, because HEC is chemically stable against the iron ion etc. in the water. They are also fairly stable against the high shear and, even when the same solution is used repeatedly, their properties are scarcely changed.

3. Experimental results

Figure 2 shows a graph of resistance coefficient vs. Reynolds number for the pipe flow of various HEC solutions. HEC solutions of these concentrations give the non-Newtonian viscosity and so the ordinate takes the generalized Reynolds number

$$R_e^* = \rho V^2 a(2a)/m \left( \frac{3n+1}{4n} \right)^\frac{1}{n}$$

based on the power law model

$$\tau_\tau = -m \left( \frac{\partial u_s}{\partial r} \right)^{n-1},$$

where $\rho$ is the solution density, $V$ is the mean flow velocity, $a$ is the pipe radius, and $m,n$ are material constants of the fluids. For comparison Fig.3 gives the resistance coefficient of PEO solutions which were reported in the previous paper. According to Fig.2 the data of the HEC solutions exist on the straight line of laminar flow just up to the transition and do not show the anomaly that the resistance coefficient departs gradually upward from the line $\lambda = 64/R_e$ as seen in the preparing region of the PEO solutions. In this figure we can also see the transition to turbulent flow delay as the concentration increases.

Figure 4 gives the pressure fluctuation of the HEC solutions in the flow region before transition. We can see the fluctuation is negligibly small in the flow region of low velocity speed (Fig.4(a)), but it becomes apparent at a Reynolds number ($R_e^* = 1770$) which is less than the critical one for water ($2300$) (Fig.4(b)), and its amplitude gradually increases till the transition occurs (Fig.4(c),(d)). This tendency is

![Fig.2 Resistance coefficient ($\lambda$) vs. generalized Reynolds number ($R_e^*$) for HEC water solutions](image)

![Fig.3 Resistance coefficient ($\lambda$) vs. Reynolds number ($R_e$) for PEO water solutions](image)

![Fig.4 Pressure fluctuations for 0.2% HEC solution](image)
similar to the case of the PEO solutions. Hence for
HEC solutions, even if the anomaly of the resistance
coefficient is not recog-
nized, velocity fluctu-
ations may exist in a pipe
flow as in the case of PEO
solutions. Since for
water the pressure fluctua-
tions are hardly recog-
nized and hence the
velocity fluctuations are
thought to be very small,
these fluctuations for HEC
and PEO solutions may be
produced by the amplifi-
cation of small fluctua-
tions by the elastic
properties of fluid.
The effect of velocity
fluctuations on the
resistance coefficient is
discussed in the following
section.

Figure 5 shows a
photograph of the jet
appearances of 0.2\% HEC
solutions. Matching the
jet appearances to the
fluctuations of static
pressure, we can guess
that when pressure
fluctuations are not
observed in the region of
low velocity, the anomaly
does not appear in jets
(Fig.5(a)), but once the
pressure fluctuations begin to occur, the jet
also shows the anomaly
such as winding or crook-
ing (Fig.5(b)-(e)).

Further detailed obser-
vation shows that the
irregularity of the jets
enlarges beyond $R*=2500$
(Fig.5(e)) and that when
the flow becomes tur-
bulent after transition, the
jet appearance also be-
comes very similar to that
of water (See Fig.4(f) and
Ref.1(1)). These phenomena
are almost the same in
both cases of PEO and HEC
solutions, but in detail
they are different. That
is, as soon as the pres-
sure begins to fluctuate,
the jet of HEC solution
shows windings, but the
jet of PEO solution yields
only waves like sine curve
on its surface in the
first stage of pressure
fluctuating region and it is at Reynolds
numbers above 2500 that the jet gives such
winding phenomena on the whole as in the
jet of HEC solutions. Moreover, the jet
length between the pipe exit and the point
where the jet breaks into droplets becomes
short once the jet anomaly occurs for
dilute PEO solutions, but such phenomenon
of the jet length does not appear for HEC
solutions. The cause of the differences
between PEO and HEC solutions in jets is
not clear so far, but the HEC solutions

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**Fig.5** Photographs of jet for 0.2\% HEC solution
have about 4 times as high viscosity as the PEO solutions and this fact may cause the differences. Thus, indeed, detailed examination on the jet anomalies of PEO and HBO solutions gives the differences between them, but they agree in the fact that once the pressure begins to fluctuate, the jet gives some anomaly in appearance. Therefore, observing the appearances of issuing jet from the pipe exit, one can approximately guess in what condition the pipe flow is, laminar, turbulent or whether some fluctuations exist in the pipe flow or not.

4. Analysis

Considering that some important problems are not settled yet on the analysis of stability problem of the Poiseuille flow of Newtonian fluids, we shall meet many difficulties in the exact theoretical analysis of the transition region of pipe flow for viscoelastic fluids which display very complicated phenomena even in simple flow fields. Hence, hereafter, standing on some physical viewpoints and experimental results, we will make an approximate theoretical analysis of the anomaly of the resistance coefficient of pipe flow before transition for the viscoelastic fluids. As stated in the previous paper, we may roughly consider the preparing region before transition as follows: namely, in the pipe flow before transition disturbances may be induced by any causes and affected by the viscoelasticity the fluid possess, and in this case the viscosous behavior may work to suppress disturbances, whereas the elastic one will play a role to increase disturbances with a frequency, say, like resonance for an elastic substance (of course, it does not increase infinitely but balances at a certain value), and further these disturbances will interact with the main flow and finally increase the value of resistance coefficient.

Now we analyze the problem based on the above mentioned physical viewpoints. At first, the following assumptions are made.

(1) The fluid treated in this analysis possesses a Newtonian viscosity, that is, the viscosity function is constant regardless of shear rates.

(2) The fluid fits the Denny model.

(3) The disturbances added to the main flow are axiymmetric.

The Denny model showing the Newtonian viscosity is given by the following equation.

\[ \tau_{ij} + \tau_{ij} \left( \frac{1}{2} \right) \left( s^{(-3/2)} \right) \frac{\partial \tau_{ij}}{\partial t} = 2 \mu \varepsilon_{ij} \] ........................(1)

where \( \tau_{ij} \) are the components of deviatoric stress tensor and rate of strain tensor respectively, and besides we have \( \varepsilon_{ij} = g^{ik} g^{jl} e_{kl} \) ........................................(2)

\[ e_{kl} = \frac{1}{2} \left( e_{k,l} + e_{l,k} \right) \] .............................................(3)

Also, \( \partial / \partial t \) shows the convected derivative, and \( \Pi \) is an invariant of \( \varepsilon_{ij} \) and they can be expressed as

\[ \frac{\partial \tau_{ij}}{\partial t} = \frac{\partial \tau_{ij}}{\partial t} + \rho \tau_{ij,k} - \nu_{ij,k} - \nu_{ij,k} - \nu_{ij,k} \] ....................................(4)

\[ \Pi = 4 \tau_{ij} \sigma_{ij} \] .............................................(5)

where \( \sigma_{ij} \) is the metric tensor and the comma, as seen in \( n_{ij} \), etc. shows the covariant derivative. Furthermore, \( \mu, \tau, s \) are material constants, especially \( \mu \) having the same dimension as the viscosity of Newtonian fluids, \( \tau \) having \( \tau^{(-1)} \) and \( \sigma \) having no dimension. In the case \( s = 2 \), the above equation agrees with the Maxwell model. Since Eq.(1) contains the convected derivative of the stress tensor \( \tau_{ij} \), it is rather difficult and complicated mathematically to use it simultaneously with the equations of motion to analyze a problem of flow. Hence, in this paper, the next procedure is adopted to obtain the stress tensor as an explicit function of \( \varepsilon_{ij} \). That is, since the numerical value of \( \tau \) is ordi-

narily less than unity, it is expanded in a series of \( \tau \)

\[ \tau^{ij} = \tau_{ij} + \tau_{ij} + \tau_{ij} + \tau_{ij} + \ldots \] ........................................(6)

Substituting Eq.(6) into Eq.(1) to get

\[ (\tau_{ij} + \tau_{ij} + \tau_{ij} + \ldots) + \tau_{ij} \left( \frac{1}{2} \right) \left( s^{(-3/2)} \right) \frac{\partial \tau_{ij}}{\partial t} \times (\tau_{ij} + \tau_{ij} + \tau_{ij} + \ldots) = 2 \mu \varepsilon_{ij} \] ......................................(7)

and equating the like powers of \( \tau \) to decide \( \tau_{ij} \), \( \tau_{ij} \), \( \tau_{ij} \)

we obtain from Eq.(6)

\[ \tau_{ij} = 2 \mu \varepsilon_{ij} - 2 \mu \left( \frac{1}{2} \right) \left( s^{(-3/2)} \right) \frac{\partial \tau_{ij}}{\partial t} + 2 \tau \mu \left( \frac{1}{2} \right) \left( s^{(-3/2)} \right) \frac{\partial \tau_{ij}}{\partial t} + \left( \frac{1}{2} \right) \left( s^{(-3/2)} \right) \frac{\partial \tau_{ij}}{\partial t} \] ..........................(8)

In this analysis we will adopt Eq.(8) as a rheological equation of state. As the flow is assumed to be axiymmetric, that is, \( n_{ij} = 0 \), \( \partial / \partial \theta = 0 \), the physical components of the convected derivatives in cylindrical coordinates become
\[
\frac{\partial e_{rr}}{\partial t} + v_r \frac{\partial e_{rr}}{\partial r} + v_z \frac{\partial e_{rr}}{\partial z} - 2 \left( e_{rr} \frac{\partial v_r}{\partial r} + e_{rz} \frac{\partial v_z}{\partial z} \right) = \frac{\partial e_{rr}}{\partial t} + r^2 v_r \frac{\partial}{\partial r} \left( e_{rr} \right) + v_z \frac{\partial e_{rr}}{\partial z} \]

\[
\frac{\partial e_{zz}}{\partial t} + v_r \frac{\partial e_{zz}}{\partial r} + v_z \frac{\partial e_{zz}}{\partial z} - 2 \left( e_{rr} \frac{\partial v_r}{\partial r} + e_{rz} \frac{\partial v_z}{\partial z} \right) = \frac{\partial e_{zz}}{\partial t} + r^2 v_r \frac{\partial}{\partial r} \left( e_{zz} \right) + v_z \frac{\partial e_{zz}}{\partial z} \]

\[
\frac{\partial e_{tr}}{\partial t} + v_r \frac{\partial e_{tr}}{\partial r} + v_z \frac{\partial e_{tr}}{\partial z} = \frac{\partial e_{tr}}{\partial t} + \tau_{rr} \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rr}}{\partial z} \]

\[
\frac{\partial e_{ts}}{\partial t} + v_r \frac{\partial e_{ts}}{\partial r} + v_z \frac{\partial e_{ts}}{\partial z} = \frac{\partial e_{ts}}{\partial t} + \tau_{rr} \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rr}}{\partial z} \]

\[
\frac{\partial e_{tr}}{\partial t} = 0, \quad \frac{\partial e_{ts}}{\partial t} = 0
\]

Similarly, \( \frac{\partial e^{ij}}{\partial t} \) is obtained by replacing \( e^{ij} \) by \( \frac{\partial e^{ij}}{\partial t} \) in Eq.(9). Also, the rate of strain tensor \( e^{ij} \) is expressed as

\[
e_{rr} = \frac{\partial v_r}{\partial r}, \quad e_{zz} = \frac{v_z}{r}, \quad e_{rz} = \frac{\partial v_z}{\partial z}, \quad e_{rs} = \frac{1}{2} \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right), \quad e_{sq} = e_{sz} = 0
\]

Furthermore, the equations of motion in the \( r \) and \( z \) directions are written as

\[
\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right) = - \frac{\partial P}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_{rr} \right) - \frac{\tau_{zz}}{r} \frac{\partial}{\partial z} \frac{\partial v_r}{\partial z} \]

\[
\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial P}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_{rr} \right) + \frac{\tau_{zz}}{r} \frac{\partial}{\partial z} \frac{\partial v_z}{\partial z} \]

and the equation of continuity is given by

\[
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0
\]

Now, assuming that the velocity fluctuations \( v'_r, v'_z, v'_r = 0 \) are added to the mean velocity profile \( V_r = \bar{V}_r \), we have

\[
v_r = \bar{V}_r + v'_r, \quad v_z = \bar{V}_z, \quad v'_z = 0
\]

Substitution into Eqs.(11), (12), (13) (where the prime is omitted for simplification) then yields

\[
\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} + V_z \frac{\partial v_r}{\partial z} \right) = - \frac{\partial P}{\partial r} + \frac{\tau_{rr}}{r} + \frac{\tau_{rr}}{r} + \frac{\partial \tau_{rr}}{\partial z} \]

\[
\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} + V_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial P}{\partial z} + \frac{\tau_{rr}}{r} + \frac{\tau_{rr}}{r} + \frac{\partial \tau_{rr}}{\partial r} \]

\[
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0
\]

In this case, we have the following equations,

\[
e_{rr} = \frac{\partial v_r}{\partial r}, \quad e_{zz} = \frac{v_z}{r}, \quad e_{rz} = \frac{\partial v_z}{\partial z}, \quad e_{rs} = \frac{1}{2} \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right), \quad e_{sq} = 0, \quad e_{sz} = 0
\]

and then Eq.(9) becomes

\[
\frac{\partial e_{rr}}{\partial t} = \frac{\partial^2 v_r}{\partial r^2} + v_r \frac{\partial^2 v_r}{\partial r^2} + \left( v_z + V_z \right) \frac{\partial^2 v_r}{\partial z^2} - \left( \frac{\partial v_r}{\partial r} \right)^2 + \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} + \frac{\partial V_z}{\partial r} \right) \frac{\partial v_r}{\partial z}
\]

\[
\frac{\partial e_{zz}}{\partial t} = \frac{\partial v_z}{\partial r} + v_r \frac{\partial v_z}{\partial r} + \left( v_z + V_z \right) \frac{\partial^2 v_z}{\partial r^2} - \left( \frac{\partial v_z}{\partial r} + \frac{\partial V_z}{\partial r} \right) \frac{\partial v_z}{\partial r}
\]

\[
\frac{\partial e_{tr}}{\partial t} = \frac{\partial v_r}{\partial r} + v_r \frac{\partial v_z}{\partial r} + \left( v_z + V_z \right) \frac{\partial^2 v_z}{\partial r^2} - \left( \frac{\partial v_z}{\partial r} + \frac{\partial V_z}{\partial r} \right) \frac{\partial v_z}{\partial r}
\]

\[
\frac{\partial e_{ts}}{\partial t} = \frac{\partial v_z}{\partial r} + v_r \frac{\partial v_z}{\partial r} + \left( v_z + V_z \right) \frac{\partial^2 v_z}{\partial r^2} - \left( \frac{\partial v_z}{\partial r} + \frac{\partial V_z}{\partial r} \right) \frac{\partial v_z}{\partial r}
\]

and the physical components of \( \frac{\partial e^{ij}}{\partial t} \) are found similarly. On the other hand, the invariant of the strain rate tensor gives

\[
\frac{1}{2} \Pi = 2 \left( \frac{\partial v_r}{\partial r} \right)^2 + 2 \left( \frac{v_z}{r} \right)^2 + \frac{1}{2} \left( \frac{\partial v_z}{\partial z} + \frac{\partial (V_z + v'_z)}{\partial r} \right)^2
\]
Considering \( v_r, v_r \ll V_r \), we find
\[
\frac{1}{2} \Pi \approx \left( \frac{\partial V_r}{\partial r} \right)^2 \left[ 1 + 2\left( \frac{\partial v_r}{\partial z} + \frac{\partial v_r}{\partial r} \right) \right] \frac{\partial V_r}{\partial r}
\]
and Eq. (20) is expressed as
\[
\left( \frac{1}{2} \Pi \right)^{(s-1)/2} \frac{\partial V_r}{\partial r} \left[ 1 + \frac{2\left( \frac{\partial v_r}{\partial z} + \frac{\partial v_r}{\partial r} \right)}{\frac{\partial V_r}{\partial r}} \right] \left( \frac{1}{2} \Pi \right)^{(s-1)/2} \frac{\partial V_r}{\partial r} \left[ \frac{\partial V_r}{\partial r} + (s-2) \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_r}{\partial r} \right) \right]
\]
Thus, the physical components of the deviatoric stress tensor in cylindrical coordinates are given as
\[
\tau_{rr} = 2\mu \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_r}{\partial r} \right) \left( \frac{1}{2} \Pi \right)^{(s-1)/2} \frac{\partial V_r}{\partial r} + 2\tau \left[ \frac{1}{2} \Pi \right] \left( \frac{\partial V_r}{\partial r} \right) \left( \frac{\partial V_r}{\partial r} \right)
\]
\[
\tau_{zz} = 2\mu \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_r}{\partial r} \right) \left( \frac{1}{2} \Pi \right)^{(s-1)/2} \frac{\partial V_r}{\partial r} + 2\tau \left[ \frac{1}{2} \Pi \right] \left( \frac{\partial V_r}{\partial r} \right) \left( \frac{\partial V_r}{\partial r} \right)
\]
\[
\tau_{zr} = \mu \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_r}{\partial r} \right) \left( \frac{1}{2} \Pi \right)^{(s-1)/2} \frac{\partial V_r}{\partial r} + 2\tau \left[ \frac{1}{2} \Pi \right] \left( \frac{\partial V_r}{\partial r} \right) \left( \frac{\partial V_r}{\partial r} \right)
\]
Now, assuming a type of \( v_r, v_r \), we can in principle discuss the stability problem of the pipe flow of viscoelastic fluid by Eqs. (15), (16), (19), (23), but in reality this procedure may be difficult mathematically. Then, to explain the anomaly of the resistance coefficient in the preparing region, we treat the problem as follows. The resistance coefficient \( \lambda \) is defined as
\[
\lambda = -\frac{\partial P_m}{\partial z} \left( \frac{1}{2} \rho V^2 \right) \frac{1}{2a}
\]
where \( P_m \) is the measured stress at the wall, \( V \) is the mean uniform velocity and \( a \) is the pipe radius. For Newtonian fluids \( P_m \) is equal to the isotropic pressure \( P \) but for viscoelastic fluids it is given as
\[
P_m = \left[ P - \tau_{rr} \right]
\]
and Eq. (24) becomes
\[
\lambda = \left[ \frac{\partial P}{\partial z} + \frac{\partial \tau_{rr}}{\partial z} \right] \left( \frac{1}{2} \rho V^2 \right) \frac{1}{2a}
\]
where \( \lambda \) expresses the given resistance coefficient. Since at the wall \( v_z = 0, v_r = 0, V_z = 0 \), Eq. (16) yields
\[
\left[ \frac{\partial P}{\partial z} \right] = \left[ -\frac{\partial \tau_{rr}}{\partial r} - \frac{\tau_{rr}}{r} - \frac{\partial \tau_{rr}}{\partial z} \right]
\]
Hence, the resistance coefficient \( \lambda \) is given by the following expression.
\[
\lambda = \left[ \frac{\partial \tau_{rr}}{\partial r} - \frac{\tau_{rr}}{r} + \frac{\partial \tau_{rr}}{\partial z} \right] \left( \frac{1}{2} \rho V^2 \right) \frac{1}{2a}
\]
Now, we assume the stream function of the fluctuations as
\[
\psi = \sum \phi_i(r) \cos(\alpha_i z - \omega t)
\]
where the term \( \phi_i(r) \) giving the damping or amplification is omitted for simplification. If we discuss the laminar stability problem in which we must consider whether some given fluctuations are damped or amplified, it plays an important role. In this analysis, however, we are considering the effect that the existing fluctuations give on the timely averaged pressure loss, and so we do not take into consideration the effect of their amplification or damping with time. Thus, we get
\[
v_z = -\frac{1}{r} \frac{\partial \psi}{\partial r} = -\frac{1}{r} \sum \phi'_i(r) \cos(\alpha_i z - \omega t)
\]
\[
v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} = -\frac{1}{r} \sum \alpha_i \phi_i(r) \sin(\alpha_i z - \omega t)
\]
and from the boundary conditions at \( r = a \),
\[
v_z = 0, v_r = 0, \text{ we have}
\]
\[
\phi(a) = 0, \phi'(a) = 0
\]
and obtain
\[
\left[ \frac{\partial \tau_{rr}}{\partial r} \right] = \left[ \frac{\partial \tau_{rr}}{\partial z} \right] = 0
\]
Considering Eq. (32), we calculate \( \left[ \partial \tau_{rr}/\partial r \right]_s, \left[ \tau_{rr}/r \right]_s, \) etc. by Eq. (23) and obtain
\[
\frac{[\tau_{rr}]}{[r]}_s = \frac{1}{a} \left[ \frac{\mu (V_z + v_z)}{V_z + v_z} - 2\tau \frac{[V_z + v_z]}{[r]} \right]_s + \frac{2}{2} \frac{(v_z + v_z)}{[r]}_s
\]
\[
\frac{\partial \tau_{rr}}{\partial r} = \mu (V_{r,r} + v_{r,r}) - 2 \pi \mu \left( \frac{1}{2} v_{r,r} + \frac{1}{2} v_{r,r} - \frac{3}{2} (v_{r,r} + V_{r,r} + (s-2)v_{r,r}) \right) - 2 \pi \mu \left( \frac{1}{2} v_{r,r} + \frac{1}{2} v_{r,r} - \frac{3}{2} (v_{r,r} + V_{r,r}) \right) - 2 \pi \mu \left( \frac{1}{2} v_{r,r} + \frac{1}{2} v_{r,r} - \frac{3}{2} (v_{r,r} + V_{r,r}) \right) - 2 \pi \mu \left( \frac{1}{2} v_{r,r} + \frac{1}{2} v_{r,r} - \frac{3}{2} (v_{r,r} + V_{r,r}) \right)
\]

In the above equations the comma means the partial derivative with respect to the independent variable followed by it. That is, \(\partial /\partial r = v_{r,r}, \partial V_{r} /\partial r = v_{r,r}, \) etc.

In Eqs. (35), (34), (33) the functional form of the mean velocity profile \(V_{r} = V_{r}(r)\) is still unknown. The phenomenon is, in reality, that the fluctuations will interact with the main flow and the whole flow may become so complicated that it is very difficult simultaneously to solve both the fluctuations and the main flow velocity as unknowns. Usually in this analysis we need not the velocity profile itself but the values of \(\partial V_{r}/\partial r\) and \(\partial^{2} V_{r}/\partial r^{2}\), so we assume that the fluctuations give only negligibly small effect on \(\partial V_{r}/\partial r\), etc. and that the values of these quantities are the same as the Poiseuille flow. That is, we assume

\[
\frac{\partial V_{r}}{\partial r} = \frac{4V}{a}, \quad \frac{\partial^{2} V_{r}}{\partial r^{2}} = \frac{4V}{a^{2}} \quad \cdots (36)
\]

Among many fluctuations with various phases and frequencies, those which are amplified by the viscoelastic force and give some effect on the resistance coefficient are now assumed to be represented by a fluctuation to simplify the problem. Namely Eq. (30) is replaced by

\[
\begin{align*}
v_{r} &= -\frac{1}{r} \psi(r) \cos(\alpha z - \omega t) \\
v_{r} &= -\frac{1}{r} \phi(r) \sin(\alpha z - \omega t)
\end{align*}
\]

If the following expressions are utilized,

\[
\cos(\alpha z - \omega t) = \sin(\alpha z - \omega t) = \frac{1}{2} \quad \cdots (37)
\]

and the time average of \(\cos(\alpha z - \omega t) \cos(\alpha z - \omega t)\) \(\times \sin(\alpha z - \omega t) , \cos(\alpha z - \omega t)\) is taken to be zero, then substituting Eqs. (36), (30)' into Eqs. (35), (34), (33) and taking the

\[
\begin{align*}
\frac{\partial^{2} \tau_{rr}}{\partial r^{2}} &= -\mu \frac{4V}{a^{2}} - \pi \mu \frac{2}{a} \\
\times \frac{\phi''''}{a} + \frac{2\phi'''}{a} + \frac{2\phi''}{a} + \frac{\phi'}{a} \quad \cdots (38)
\end{align*}
\]

where \(\phi''', \phi''''\) represent \([\partial^{3} \phi(r)/\partial r^{3}]_{s}\), \([\partial^{2} \phi(r)/\partial r^{2}]_{s}\) respectively. Therefore from Eqs. (28) and (30) we have

\[
\lambda = 64 + \pi \mu \frac{2V^{2}}{R_{s}} \frac{1}{12} + \frac{1}{12} \frac{1}{12} \frac{1}{12} \frac{1}{12}
\]

For the Newtonian fluid \(\tau = 0\), and the above equation become \(\lambda = 64/R_{s}\), which shows the one for the Poiseuille flow. For the viscoelastic fluids the second term on the right hand side occurs as the increment of the resistance coefficient \(\lambda\). Now, to advance the analysis further, we must know \(\alpha, \omega, \mu\),

\[
\begin{align*}
\frac{\partial \tau_{rr}(s)}{\partial r} &= -\mu \frac{4V}{a^{2}} - \pi \mu \frac{2}{a} \\
\times \frac{\phi''''}{a} + \frac{2\phi'''}{a} + \frac{2\phi''}{a} + \frac{\phi'}{a} \quad \cdots (38)
\end{align*}
\]
\( \psi''', \psi''' \) as the functional form of \( V \), but this is very difficult as stated before. Hence from now on we make some assumptions based on such physical viewpoints or experimental results as mentioned in the beginning of this section. At first we assume the wave length of fluctuation \( 2\pi/\alpha \) is nearly the same as the pipe diameter. Also on the analogy of the resonance of elastic substances, the frequency of fluctuation \( \omega \) is assumed to become high as the elastic force of the fluid increases. Since the elastic force of the fluid usually enlarges with the shear rate, it is considered that \( \omega \) is approximately proportional to the shear rate \( 4V/a \). That is,

\[
\alpha = \frac{C_1}{a}, \quad \omega = C_2 \frac{4V}{a}
\]

where \( C_1, C_2 \) are constants. Also, we assume that the wall shear rate of the fluctuation \( \phi'''/a \) does not very much differ from \( \phi'' \) which has the same dimension. Namely, we assume

\[
\phi'''' \approx \phi''/a
\]

According to Eqs. (40) and (41), Eq. (59) becomes

\[
\lambda = \frac{64}{R_e} + \frac{\tau^2}{\mu} C_1 C_2 \frac{(13-8s)}{2} \\
+ C_2 (9-3s)(2-s) \left( \frac{\phi'''}{V} \right) \left( \frac{4V}{a} \right)^{1-s} \frac{1}{\rho \alpha V} \\
= \frac{64}{R_e} + C \left( \frac{\phi'''}{V} \right)^{1-s} N_r \frac{N_r}{R_e}
\]

where

\[
C = \frac{1}{2} \left[ \frac{13-8s}{2} C_1 C_2 + (9-3s)(2-s) C_2 \right]
\]

is a function of the material constant \( s \). The value of \( C \) might possibly become minus in the range \( 1.62<s<2 \), but according to the data in Fig. 6 of the first normal stress differences taken so far, \( s \) is less than 1.5 in most cases of the range of high shear rates. Hence we may think \( C>0 \) in the usual cases and that \( \lambda \) is larger than \( \lambda = 64/R_e \).

In Eq. (42) \( R_e = 2\alpha V/\mu \) is the Reynolds number, \( N_r = 2\alpha (4V/a)^{1-s} \) represents the ratio of elastic force to viscous one and is referred to as the Weissenberg number. The above equation (42) shows that the increment of \( \lambda \) is larger as \( N_r \) increases, i.e., \( R_e \) and \( \phi'''/V \) are unaltered in value. Rearrangement of Eq. (42) gives

\[
\lambda = \frac{64}{R_e} + 4C \left( \frac{2\alpha}{a^2} \right)^{1-s} \phi'''' \left( \frac{V}{R_e} \right)^{2s-3} 
\]

Here it is thought that the dimensionless velocity of fluctuation \( \phi'''/V \) is inversely proportional to the relaxation time of the solution and becomes larger as the viscosity of the solution is lower or the diameter is larger, so we may give

\[
\phi'''' \propto \frac{1}{\tau} \left( \frac{a^2}{2\alpha} \right)^{1-s}
\]

Then Eq. (43) becomes

\[
\lambda = \frac{64}{R_e} + K R_e^{2s-3}
\]

where \( K \) is a function of material constant \( s \) and should be decided by experiment.

5. Comparison between experimental data and theoretical results

For comparison between experimental data and theoretical results the material constants of the rheological equation of state must be known beforehand, and for this purpose both the shear stress and the first normal stress difference must be measured in a certain range of shear rates. The measurement of shear stress is comparatively easy but there are a few unsettled problems in the measurement of the normal stress differences especially for the dilute polymer solutions where the normal stress differences become small.
stresses have small values. Up to date the most widely used instrumentation to measure the normal stress differences of dilute polymer solutions is to utilise the thrust of jet, and the authors measured the first normal stress differences by this method. See reference (4) for this apparatus and the experimental procedure. These experimental data are presented in Figs.6 and 7.

By Eq. (42), it was shown that when both \( R_e \) and \( \phi / V \) are constant in the two flows, the increment of \( \lambda \) is large as \( N \) increases. To ascertain this, a graph of \( N \) vs. \( 4/V \) is drawn both for 5 ppm PEO and 0.2% HEC solution in Fig.8.

\[ \lambda = \frac{64}{R_e} + K' \]  

(47)

In Fig.9 the line of this equation is also shown. We see both equations are almost the same up to \( R_e = 2 \times 10^4 \), but in the range of larger Reynolds numbers Eq. (46) is better fit to the data as seen in Fig.10. Figure 10 gives the data with various concentrations and pipe diameters. The normal stress differences are not measured for all the solutions presented and so all values of \( s \) are not known but according to Oliver (8) the 100 ppm PEO solution has the value of \( s \) equal to 1.41, which is not different from that of 5 ppm PEO solution. Like this, the value of \( s \) may not very largely as the concentration changes. Therefore, we may think Eq. (46) to be comparatively universal for various kinds of PEO solutions and pipe diameters.

![Fig.8 Weissenberg number](image)

According to this figure \( N \), of HEC solution is about a quarter of that of PEO solution and hence referring to Eq. (42), we can conjecture that the increment of \( \lambda \) in HEC solution is about 1/16 times as large as in PEO solution. Of course the comparison is not strict because the values of \( s \) are different in both solutions and moreover the HEC solution exhibits the non-Newtonian viscosity, but the values of \( s \) do not differ so much in both solutions (\( s = 1.33 \) for 0.2% HEC solution and \( s = 1.41 \) for 5 ppm PEO solution) and the non-Newtonian viscosity is not pronounced (\( \pi = 0.85 \)), and so the approximate comparison between the two fluids may be possible.

Because of the fact mentioned above, it is thought that the increment of \( \lambda \) for HEC solutions is very little, even if some fluctuations are recognized in static pressure, as seen from Figs.2 and 3. Moreover from Fig.7 we can decide material constants based on the Denn model as \( s = 1.41 \), \( \tau = 0.26 \cdot e^{-1} \), \( n = 10.0, \mu = 0.01 \text{ cm/s}^2 \). Then Eq. (45) becomes, if we take \( K = 0.026 \).

\[ \lambda = \frac{64}{R_e} + 0.026 \]  

(46)

and this equation shows approximate agreement with the experimental data as seen in Fig.9. In the previous paper (4), the authors found the expression of \( \lambda \) experimentally as

![Fig.9 Resistance coefficient (\( \lambda \)) vs. Reynolds number (\( R_e \)) for 5 ppm PEO solution](image)

![Fig.10 Resistance coefficient (\( \lambda \)) vs. Reynolds number (\( R_e \)) for various PEO solutions. See Ref. (1) for the data of 0.132 cm and 0.256 cm diameters and Ref. (5) for the data of 2.51 cm - 4.125 cm diameters.](image)
6. Conclusions

Transitional region of pipe flows of dilute polymer solutions is investigated. 
(1) Experiment is made with various HEC (hydroxyethyl cellulose) water solutions and the following results are obtained. Namely, anomaly of the resistance coefficient is very small in the flow region before transition, though it is comparatively large for FEO solutions as reported in the previous paper. In this flow region a kind of fluctuations of static pressure is recognized and once the fluctuations occur, the appearance of issuing jet shows anomalous winding or crooking effect. These phenomena on the static pressure and the jet appearance are similar to those of FEO solutions.

(2) An approximate analysis is made on the anomaly of the resistance coefficient shown in the flow region before transition. That is, the Derr model is adopted as the rheological equation of state and used simultaneously with the equations of motion, the velocity fluctuations which satisfy the continuity equation are added to the main flow, and the value of the resistance coefficient is obtained. The results show that the resistance coefficient becomes larger than that of Poiseuille flow and that if Reynolds number and the fluctuation do not change, the rate of its increase enlarges as Weissenberg number increases. Comparison between analytical results and experimental data shows approximate agreement.

References